Dipole approach with Pomeron loops: asymptotic solution

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Low x WS, Sinaia, Romania, June 29 - July 2, 2005

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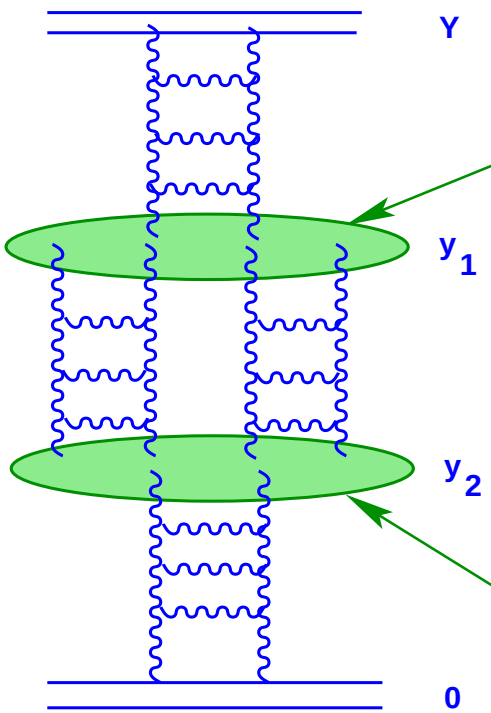
Outline:

- **Main ideas, tools and assumptions;**
- **Generating functional with Pomeron loops and Markov's processes;**
- **Asymptotic solution for the toy-model;**
- **General solution;**
- **The high energy asymptotic behaviour:**
  - **NO geometric scaling;**
  - **NO black disc;**
- **Problems, ideas, solutions ...** $\equiv$ **bright future;**

Ideas, tools and assumptions

Ideas:

- **BFKL Pomeron Calculus**



Γ (1 –> 2)

In Reggeon Calculus

Γ (1 –> 2) = Γ (2–> 1)

In probabilistic interpretation we need to find a correct normalization for Γ(2–> 1)

Γ (2 –> 1)

(Salam, Mueller & Salam,
Iancu & Mueller,
Mueller, Munier & Shoshi,
Iancy & Triantafyllopoulos,
Mueller, Shoshi & Wong,
E.L & Lublinsky,
Kovner & Lublinsky,
Rembiesta & Stasto,
.....)

- **Probabilistic interpretation:**

  " Reggeon field theory is equivalent to a chemical process where a radical can undergo diffusion, absorption, recombination, and autocatalytic production. Physically, these "radicals" are wee partons (colour dipoles)."

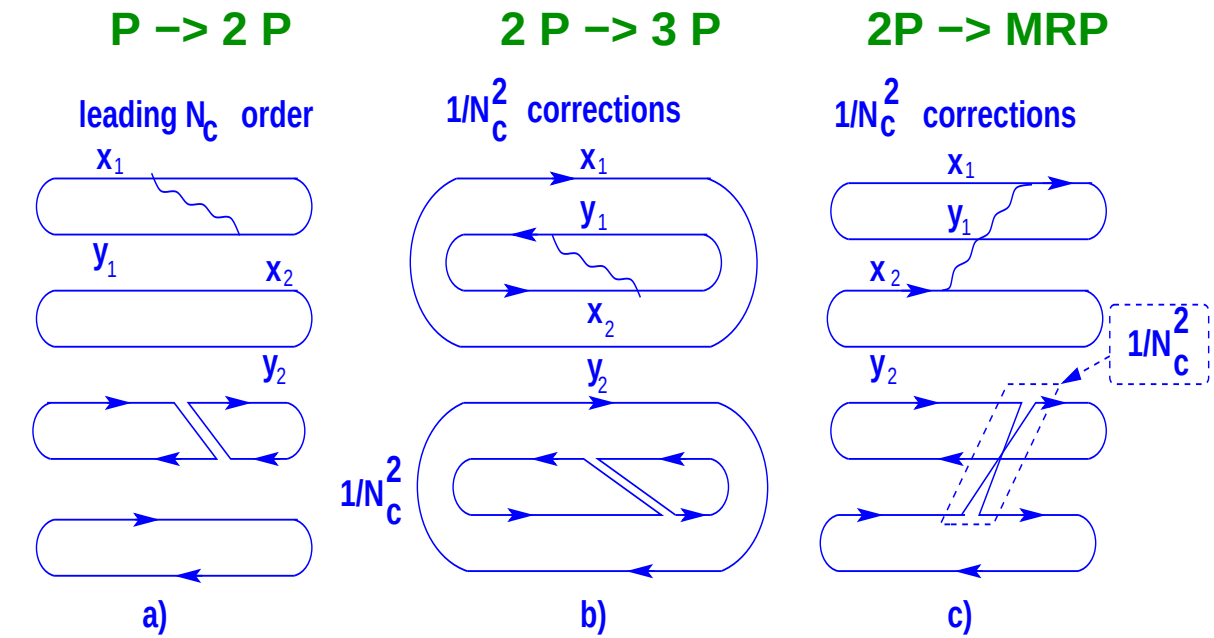
  (P. Grassberger & K. Sundermeyer, 1978: "Reggeon Field Theory and Markov processes")

- **Colour dipoles:**

  - **the wee partons of the BFKL Pomeron;**
  - **the correct degrees of freedom at high energy.**

Assumptions:

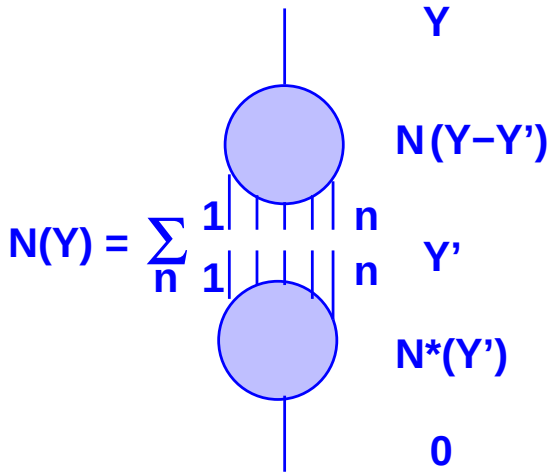
- **The intercept of multi (n) -Reggeon Pomeron is smaller than the intercept of n-BFKL Pomeron exchange**



1 BFKL Pomeron topology **2 BFKL Pomeron topology** **Multi Reggeon Pomeron**

$$\Delta_2 \gg \Delta_{MRP}$$

- ## t-channel unitarity constraints



Unitarity constraint:

$$N(\infty) = \sum_{n=1}^{\infty} \left(\frac{-1}{\kappa}\right)^n n!\, \rho_n^p(\infty)\ \rho_n^t(\infty)$$

$$\rho_n \equiv \partial^n Z / \partial u^n |_{u=1}$$

Tools:

- **Generating Functional** (Mueller-1994)

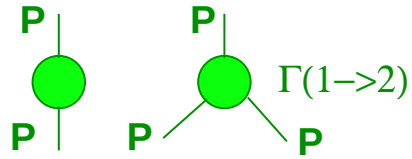
Generating Functional and Markov's processes

- $$Z(Y;[u]) = \sum_i \int P_n(...x_i, y_i...) \prod_{i=1}^{n} u(x_i, y_i)\, d^2x_i\; d^2\, y_i$$

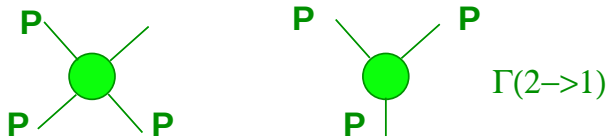
Initial conditions:

- $Z(Y, [u = 1]) = 1$;
- $Z(Y = Y_0, [u]) = u$;

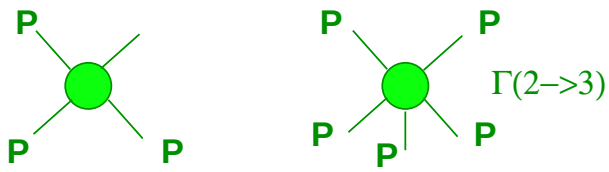
- $\frac{\partial P_n}{\partial Y} = -\sum_i \Gamma(1 \to 2) \otimes (P_n(...x_i, y_i...) - P_{n-1}(...x_i, y_i...))$



$-\sum_{i,k} \Gamma(2 \to 1) \otimes (P_n(...x_i, y_i, ..., x_k, y_k, ...) - P_{n+1}(...x_i, y_i, ..., x_k, y_k, ...))$



$-\sum_{i,k} \Gamma(2 \to 3) \otimes (P_n(...x_i, y_i, ..., x_k, y_k, ...) - P_{n-1}(...x_i, y_i, ..., x_k, y_k, ...))$



The typical death-birth process (Markov's chain)

$$\frac{\partial Z(Y;[u])}{\partial Y} = \chi[u]\, Z(Y;[u])$$

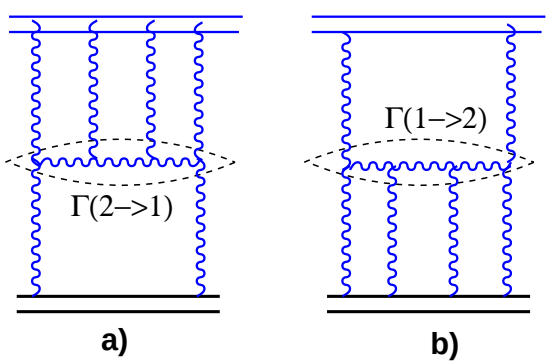
$$\chi[u] =$$

$$\Gamma(1 \to 2) \otimes (-u(x,y) + u(x,z)u(z,y))\, \frac{\delta}{\delta u(x,y)} +$$

$$\Gamma(2 \to 1) \otimes (u(x,y) - u(x_1,y_1)u(x_2,y_2))\, \frac{1}{2}\frac{\delta^2}{\delta u(x_1,y_1)\ \delta u(x_2,y_2)} +$$

$$\Gamma(2 \to 3) \otimes (1 - u(x_1,y_2))\,(u(x_2,y_1) - u(z,y_1)\, u(x_2,z))\frac{1}{2}\frac{\delta^2}{\delta u(x_1,y_1)\ \delta u(x_2,y_2)}$$

(Iancu & Triantafyllopoulos - 2004)



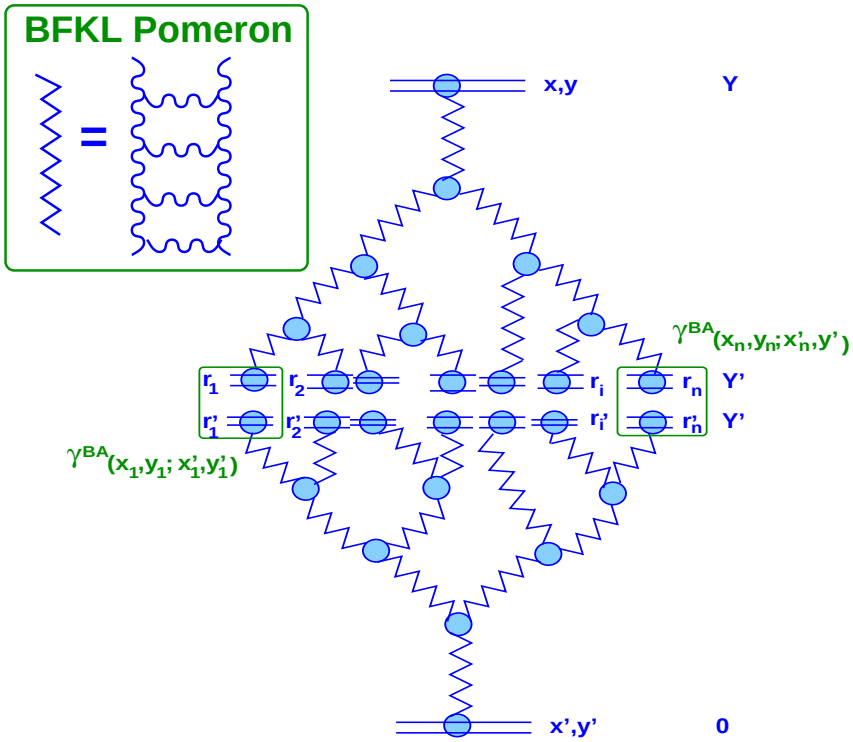
$$\Gamma_{2\to 1}\,(x_1, y_1 + x_2, y_2 \to x, y) = \frac{2N_c^2}{\bar{\alpha}_S^2}$$

$$\Delta_x \Delta_y \int d^2x_1'\, d^2y_1'\, d^2x_2'\, d^2y_2' \times \Gamma_{1\to 2}\,(x', y' \to x_1', y_1' + x_2', y_2')$$
$$\gamma^{BA}(x_1, y_1; x_1', y_1')\, \gamma^{BA}(x_2, y_2; x_2', y_2')$$

where

$$\gamma^{BA}(x, y; x', y') = \frac{\bar{\alpha}_S^2}{2\, N_c^2}\, \frac{1}{16\pi^2}\, \ln^2\left(\frac{(x-x')^2 (y-y')^2}{(x-y')^2\, (y-x')^2}\right)$$

Unitarity



$$\boxed{N(Y;[\gamma_i])} = -\int \sum_n \frac{(-1)^n}{n!} \prod_{i=1}^n d^2x_i\, d^2y_i\, \gamma(x_i,y_i)\, \frac{\delta}{\delta u_i} Z(Y,[u_i])|_{u_i=1}$$

$$= -\int \sum_n \frac{(-1)^n}{n!} \prod_{i=1}^n d^2x_i\, d^2y_i\, \gamma(x_i,y_i)\, \rho_n(x_1,y_1,\ldots,x_n,y_n;Y_0)$$

$$\boxed{N(x,y;x',y';Y)} = \sum_n (-1)^n\, n! \int \prod_{i=1}^n d^2x_i\, d^2y_i\, d^2x'_i\, d^2y'_i\, \gamma^{BA}(x_i,y_i;x'_i,y'_i)$$

$$\boxed{\rho_n^p(x,y;x_1,y_1,\ldots,x_n,y_n;Y-Y')} \times$$

$$\boxed{\rho_n^t(x',y';x'_1,y'_1,\ldots,x'_n,y'_n;Y')}$$

(Iancu & Mueller-03, Kozlov & E.L.-04, Lublinsky and E.L. -05)

Toy model:

$$\frac{\partial Z}{\partial Y} = -\Gamma(1 \to 2)\; u(1-u)\,\frac{\partial Z}{\partial u} +$$

$$\Gamma(2 \to 1)\; u(1-u)\;\frac{\partial^2 Z}{(\partial u)^2} + \Gamma(2 \to 3) u\,(1-u)^2 \frac{\partial^2 Z}{(\partial u)^2}$$

$$\Gamma(1 \to 2) \propto \bar{\alpha}_S\,; \quad \Gamma(2 \to 1) \propto \bar{\alpha}_S^3/N_c^2; \quad \Gamma(2 \to 3) \propto \bar{\alpha}_S/N_c^2;$$

$$\boxed{\frac{\Gamma(1\to 2)}{\Gamma(2\to 1)} \equiv \kappa \propto \frac{N_c^2}{\bar{\alpha}_S^2} \gg 1} \;; \quad \boxed{\frac{\Gamma(1\to 2)}{\Gamma(2\to 3)} \equiv \tilde{\kappa} \propto N_c^2 \gg 1}$$

Semi-classical approach :

Asymptotic solution:

Semi-classical approach:

- $$Z(Y=\infty;u) \equiv Z_{asymp}(u) = e^{\Phi(\infty;u)} \;;$$
- $$\frac{\partial^2 Z_{asymp}(u)}{(\partial u)^2} \Longrightarrow (\Phi'_u)^2 \; e^{\Phi(\infty;u)}$$

Equation:

- $$0 = -u(1-u)\,\Phi'_u(\infty;u) \; + \; u(1-u)\left(\frac{1}{\kappa} + \frac{1}{\tilde{\kappa}}\,u\right)\;(\Phi'_u(\infty;u))^2$$

Solution:

- $$\Phi(\infty, u) = - \tilde{\kappa} \ln\left(1 + \frac{\kappa}{\tilde{\kappa}}(1-u)\right)$$

- $$Z(\infty, u) = \left(1 + \frac{\kappa}{\tilde{\kappa}}(1-u)\right)^{-\tilde{\kappa}}$$

- $$\boxed{\rho_n = \frac{(-1)^n}{n!} \frac{\Gamma(\tilde{\kappa}+n)}{\Gamma(\tilde{\kappa})} \left(\frac{\kappa}{\tilde{\kappa}}\right)^n}$$

- $$\boxed{N(\infty)} = \sum_n \frac{1}{n!} \frac{\Gamma^2(\tilde{\kappa}+n)}{\Gamma^2(\tilde{\kappa})} \left(-\frac{1}{\kappa(1-u_0)^2}\right)^n \equiv$$

$$\equiv {}_2F_0\left(\tilde{\kappa}, \tilde{\kappa}, -\frac{1}{\kappa(1-u_0)^2}\right) <$$

$$< 1 - \exp\left(-\frac{1}{\kappa(1-u_0)^2}\right) + \exp\left(-\frac{1}{\kappa(1-u_0)^2}\right) Ei(-\kappa(1-u_0)^2)$$

with

$$1 - u_0 = -\frac{\tilde{\kappa}}{\kappa}$$

which can be found from

- $\Gamma(2 \to 1)\, u_0(1 - u_0) = -\Gamma(2 \to 3)\, u_0(1 - u_0)^2$

Approaching the asymptotic solution:

Assumptions:

$$\Phi(Y; u) = \Phi(\infty, u) + \Delta\Phi(Y, u) \text{ with } \Delta\Phi(Y, u) \ll \Phi(\infty, u)$$

Equation (Liouville equation) :

$$\bullet \quad \frac{\partial \Delta\Phi(\mathcal{Y}Y, u)}{\partial \mathcal{Y}} = - u(1-u)\frac{\partial \Delta\Phi(\mathcal{Y}, u)}{\partial u}$$

Initial conditions :

$$\bullet \quad \Delta\Phi(\mathcal{Y}=0, u) \; = \; \tilde{\kappa}\, \ln\left(1 + \frac{\kappa}{\tilde{\kappa}}(1-u)\right) \; + \; \ln u$$

Solution :

$$\bullet \quad Z(\mathcal{Y}; u) = \frac{u e^{\mathcal{Y}}}{1 - u + u\, e^{\mathcal{Y}}}\left(1 + \frac{\kappa}{\tilde{\kappa}}(1-u)\right)^{-\tilde{\kappa}} \left(1 + \frac{\kappa}{\tilde{\kappa}}\frac{1-u}{1-u+u\,e^{\mathcal{Y}}}\right)^{\tilde{\kappa}}$$

Lessons for searching a general solution:

- We can use semi-classical approach based on large $\kappa \gg 1$ and $\tilde{\kappa} \gg 1$;

- The first step is to find the asymptotic solution as the solution to the master equation with the l.h.s. equal to zero;

- The corrections to the asymptotic solution can be considered small and the linear Liouville equation can be written for them;

- The role of the $2 \longrightarrow 1$ process is very interesting: this process suppresses the small values of $1 - u \leq \tilde{\kappa}/\kappa$ in such a way that these $1 - u$ can be neglected;

- The inclusion of $\Gamma(2 \longrightarrow 3)$ is very essential and this vertex determines the high energy behaviour of the amplitude while $\Gamma(2 \longrightarrow 1)$ should be taken into account only for calculating the value of u_0 at which $Z(u = u_0; Y) = 1$;

General consideration :

Asymptotic solution for $\Gamma(2 \to 1) = 0$

- **Semi-classical approximation + guess for the form of the solution:**

$$\spadesuit \quad Z(Y; [u_i = 1 - \gamma_i]) \equiv \exp\left(\Phi(Y; [\gamma_i])\right);$$

$$\spadesuit \quad \frac{\delta^2 \Phi}{\delta\gamma(x_1, y_1)\delta\gamma(x_2, y_2)} \ll \frac{\delta\Phi}{\delta\gamma(x_1, y_1)}\frac{\delta\Phi}{\delta\gamma(x_2, y_2)}$$

$$\spadesuit \quad \Phi\left(\infty; [\gamma_i]\right) = -\tilde{\kappa}\ \ln\left(\frac{\int d^2x\, d^2y\, \Theta(x, y)\, \gamma(x, y)}{\int d^2x\, d^2\, \Theta(x, y)}\right)$$

$$= -\tilde{\kappa} \ln\left(\frac{\mathcal{F}}{N}\right)$$

- **Asymptotic equation:**

$$0 = -\tilde{\kappa} \int d^2x\, d^2y\, d^2z\, K(x,y;z)\, (\gamma(x,y) - \gamma(x,z)\,\gamma(z,y))\, \frac{\Theta(x,y)}{\mathcal{F}} -$$

$$- \tilde{\kappa} \int d^2x_1\, d^2y_1\, d^2x_2\, d^2y_2\, d^2z\, K(x_1,y_2;z) \times$$

$$\times\, (\gamma(x_1,y_2)\,\gamma(x_2,y_1) - \gamma(x_2,y_1)\,\gamma(x_1,z)\,\gamma(z,y_2))\, \frac{\Theta(x_1,y_1)}{\mathcal{F}}\, \frac{\Theta(x_2,y_2)}{\mathcal{F}}$$

with

$$K(x,y;z) = \frac{(x-y)^2}{(x-z)^2\,(z-y)^2}$$

Integrating over x_2 and y_1 and changing the notation (x_1, y_2) to (x, y) in the second term we see that this equation is satisfied

- **Initial condition:**

$$Z(\infty, [\gamma_i]) = \left(\frac{\int \; d^2x \, d^2y \; \Theta(x,y) \; \gamma(x,y)}{N} \right)^{-\tilde{\kappa}}$$

$Z(\infty, [\gamma = 0]) \; = 1$ **but,** $Z(\infty, [\gamma = 0])|_{\gamma \to 0} \Longrightarrow \infty$ **?!**

Including $\Gamma(2 \to 1)$ we have:

$Z(\infty, [\gamma = \gamma_0]) \; = 1$ **and** $Z(\infty, [\gamma_i]) = \left(\frac{\mathcal{F}([\gamma_i])}{\mathcal{F}([\gamma_0])} \right)^{-\tilde{\kappa}}$

- **Scattering amplitude:**

- $$\rho_n(x_1, y_1; \ldots, x_n, y_n; Y = \infty) = \prod_{i=1}^{n} \frac{(-1)^n}{n!} \frac{\Gamma(-\tilde{\kappa})}{\Gamma(-\tilde{\kappa} - n)} \frac{\Theta(x_i, y_i)}{\mathcal{F}([\gamma_{0,i}])}$$

$$\boxed{\text{N (x,y;x',y')}} =$$

$$_2F_0\left(\tilde{\kappa}, \tilde{\kappa}, -\frac{\int d^2\hat{x}\, d^2\hat{y}\, d^2\bar{x}\, d^2\bar{y}\, \Theta(\hat{x}, \hat{y})\, \Theta(\bar{x}, \bar{y})\, \gamma^{BA}(\hat{x}, \hat{y}; \bar{x}, \bar{y})}{\mathcal{F}^2([\gamma_{0,i}])}\right)$$

- $\gamma_0(x,y)$:

Equation:

$$\Gamma(2\to 1)\bigotimes\gamma_0(x,y)=\Gamma(2\to 3)\bigotimes\gamma_0(x_1,y_2)\;\gamma_0(x_2,y_1)$$

$$\frac{1}{\kappa}\int d^2x\,d^2y\;\gamma_0(x,y)\,\Delta_x\,\Delta_y\int d^2\,z\,K(x,y;z)\,\gamma^{BA}(x,z;x_1,y_1)\,\gamma^{BA}(z,y;x_2,y_2)=$$

$$=\frac{1}{2\tilde{\kappa}}\;\gamma_0(x_1,y_2)\;\gamma_0(x_2,y_1)\;\int d^2z\;\left(K(x_2,y_1;z)\;+\;K(x_1,y_2;z)\right)$$

Solution:

$$\gamma_0(x,y)=\frac{1}{16\pi^2}\frac{\tilde{\kappa}}{\kappa}\;\ln^2\frac{x^2}{y^2}$$

- **Asymptotic amplitude:**

$$\boxed{\text{N (x,y;x',y')}} = {}_2F_0\left(\tilde{\kappa}, \tilde{\kappa}, -\frac{32\,\kappa}{\tilde{\kappa}^2}\right) < 1 - e^{-\frac{32\kappa}{\tilde{\kappa}^2}}$$

Corrections to the asymptotic solution

$$\Phi(Y; [\gamma_i]) = \Phi(\infty; [\gamma_i]) + \Delta\Phi(Y; [\gamma_i])$$

- **Liouville equation for $\Delta\Phi$:**

$$\frac{\partial \Delta\Phi(Y;[\gamma_i])}{\partial \mathcal{Y}} =$$

$$-\int d^2x\, d^2y\, d^2z\, K(x,y;z)\, \left(\gamma(x,y) - \gamma(x,z)\,\gamma(z,y)\right) \frac{\delta \Delta\Phi(Y;[\gamma_i])}{\delta\gamma(x,y)} +$$

$$+\, 2\, \frac{1}{\mathcal{F}(\infty,[\gamma_{0,i}])} \int d^2x_1\, d^2y_1\, d^2x_2\, d^2y_2\, d^2\, K(x_1,y_2;z) \times$$

$$\left(\gamma(x_1,y_2)\,\gamma(x_2,y_1) - \gamma(x_2,y_1)\gamma(x_1,z)\,\gamma(z,y_2)\right) \frac{\delta \Delta\Phi(Y;[\gamma_i])}{\delta\gamma(x_1,y_1)}$$

- **Assumptions:**

- $\Delta\Phi(Y;[\gamma_i]) = \Delta\Phi([\gamma_i(\mathcal{Y},x,y)])$;

- $\gamma(x,y) = \gamma_0(x,y) \;+\; \Delta\gamma(x,y)$ **with** $\Delta\gamma \ll \gamma_0$.

- **Liouville equation for $\Delta\gamma$:**

$$\frac{\partial \Delta\gamma(\mathcal{Y}; x, y)}{\partial \mathcal{Y}} = - \int d^2 z \; K(x, y; z) \times$$
$$(\Delta\gamma(\mathcal{Y}; x, z) + \Delta\gamma(\mathcal{Y}; z, y) - \Delta\gamma(\mathcal{Y}; x, y))$$

- **Solution of the Liouville equation:**

$$\Delta\gamma(Y; x, y) = \Delta\gamma(Y = 0; x, y) \times$$
$$\exp\left(-\frac{1 - \gamma_{cr}}{2\,\chi(\gamma_{cr})} \; \ln^2((x - y)^2 \, Q_s^2(Y))\right)$$

- **Approaching the asymptotic solution :**

$$R \equiv \frac{N(x,y;x'y';\infty) - N(x,y;x'y';Y)}{N(x,y;x'y';\infty)} =$$

$$\propto \quad \exp\left(-\bar{\alpha}_S \frac{1-\gamma_{cr}}{8\chi(\gamma_{cr})} Y^2\right);$$

$$R_{I-M} \propto \quad \exp\left(-\bar{\alpha}_S \frac{1-\gamma_{cr}}{4\chi(\gamma_{cr})} Y^2\right);$$

$$R_{B-K} \propto \quad \exp\left(-\bar{\alpha}_S \frac{1-\gamma_{cr}}{2\chi(\gamma_{cr})} Y^2\right).$$

Main surprises of asymptotic solution

- The asymptotic solution is not unity. Therefore, the high energy scattering looks as a gray disc but not a black disc which was expected;

- We do not expect a geometrical scaling for our solution ;

- The process of dipole interaction $\Longrightarrow$ a typical Markov's process;

- Equations $\Longrightarrow$ new Monte Carlo code whith the Pomeron loops;

- **The entire dynamics can be found, neglecting the process of merging of two dipoles;**

- **We need $2 \longrightarrow 1$ process only to determine the value of γ_0 but it does not affect any qualitative properties of the asymptotic behaviour of the scattering amplitude at high energy.**

Bright future

- **We hope this talk is the convincing illustration that we are on the right track;**

- **Life becomes much more difficult but we overcome the simple non-linear equation for the mean field approximation;**

- **We have a remarkable progress in understanding, in solution, and in practical realization of high theoretical ideas.**