BFKL equation and anomalous dimensions in $N = 4$ SUSY

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1 High energy amplitudes

Elastic scattering

$$A + B \to A' + B'$$

Mandelstam invariants

$$s = (p_A + p_B)^2 \,,\; t = (p_A - p_{A'})^2 \,,\; u = (p_A - p_{B'})^2 \,,$$

$$s + t + u = 4m^2 \,,\; p_A^2 = p_B^2 = p_{A'}^2 = p_{B'}^2 = m^2$$

Regge kinematics in the s-channel

$$s = 4E^2 \gg -t = |q|^2 \approx E^2\,\theta^2 \,,\; q = p_A - p_{A'}$$

Sudakov decomposition of q

$$q = q_\perp + \frac{m^2 + |q|^2}{s}(p_A - p_B) \approx q_\perp$$

Impact parameter representation

$$A(s,t) = -2is \int d^2\rho\, e^{i\vec{q}\vec{\rho}} \left(e^{i\delta(s,\vec{\rho})} - 1\right) \,,\; \sigma_t = \frac{\Im A(s,0)}{s}$$

Total and elastic cross-sections

$$\sigma_t = 2\int d^2\rho\, \Re\left(1 - e^{i\delta(s,\vec{\rho})}\right) \,,\; \sigma_{el} = \int d^2\rho\, |1 - e^{i\delta(s,\vec{\rho})}|^2$$

2 Regge model

Mellin representation

$$A_p(s,t) = \int_{a-i\infty}^{a+i\infty} \frac{d\,j}{2\pi i}\, \xi_p(j)\, f_p(j,t)\, s^j \ , \quad a > 0$$

Signature factor

$$\xi_p(j) = -\frac{e^{-i\pi j} + p}{\sin\, \pi j} \ , \ p = \pm 1$$

Regge poles in t-channel partial waves

$$f_p(j,t) = \frac{\gamma_A(t)\, \gamma_B(t)}{j - j_p(t)}$$

Linear Regge trajectories

$$j_p(t) = j_p(0) + \alpha'_p t \ , \ j_p(m_r^2) = 2r + \frac{1-p}{2} \ , \ (r = 0, 1, ...)$$

Pomeranchuk trajectory

$$j_P(t) = 1 + \Delta_P + \alpha'_P t \ , \ p = 1 \ , \ \Delta_P \ll 1$$

Pomeron contributions

$$A_P(s,t) \approx i\, \gamma_A(t)\, \gamma_B(t)\, s^{j_P(t)} \ , \ \sigma_t^{AB} = \gamma_A(0)\, \gamma_B(0)\, s^{\Delta}$$

3 Production amplitudes

Inelastic scattering

$$A + B \to A' + B' + d_1 + d_2 + ... + d_n$$

Multi-Regge kinematics

$$s \gg s_r = (k_{r-1} + k_r)^2 \gg |q_r|^2 \,, \;\; k_r = q_r - q_{r+1}$$

Sudakov decomposition of particle momenta

$$k_r = k_r^{\perp} + \beta_r \, p_A + \alpha_r \, p_B \,, \;\; \beta_r = \frac{k_r p_B}{p_A p_B} \,, \;\; \alpha_r = \frac{k_r p_A}{p_A p_B}$$

Transversality of momentum transfers

$$q_r = q_r^{\perp} + \beta_r p_A - \alpha_{r-1} p_B \,, \;\; (q_r)^2 = -|q_r|^2$$

Kinematical constraint

$$\prod_{r=1}^{n+1} s_r = s \prod_{r=1}^{n} \left(m_r^2 + |k_r|^2\right)$$

One particle production amplitude

$$M_{2\to 3} = (-s)^{j_2} (-s_1)^{j_1 - j_2} f_L + (-s)^{j_1} (-s_2)^{j_2 - j_1} f_R + ...$$

4 Gluon reggeization

QCD Born amplitude in he Regge limit

$$M_{AB}^{A'B'}(s,t)|_{Born} = g\,T^c_{A'A}\,\delta_{\lambda_{A'}\lambda_A}\,\frac{2s}{t}\,g\,T^c_{B'B}\,\delta_{\lambda_{B'}\lambda_B}$$

Leading logarithmic approximation

$$M_{AB}^{A'B'}(s,t) = M_{AB}^{A'B'}(s,t)|_{Born}\,s^{\omega(t)},$$

$$\alpha_s \ln s \sim 1\,,\ \alpha_s = \frac{g^2}{4\pi} \ll 1$$

Gluon Regge trajectory in LLA

$$\omega(-|q|^2) = -\frac{\alpha_s N_c}{4\pi^2}\int d^2k\,\frac{|q|^2}{|k|^2|q-k|^2} \approx -\frac{\alpha_s N_c}{2\pi}\,\ln\frac{|q^2|}{\lambda^2}$$

Gluon production amplitude in LLA

$$M_{2\to 1+n} \sim \frac{s_1^{\omega_1}}{|q_1|^2}\,gT^{d_1}_{c_2c_1}C(q_2,q_1)\,\frac{s_2^{\omega_2}}{|q_2|^2}...C(q_n,q_{n-1})\,\frac{s_n^{\omega_n}}{|q_n|^2}$$

Multi-Regge kinematics

$$s \gg s_r = (k_{r-1}+k_r)^2 \gg |q_r|^2\,,\ C(q_{r+1},q_r) = \frac{q_{r+1}q_r^*}{k_r^*}$$

5 BFKL equation (1975)

Impact parameter coordinates and momenta

$$\rho_k = x_k + iy_k\,,\ \rho_k^* = x_k - iy_k\,,\ p_k = i\frac{\partial}{\partial\rho_k}\,,\ p_k^* = i\frac{\partial}{\partial\rho_k^*}$$

Balitsky-Fadin-Kuraev-Lipatov equation

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\,,\ \Delta = -\frac{\alpha_s N_c}{2\pi}\,E$$

BFKL Hamiltonian

$$H_{12} = \ln\,|p_1 p_2|^2 + \frac{1}{p_1 p_2^*}(\ln|\rho_{12}|^2)p_1 p_2^*$$

$$+\frac{1}{p_1^* p_2}(\ln|\rho_{12}|^2)p_1^* p_2 - 4\psi(1)\,,\ \rho_{12} = \rho_1 - \rho_2$$

Möbius invariance and conformal weights (L. (1986))

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d}\,,$$

$$m = \gamma + n/2\,,\ \widetilde{m} = \gamma - n/2\,,\ \gamma = 1/2 + i\nu$$

Bartels-Kwiecinski-Praszalowicz equation (1980)

$$E\,\Psi(\vec{\rho}_1,...,\vec{\rho}_n) = H\,\Psi(\vec{\rho}_1,...,\vec{\rho}_n)\,,\ H = \sum_{k<l}\frac{\vec{T}_k\vec{T}_l}{-N_c}\,H_{kl}$$

6 Integrability of BKP model

Holomorphic factorization for $N_c \to \infty$ (L. (1988))

$$\Psi(\vec{\rho}_1, \vec{\rho}_2, ..., \vec{\rho}_n) = \sum_{r,s} a_{r,s}\, \Psi_r(\rho_1, ..., \rho_n)\, \Psi_s(\rho_1^*, ..., \rho_n^*)$$

Duality symmetry (L. (1998))

$$\rho_{r,r+1} \to p_r \to \rho_{r-1,r}$$

Simplest integral of motion

$$A = q_n = \rho_{12}\rho_{23}...\rho_{n1}\, p_1 p_2 ... p_n\,, \;\; [h, A] = 0$$

Transfer and monodromy matrices (L. (1993))

$$T(u) = tr\, t(u)\,, \;\; t(u) = L_1 L_2 ... L_n = \sum_{r=0}^{n} u^{n-r}\, q_r\,,$$

$$L_k = \begin{pmatrix} u + \rho_k\, p_k & p_k \\ -\rho_k^2\, p_k & u - \rho_k\, p_k \end{pmatrix}\,, \;\; \hat{l} = u\,\hat{1} + i\,\hat{P}$$

Yang-Baxter equation (L. (1993))

$$t^{s_1}_{r'_1}(u)\, t^{s_2}_{r'_2}(v)\, l^{r'_1 r'_2}_{r_1 r_2}(v-u) = l^{s_1 s_2}_{s'_1 s'_2}(v-u)\, t^{s'_2}_{r_2}(v)\, t^{s'_1}_{r_1}(u)$$

7 Effective action (1995)

Effective action local in rapidity y

$$S = \int d^4x \left(L_0 + L_{ind}\right) \ , \ |y - y_0| < \eta \, , \ y = \frac{1}{2} \ln \frac{\epsilon_k + |k|}{\epsilon_k - |k|}$$

QCD Lagrangian

$$L_0 = i\bar{\psi}\hat{D}\psi + \frac{1}{2} Tr\, G^2_{\mu\nu}, \ D_\mu = \partial_\mu + g v_\mu, \ G_{\mu\nu} = \frac{1}{g}[D_\mu, D_\nu]$$

Induced contribution

$$L_{ind} = L^k_{ind} + L^{GR}_{ind} \, , \ L^k_{ind} = -\partial_\mu A^a_+ \partial_\mu A^a_- , \ \partial_\mp A_\pm(x) = 0$$

Gluon-Reggeon couplings

$$L^{GR}_{ind} = -\frac{1}{g}\partial_+ \, P \exp\left(-\frac{1}{2}\int_{-\infty}^{x^+} v_+(x')d(x')^+\right) \partial^2_\sigma A_- -$$

$$\frac{1}{g}\partial_- \, P \exp\left(-\frac{1}{2}\int_{-\infty}^{x^-} v_-(x')d(x')^+\right) \partial^2_\sigma A_+ =$$

$$Tr\left(v_+ - g v_+ \frac{1}{\partial_+} v_+ + g^2 v_+ \frac{1}{\partial_+} v_+ \frac{1}{\partial_+} v_+ - ...\right) \partial^2_\sigma A_-$$

$$+ \left(v_- - g v_- \frac{1}{\partial_-} v_- + g^2 v_- \frac{1}{\partial_-} v_- \frac{1}{\partial_-} v_- - ...\right) \partial^2_+ A_+$$

8 AdS/CFT correspondence

Maldacenna guess for $N = 4$ SUSY

$$YM\,(D=4) \;=\; String\,(D=10)$$

String compactification

$$\Omega = AdS_5 \times S_5$$

Anti-de Sitter space

$$-X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = -R^2$$

5-sphere

$$X_5^2 + X_6^2 + X_7^2 + X_8^2 + X_9^2 + X_{10}^2 = R^2$$

Polyakov correspondence

$$E^2|_{D=10} = (j+\Gamma)^2 - 4\,|_{D=4}\,,\;\; \Gamma = -2\gamma$$

Anomalous dimensions of local operators

$$\frac{\partial}{\partial \ln(Q^2)} < |O_i^j| >= \gamma_{ik}^j < |O_k^j| >\,,\;\; \alpha(Q^2) = const$$

9 Pomeron in $N = 4$ SUSY

BFKL kernel in two loops (F., L. (1998))

$$\omega = 4\,\hat{a}\;\chi(n,\gamma) + 4\;\hat{a}^2\,\Delta(n,\gamma)\,,\;\; \hat{a} = g^2 N_c/(16\pi^2)\,,$$

$$\chi(n,\gamma) = 2\Psi(1) - \Psi(\gamma + |n|/2) - \Psi(1 - \gamma + |n|/2)$$

Non-analytic terms in QCD (K.,L. (2000))

$$\Delta_{QCD}(n,\gamma) = c_0\delta_{n,0} + c_2\delta_{n,2} + ...$$

Hermitian separability in $N = 4$ SUSY (K.,L. (2000))

$$\Delta(n,\gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M) + \rho(M^*)}{2\hat{a}/\omega}\,,\; M = \gamma + \frac{|n|}{2}\,,$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2)\,,\; \beta'(z) = \frac{1}{4}\left[\Psi'\left(\frac{z+1}{2}\right) - \Psi'\left(\frac{z}{2}\right)\right]$$

Maximal transcedentality (K.,L. (2002))

$$\phi(M) = 3\zeta(3) + \Psi''(M) - 2\Phi(M) + 2\beta'(M)\Big(\Psi(1) - \Psi(M)\Big)$$

$$\Phi(M) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+M}\left(\Psi'(k+1) \;-\; \frac{\Psi(k+1) - \Psi(1)}{k+M}\right)$$

10 Pomeron and graviton

BFKL Pomeron in a diffusion approximation

$$j = 2 - \Delta - D\,\nu^2$$

Anomalous dimension of twist-2 operators

$$\gamma = 1 + \frac{j-2}{2} + i\nu$$

Constraint from the conservation of $T_{\mu\nu}$

$$\gamma = (j-2)\left(\frac{1}{2} - \frac{1/\Delta}{1+\sqrt{1+(j-2)/\Delta}}\right)$$

AdS/CFT for the graviton Regge trajectry

$$j = 2 + \frac{\alpha'}{2}\,t\,,\ t = E^2/R^2\,,\ \alpha' = \frac{R^2}{2}\,\Delta$$

Gubser, Klebanov and Polyakov prediction

$$\gamma_{|z,j\to\infty} = -\sqrt{j-2}\,\Delta^{-1/2}_{|j\to\infty} = \sqrt{\pi j}\,z^{1/4}$$

Pomeron intercept at large α (K.,L.,O.,V.)

$$j = 2 - \Delta\,,\ \Delta = \frac{1}{\pi}\,z^{-1/2} \approx \frac{\sqrt{3}}{2\pi}z^{-1/2}\,,$$

See also: Polchinsky, Strassler, Brower, C-I Tan

11 Anomalous dimensions

Parton distributions

$$f_a(x,Q^2) \;=\; \int_{k_\perp^2<Q^2} dk_\perp^2\, \varphi_a(x,k_\perp^2)\;,\;\; a=g,q,s$$

Mellin transformations

$$f_a(j,Q^2) = \int_0^1 d\,x\; x^{j-1}\, f_a(x,Q^2)$$

DGLAP equation and anomalous dimensions

$$\frac{d}{d\ln Q^2} f_a(j,Q^2) = \sum_b \gamma_{ab}(j) f_b(j,Q^2)$$

Light-cone components

$$O^a = \tilde{n}^{\mu_1}...\tilde{n}^{\mu_j}\, O^a_{\mu_1,...,\mu_j}\;,\;\; \tilde{\mathcal{O}}^a = \tilde{n}^{\mu_1}...\tilde{n}^{\mu_j}\, \tilde{\mathcal{O}}^a_{\mu_1,...,\mu_j}$$

Mixed projections

$$\tilde{n}^{\mu_1}...\tilde{n}^{\mu_{1+\omega}}\, O^a_{\mu_1,...,\mu_{1+\omega},\sigma_1,...,\sigma_{|n|}}\, l_\perp^{\sigma_1}...l_\perp^{\sigma_{|n|}}$$

Eigenvalues of the BFKL equation

$$\omega = j-1 = \omega(\gamma\,,|n|)\,,\;\; m=\gamma+\frac{n}{2}\,,\; \tilde{m}=\gamma-\frac{n}{2}$$

12 One-loop anomalous dimension in $N = 4$ SUSY

Anomalous dimension matrix (L. (1997))

$$\gamma_{gg} = -\frac{8}{j-1} + \frac{8}{j} - \frac{8}{j+1} + \frac{8}{j+2} + 8S_1(j)\,,\ \gamma_{q\varphi} = -\frac{16}{j}\,,$$

$$\gamma_{gq} = -\frac{8}{j-1} + \frac{8}{j} - \frac{4}{j+1}\,,\ \gamma_{g\varphi} = -\frac{8}{j-1} + \frac{8}{j}\,,$$

$$\gamma_{qg} = -\frac{16}{j} + \frac{32}{j+1} - \frac{32}{j+2}\,,\ \gamma_{qq} = -\frac{16}{j} + \frac{16}{j+1} + 8S_1(j),$$

$$\gamma_{\varphi g} = -\frac{24}{j+1} + \frac{24}{j+2}\,,\ \gamma_{\varphi q} = -\frac{12}{j+1}\,,\ \gamma_{\varphi\varphi} = 8S_1(j),$$

$$\widetilde{\gamma}_{gg} = -\frac{16}{j} + \frac{16}{j+1} + 8S_1(j)\,,\ \widetilde{\gamma}_{gq} = -\frac{8}{j} + \frac{4}{j+1}\,,$$

$$\widetilde{\gamma}_{qg} = \frac{16}{j} - \frac{32}{j+1}\,,\ \widetilde{\gamma}_{qq} = \frac{8}{j} - \frac{8}{j+1} + 8S_1(j)$$

Diagonalization in the Born Approximation

$$\begin{vmatrix} S_1(j-2) & 0 & 0 \\ 0 & S_1(j) & 0 \\ 0 & 0 & S_1(j+2) \end{vmatrix}, \begin{vmatrix} S_1(j-1) & 0 \\ 0 & S_1(j+1) \end{vmatrix}$$

Integrable Heisenberg spin model (L. (1997))

13 Maximal transcedentality

Most transcendental functions (K.,L. (2002))

$$\gamma(j) = \hat{a}\gamma_1(j)+\hat{a}^2\gamma_2(j)+\hat{a}^3\gamma_3(j)+\ldots\,,\;\; \gamma_1(j+2) = -4S_1(j)$$

Two-loop dimension (K.,L.,V. (2003))

$$\frac{\gamma_2(j+2)}{8} = 2S_1\left(S_2+S_{-2}\right) - 2S_{-2,1} + S_3 + S_{-3}$$

Three-loop dimension (K.,L.,O.,V. (2004))

$$\gamma_3(j+2)/32 = -12\,\left(S_{-3,1,1}+S_{-2,1,2}+S_{-2,2,1}\right)$$

$$+6\,\left(S_{-4,1}+S_{-3,2}+S_{-2,3}\right) - 3\,S_{-5} - 2\,S_3\,S_{-2} - S_5$$

$$-2\,S_1^2\left(3\,S_{-3}+S_3-2\,S_{-2,1}\right) - S_2\left(S_{-3}+S_3-2\,S_{-2,1}\right)$$

$$+24\,S_{-2,1,1,1} - S_1\left(8S_{-4}+S_{-2}^2+4S_2S_{-2}+2S_2^2\right)$$

$$-S_1\left(3S_4-12S_{-3,1}-10S_{-2,2}+16S_{-2,1,1}\right)\,,$$

$$S_a(j) = \sum_{m=1}^{j}\frac{1}{m^a}\,,\;\; S_{a,b,c,\ldots}(j) = \sum_{m=1}^{j}\frac{1}{m^a}\,S_{b,c,\ldots}(m)\,,$$

$$S_{-a}(j) = \sum_{m=1}^{j}\frac{(-1)^m}{m^a},\; S_{-a,b,\ldots}(j) = \sum_{m=1}^{j}\frac{(-1)^m}{m^a}\,S_{b,\ldots}(m),$$

$$\overline{S}_{-a,b,c\ldots}(j) = (-1)^j S_{-a,b,\ldots}(j)+S_{-a,b,\ldots}(\infty)\Big(1-(-1)^j\Big)$$

14 Special cases

Singularities at $j = 1 + \omega \to 1$

$$\gamma_{uni}^{N=4}(j) = \hat{a}\frac{4}{\omega} + 0\hat{a}^2 + 32\zeta_3\hat{a}^3\frac{1}{\omega} - \frac{16\hat{a}^4}{\omega^4}\left(32\zeta_3 + \frac{\pi^4}{9}\omega\right)$$

DL resummation at $j + 2r = \omega \to 0$

$$\omega\gamma_{uni} = \gamma_{uni}^2 + 16\hat{a}^2(S_2 + \zeta_2 - S_1^2)$$

$$+4\,\hat{a}\left(1 - \omega S_1 - \omega^2(S_2 + \zeta_2) + \gamma^2(S_2 + S_{-2})\right)$$

Anomalous dimensions at large j

$$\gamma_{uni} = a(z)\,\ln j\,,\;\; z = \frac{\alpha N_c}{\pi} = 4\hat{a}$$

Perturbative results

$$a = -z + \frac{\pi^2}{12}\,z^2 - \frac{11}{720}\pi^4 z^3 + ...$$

Gubser, Klebanov, Polyakov prediction

$$\lim_{z\to\infty} a = -z^{1/2} + \frac{3\ln 2}{4\pi} + ...$$

Resummation

$$\widetilde{a} = -z + \frac{\pi^2}{12}\,\widetilde{a}^2 = -z + \frac{\pi^2}{12}\,z^2 - \frac{1}{72}\pi^4 z^3 + ...$$

15 Eden-Staudacher approach

Linear set of ES equations for $a(z)$ (K.,L. (2006))

$$a_{n,\epsilon} = \sum_{n'=1}^{\infty} K_{n,n'}(\epsilon)\ (\delta_{n',1} - a_{n',\epsilon})\ ,\ a(z) = \frac{2(1-a_{1,\epsilon})}{\epsilon^2},$$

$$\epsilon = \frac{1}{g\sqrt{2}}\ ,\ K_{n,n'}(\epsilon) = 2n \sum_{R=0}^{\infty} (-1)^R \frac{2^{-2R-n-n'}}{\epsilon^{2R+n+n'}}$$

$$\zeta(2R+n+n') \frac{(2R+n+n'-1)!\,(2R+n+n')!}{R!\,(R+n)!\,(R+n')!\,(R+n+n')!}$$

Transcedentality with integer coefficients

$$\gamma(\epsilon) = 8 \sum_{k=1}^{\infty} \left(-\frac{1}{4\epsilon^2}\right)^k \sum_{[s_t]} c_{[s_t]} \prod_r \zeta(s_r)\ ,\ \sum_t s_t = 2k-2$$

Another representation for $K_{n,n'}(\epsilon)$

$$\frac{\Gamma^2(\frac{n+n'+1}{2})\Gamma(\frac{n+n'}{2}+1)\Gamma(\frac{n+n'}{2})}{\pi\Gamma(n)\Gamma(n'+1)\Gamma(n+n'+1)\epsilon^{n+n'}} \sum_{k=1}^{\infty} \frac{2^{n+n'}}{k^{n+n'}} F_n^{n'}(-\frac{4}{k^2\epsilon^2})$$

Generalized hypergeometric function ($\bar{n} = \frac{n+n'}{2}$)

$$F_n^{n'}(x) =_4 F_3(\bar{n}+\frac{1}{2}, \bar{n}, \bar{n}+1, \bar{n}+\frac{1}{2}; n+1, n'+1, 2\bar{n}+1; x)$$

16 Strong coupling limit

Beisert-Eden-Staudacher equation

$$\left(\epsilon + K_0 + K_1 + \frac{2}{\epsilon} K_1 K_0\right) F(x) = \left(1 + \frac{2}{\epsilon} K_1\right) K_0 \delta(x)\,,$$

$$K_s F(x) = \int_0^\infty dx' \, \frac{t'}{e^{t'} - 1} \, K_s(x, x') \, F(x')\,,$$

$$K_s(x, y) = \frac{2}{xy} \sum_{r=1}^{\infty} (2r - 1 + s) \, J_{2r-1+s}(x) \, J_{2r-1+s}(y)$$

Essential singularity at $z = 0$ and $g \to \infty$

$$\chi_{sing}^{BES}(z) = \sum_{k=1}^{\infty} \frac{d_k}{z^k} = -\frac{i}{I_0(2\,\epsilon^{-1})} \int_L \frac{d\,z'}{2\pi i} \, \frac{\exp \frac{z'^2 - 1}{i\,\epsilon\,z'}}{z - z'}$$

Recurrent relations

$$\epsilon d_1 = 1 - i d_2\,, \;\; n\epsilon\, d_n = -i d_{n-1} - i d_{n+1}$$

Agreement with the AdS/CFT prediction (K.,L.)

$$\gamma_{sing} = \frac{2}{\epsilon} \, \frac{I_1(2\epsilon^{-1})}{I_0(2\epsilon^{-1})} \approx 2\sqrt{2}\, g - \frac{1}{2}$$

17 Meromorphic properties

BFKL prediction at $j = 1 + \omega \to 1$

$$\gamma(j) = 4\frac{\hat{a}}{\omega} + 0\frac{\hat{a}^2}{\omega} + 32\zeta_3\frac{\hat{a}^3}{\omega} - 16\left(32\zeta_3 + \frac{\pi^4}{9}\omega\right)\frac{\hat{a}^4}{\omega^4} + \ldots$$

Poles of $\gamma(j)$ at $j + r = \omega \to 0$

$$\frac{\gamma_1(j)}{4} = \frac{1}{\omega} - S_1(r+1) - \omega(S_2(r+1) + \zeta_2)) + \ldots,$$

$$\frac{\gamma_2(j)}{16} = \frac{1+(-1)^r}{2\omega^3} - S_1(r+1)\frac{1+(-1)^r}{\omega^2} - \frac{\zeta_2 + (-1)^r(\zeta_2 + 2S_2(r+1))}{2\omega} + \ldots,$$

$$\frac{\gamma_3(j)}{64} = \frac{1+(-1)^r}{\omega^5} - 2S_1(r+1)\frac{1+2(-1)^r}{\omega^4} - \frac{\frac{S_2}{2} - S_1^2 - S_{-2} + 2\zeta_2 + (-1)^r(\frac{5}{2}S_2 - 3S_1^2 + 2\zeta_2)}{\omega^3} + \ldots$$

Effective parameter of the expansion

$$x = \frac{\hat{a}}{\omega^2}$$

Different singularities for even and odd r

$$\gamma(j)_{j\to-(2k)} = \frac{\hat{a}}{\omega}\,a(x)\,,\ \gamma(j)_{j\to-(2k+1)} = \frac{4\hat{a}}{\omega} + O(\frac{\hat{a}^2}{\omega^2})$$

18 Double logarithms

Double-logarithmic approximation

$$A(s) = A_{Born}\, f(\rho)\,,\ \ f(\rho) = \sum_{k=0}^{\infty} c_k\, \rho^{2k}\,,\ \ \rho^2 = \frac{2\alpha}{\pi} \ln^2(s)$$

Example: $e\bar{e}$-backward scattering (GGLF, (1967))

$$f(\rho) = \int_{-i\infty}^{i\infty} \frac{dl}{2\pi i}\, e^{\rho\, l}\, \frac{d}{dl}\, \ln D_{-\frac{1}{4}}(l)\,,\ \ l = \frac{j}{\sqrt{2\alpha/\pi}}$$

Ansatz for ep structure functions

$$W(x, Q^2) = \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} x^{-\omega} \int_{-i\infty}^{i\infty} \left(\frac{Q^2}{\lambda^2}\right)^{\gamma} \phi_\omega(\gamma)$$

DL evolution equations in λ (KL (1986))

Positive signature in $N = 4$:

$$\gamma_+(\omega - \gamma_+) = 4\,\hat{a}$$

Negative signature in $N = 4$:

$$\gamma_-(\omega - \gamma_-) = 4\,\hat{a} - 8\frac{\hat{a}}{\omega}\gamma_8\,,$$

$$\gamma_8(\omega - \gamma_8) = 2\,\hat{a} + 2\hat{a}\frac{d}{d\omega}\gamma_8$$

19 Four-loop result (KLRSV)

$$\frac{\gamma_4}{256} = \mathbf{4S_{-7}} + \mathbf{6S_7} + 2(S_{-3,1,3} + S_{-3,2,2} + S_{-3,3,1} + S_{-2,4,1})$$
$$+3(-S_{-2,5} + S_{-2,3,-2}) + 4(S_{-2,1,4} - S_{-2,-2,-2,1} - S_{-2,1,2,-2}$$
$$-S_{-2,2,1,-2} - S_{1,-2,1,3} - S_{1,-2,2,2} - S_{1,-2,3,1}) + 5(-S_{-3,4} +$$
$$S_{-2,-2,-3}) + 6(-S_{5,-2} + S_{1,-2,4} - S_{-2,-2,1,-2} - S_{1,-2,-2,-2})$$
$$+7(-S_{-2,-5} + S_{-3,-2,-2} + S_{-2,-3,-2} + S_{-2,-2,3}) + 8(S_{-4,1,2}$$
$$+S_{-4,2,1} - S_{-5,-2} - S_{-4,3} - S_{-2,1,-2,-2} + S_{1,-2,1,1,-2}) +$$
$$9S_{3,-2,-2} - 10S_{1,-2,2,-2} + 11S_{-3,2,-2} + 12(S_{-2,2,-3} - S_{-6,1}$$
$$+S_{1,4,-2} + S_{4,-2,1} + S_{4,1,-2} - S_{-3,1,1,-2} - S_{-2,2,-2,1} -$$
$$S_{1,1,2,3} - S_{1,1,3,-2} - S_{1,1,3,2} - S_{1,2,1,3} - S_{1,2,2,-2} -$$
$$S_{1,2,2,2} - S_{1,2,3,1} - S_{1,3,1,-2} - S_{1,3,1,2} - S_{1,3,2,1} -$$
$$S_{2,-2,1,2} - S_{2,-2,2,1} - S_{2,1,1,3} - S_{2,1,2,-2} - S_{2,1,2,2} -$$
$$S_{2,1,3,1} - S_{2,2,1,-2} - S_{2,2,1,2} - S_{2,2,2,1} - S_{2,3,1,1} - S_{3,1,1,-2} -$$
$$S_{3,1,1,2} - S_{3,1,2,1} - S_{3,2,1,1}) + 13S_{2,-2,3} - 14S_{2,-2,1,-2}$$
$$+\ldots\ldots\ldots

20 Singularities in 4-loops

BFKL prediction for γ in 4 loops

$$\lim_{j\to 1}\gamma_4(j) = -16\left(32\zeta_3 + \frac{\pi^4}{9}\omega\right)\frac{\hat{a}^4}{\omega^4} + \ldots$$

Asymptotic Bethe anzatz result (KLRSV)

$$\lim_{j\to 1}\gamma_4(j) = -\frac{512}{\omega^7}$$

Dressing phase modification (Wrapping effect)

$$\zeta_3 \to \frac{47\zeta_3}{24} - \frac{S_{-3}}{4} + \frac{3S_{-2}S_1}{4} + \frac{3S_1S_2}{8} + \frac{3S_3}{8} + \frac{S_{-2,1}}{6} - \frac{17S_{2,1}}{24}$$

DL resummation at $j + 2r = \omega \to 0$

$$\gamma(j)(\omega - \gamma(j)) = 4\,\hat{a}\left(1 - \omega S_1 - \omega^2(S_2 + \zeta_2)\right)$$

$$+16\hat{a}^2(S_2 + \zeta_2 - S_1^2) + 4\,\hat{a}\,\gamma^2(j)(S_2 + S_{-2})$$

Agreement with the double-logarithmic prediction

$$\frac{1}{256}\,\gamma_4\big|_{j\to -2k+\omega} =$$

$$\frac{5}{\omega^7} - 20\frac{S_1}{\omega^6} + \frac{24\,S_1^2 - 14(S_2 + \zeta_2) + 4(S_2 + S_{-2})}{\omega^5} + \ldots$$

21 Discussion

1. Pomeron as a composite state of reggeized gluons.
2. Integrability of the BFKL dynamics.
3. Effective action for reggeized gluons.
4. Properties of the BFKL kernel in $N = 4$ SUSY.
5. Pomeron-graviton interplay.
6. Maximal transcedentality.
7. Beisert-Eden-Staudacher equation and AdS/CFT.
8. Meromorphic properties of γ.
9. BFKL predictions .
10. Double-logarithmic resummation.
11. Next-to-leading DL singularities.
12. Inconsistency of the four-loop result and wrapping.
13. Plausible guess for the anomalous dimension.