Solution to the BK equation in the saturation domain

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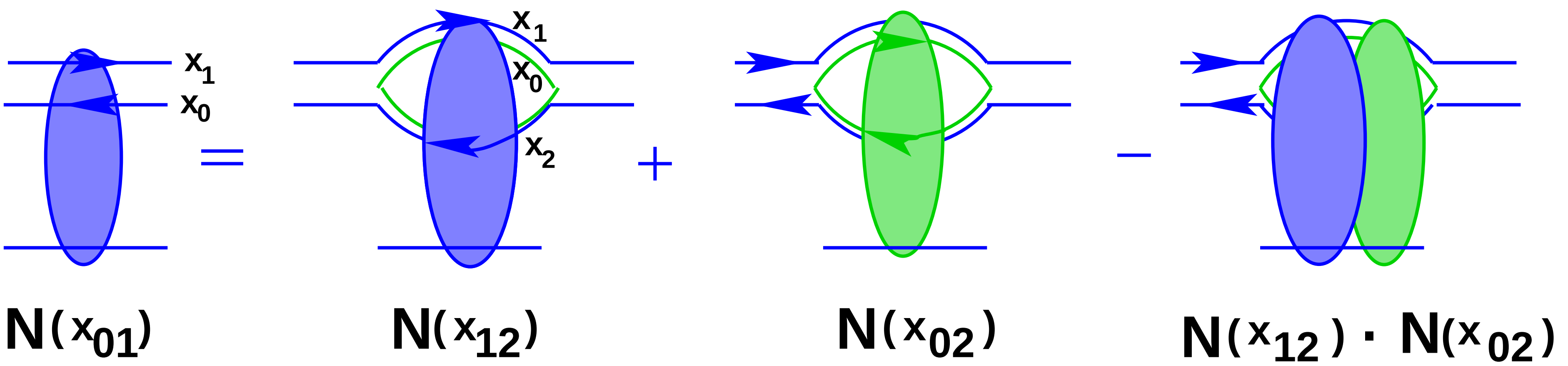
Goals:

1) Simple derivation of solution to BK equation in saturation domain.
2) Pitfalls in simplifications of the BFKL kernel in evolution equation.

Outline:

- BK equation;
- Mixed Representation, Unitarity Bound and Geometrical Scaling;
- Semi-classical approach, Two solutions;
- General Approach;
  (Simplifications of the BFKL kernel in evolution equation may lead to wrong predictions about asymptotic behavior of the scattering amplitude)
- Numerical check;
  (The new solution is solution of the non-linear homogeneous equation)
- Conclusions;

BK equation



$$\frac{\partial N(y,x_{01},b)}{\partial y} = \bar{\alpha}_S \int d^2x_2 \frac{x_{01}^2}{x_{02}^2\, x_{12}^2}\Big(2N(y,x_{12},b-\frac{1}{2}x_{02})$$

$$- N(y,x_{01},b) - N(y,x_{12},b-\frac{1}{2}x_{02})\,N(y,x_{02},b-\frac{1}{2}x_{12})\Big)$$

The physical basis for two solutions

LT Solution comes from the region $r' \sim r \gg 1/Q_s$

$\Longrightarrow$ BK reduce to the linearized evolution equation equation

LK Solution comes from the region $1/Q_s \sim r' \ll r$

$\Longrightarrow$ BK reduce to nonlinear homogeneous equation

Mixed representation of BK equation

$$N(r,y;b) \;=\; r^2 \int_0^\infty k\,dk\, J_0(k\,r)\;\tilde{N}(k,y;b)$$

$$\tilde{N}(k,y;b) \;=\; \int_0^\infty \frac{dr}{r}\, J_0(k\,r)\; N(r,y;b)$$

$$\boxed{\frac{\partial\,\tilde{N}(k,y;b)}{\partial y} \;=\; \bar{\alpha}_S\left(\chi(\hat{\gamma}(\xi))\,\tilde{N}(k,y;b) \;-\; \tilde{N}^2(k,y;b)\right)}$$

where

$$\chi\,(\hat{\gamma}) \;=\; 2\,\psi(1) - \psi(\hat{\gamma}) - \psi(1-\hat{\gamma})$$ **BFKL kernel**

and

$$\psi(\gamma) \;=\; \frac{d\ln\Gamma(\gamma)}{d\gamma}\,,$$ $\Gamma(\gamma)$ - **Euler gamma function**

$$\hat{\gamma}(\xi) \;=\; 1 \;+\; \frac{\partial}{\partial\xi}\,, \qquad \xi \;=\; \ln(k^2R^2)$$

Unitarity Bound

Due to s-channel unitarity constraint:

$$N \;\leq\; 1 \qquad \Longleftrightarrow \qquad \tilde{N} \;\leq\; \tfrac{1}{2}z:$$

Indeed,

$$\tilde{N}(k,y;b) \;=\; \int_0^\infty \frac{dr}{r}\, J_0(k\,r)\; N(r,y;b)$$

$$\approx\; \int_0^{\frac{1}{k}} \frac{dr}{r}\, N(r,y;b)$$

$$N = 1 \quad \forall \quad r > 1/Q_s$$

$$=\; \frac{1}{2}\int_0^{\ln(Q_s^2/k^2)} d\ln(1/r^2)$$

$$\leq\; \frac{1}{2}\ln(Q_s^2/k^2) \;=\; \frac{1}{2}z$$

Geometrical Scaling

- **Geometrical scaling behavior of the scattering amplitude:**

$$\tilde{N}(k, y; b) = \tilde{N}(z)$$

$$\hat{z} = \ln\left(Q_s^2(y, b)/k^2\right) = \bar{\alpha}_S \frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}(Y - Y_0) - \xi - \beta(b)$$

where

$$\gamma_{cr}: \qquad \frac{\chi(\gamma_{cr})}{1-\gamma_{cr}} = -\frac{d\chi(\gamma_{cr})}{d\gamma_{cr}}$$

- **BK equation in saturation:**

$$\bar{\alpha}_S \frac{\chi(\gamma_{cr})}{1-\gamma_{cr}} \frac{d\tilde{N}(\hat{z})}{d\hat{z}} \;=\; \bar{\alpha}_S \left(\chi(1 - f)\tilde{N}(\hat{z}) - \tilde{N}^2(\hat{z}) \right)$$

where

$$f \;\equiv\; d/d\hat{z} \;=\; -\,\partial/\partial\xi$$

- **We search solution for this equation in the form:**

$$\tilde{N}(\hat{z}) \;=\; \frac{1}{2} \int^{\hat{z}} d\, z' \left(1 \;-\; e^{-\phi(z')}\right)$$

Such $\tilde{N}(\hat{z})$ is satisfy two requirements:

– **Includes geometrical scaling behaviour:** $\tilde{N}(k, y; b) = \tilde{N}(z)$

– **Leads to asymptotic behaviour:** $\tilde{N} \;\longrightarrow\; \frac{1}{2}\, z.$

Semi-classical approach

- If ϕ is a smooth function: $\phi_{zz} \ll \phi_z \, \phi_z$

where $\qquad \phi_z \equiv d\phi/dz \quad$ and $\quad \phi_{zz} \equiv d^2\phi/(dz)^2$

then $\qquad \frac{d^n}{(dz)^n} e^{-\phi(z)} = (-\phi_z)^n \, e^{-\phi(z)}$

- BK equation in such approximation gets the form:

$$\boxed{\frac{\chi(\gamma_{cr})}{1-\gamma_{cr}} \frac{d^2\phi}{(d\hat{z})^2} = \left(1 - e^{-\phi(\hat{z})}\right) - \frac{dL(\phi_z)}{d\phi_z} \frac{d^2\phi}{(d\hat{z})^2}}$$

where $\qquad L(\phi_z) = \frac{\phi_z \, \chi(1-\phi_z) - 1}{\phi_z}$

$\frac{dL(\phi_z)}{d\phi_z}$ has double pole singularities in all integer points: $\phi_z = 1, 2, 3, \ldots$

Two solutions

- **Two analytical solutions of equation:**

$$\frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}\,\frac{d^2\phi}{(d\hat{z})^2} \;=\; \left(1 - e^{-\phi(\hat{z})}\right) - \frac{dL(\phi_z)}{d\phi_z}\,\frac{d^2\phi}{(d\hat{z})^2}$$

LT $\frac{dL(\phi_z)}{d\phi_z} \rightarrow 0$ **and** $e^{-\phi(\hat{z})} \rightarrow 0$

$$\frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}\,\frac{d^2\phi}{(d\hat{z})^2} \;=\; 1 \quad\Longrightarrow\quad \phi(\hat{z}) \;=\; \frac{1-\gamma_{cr}}{\chi(\gamma_{cr})}\,\frac{\hat{z}^2}{2}$$

LK $\frac{dL(\phi_z)}{d\,\phi_z} \rightarrow \frac{1}{(1-\phi_z)^2}$, $e^{-\phi(\hat{z})} \rightarrow 0$ **and** $\frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}\,\frac{d^2\phi}{(d\hat{z})^2} \rightarrow 0$

$$\frac{1}{(1-\phi_z)^2}\,\frac{d^2\phi}{(d\hat{z})^2} \;=\; 1 \quad\Longrightarrow\quad \phi(\hat{z}) \;=\; \hat{z} - \ln\hat{z}$$

• Precision of the new solution is: $\Delta\phi \propto 1/z^2$

indeed,

$$\frac{d^n}{(dz)^n}\, e^{-\phi(z)} \;=\; (-\phi_z)^n\, e^{-\phi(z)} \qquad \longrightarrow$$

$$\frac{d^n}{(dz)^n}\, e^{-\phi(z)} \;=\; \left((-\phi_z)^n \;-\; n\,(-\phi_z)^{n-1}\,\phi_{zz}\right)\; e^{-\phi(z)}$$

and

$$\frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}\,\frac{d^2\phi}{(d\hat{z})^2} \;=\; \left(1 - e^{-\phi(\hat{z})}\right) - \frac{dL(\phi_z)}{d\phi_z}\,\frac{d^2\phi}{(d\hat{z})^2} \qquad \longrightarrow$$

$$\frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}\,\frac{d^2\phi}{(d\hat{z})^2} = \left(1 \;-\; e^{-\phi(\hat{z})}\right) - \frac{d\,L(\phi_z)}{d\phi_z}\,\frac{d^2\,\phi}{(d\hat{z})^2} \;+\; \frac{d^2\,L(\phi_z)}{(d\,\phi_z)^2}\,\left(\frac{d^2\,\phi}{(d\hat{z})^2}\right)^2$$

$$\frac{1}{(1 \;-\; \phi_z\;)^2}\,\frac{d^2\phi}{(d\hat{z})^2} \;=\; 1 \qquad \longrightarrow$$

$$\frac{1}{(1 \;-\; \phi_z\;)^2}\,\frac{d^2\,\phi}{(d\hat{z})^2} \;=\; 1 + \frac{2}{\left(1 \;-\; \phi_{0,z}\right)^3}\left(\frac{d^2\,\phi_0}{(d\hat{z})^2}\right)^2$$

thus

$$\frac{d\Delta\phi_z}{d\hat{z}} \;=\; \frac{2}{\left(1 \;-\; \phi_{0,z}\right)}\left(\frac{d^2\,\phi_0}{(d\hat{z})^2}\right)^2 \;\rightarrow\; \frac{2}{\hat{z}^3} \qquad \Longrightarrow \qquad \Delta\phi \propto 1/z^2$$

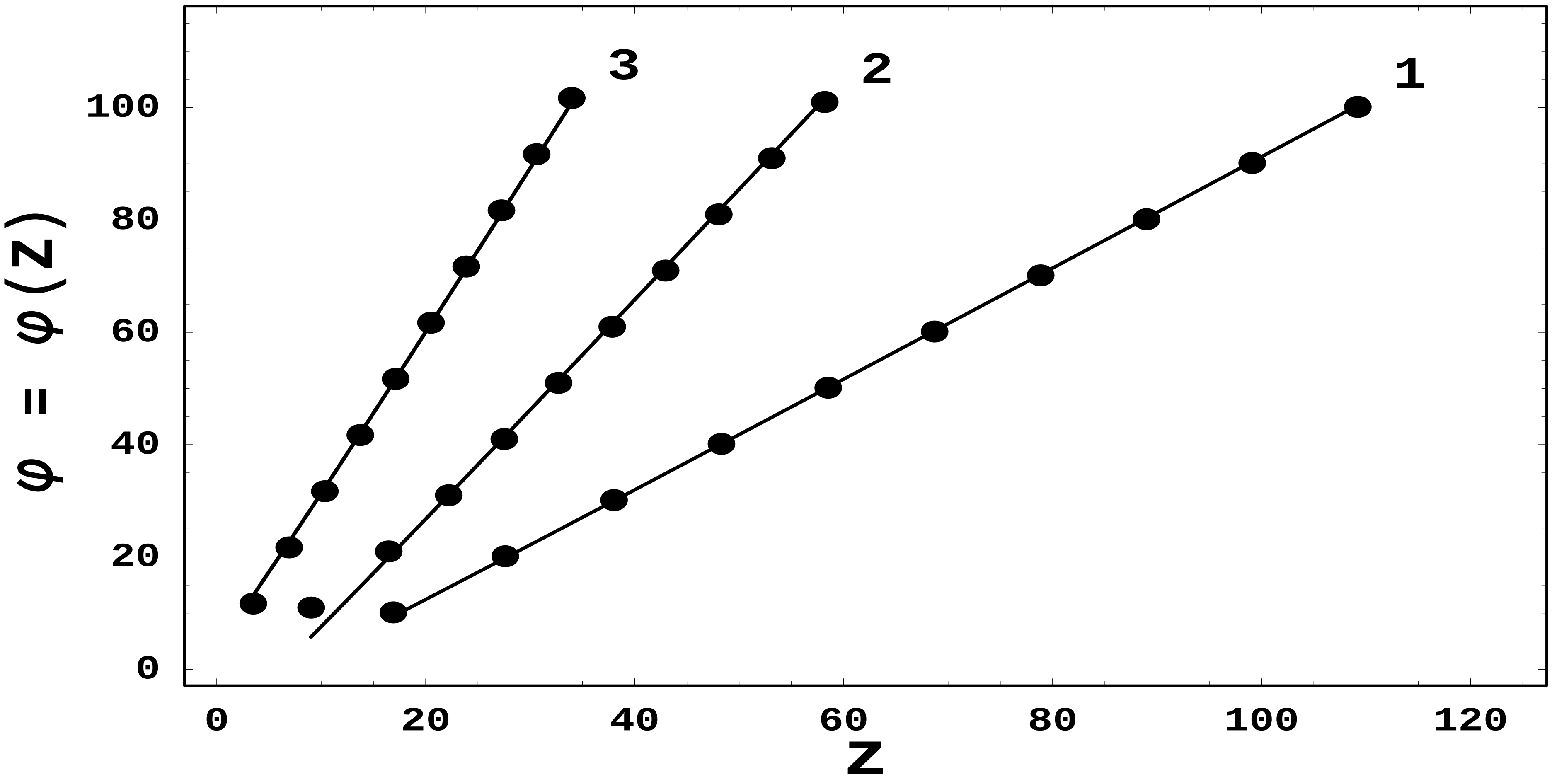
• Numerical validation of the new solution:

$$\frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}\,\frac{d^2\phi}{(d\hat{z})^2} \;=\; \left(1 - e^{-\phi(\hat{z})}\right) - \frac{dL(\phi_z)}{d\phi_z}\,\frac{d^2\phi}{(d\hat{z})^2}$$

Initial conditions for $\phi'_{z,0}$ and ϕ_0:

1 $$\frac{\frac{1}{2}\left(1-e^{-\phi_0}\right)}{N_0} \;=\; 1 - \gamma_{cr}$$

2 $$\phi'_{z,0}e^{-\phi_0} \;=\; (1 - \gamma_{cr})\,\left(1 - e^{-\phi_0}\right)$$



Points: The numerical solution as a function of the initial condition N_0: 1, 2 and 3 correspond to $N_0 = 0.1, 0.5, 0.65$, respectively.

Lines: The asymptotic solution $\phi(z) = n\, z - \ln z + \ln \ln z$ with $n = 1, 2, 3$ for curves 1,2, and 3 , respectively.

General Approach

- Since at large z we BK equation can be linearized:

$$\tilde{N}(\hat{z}) \;=\; \frac{1}{2}\int^{\hat{z}} d\,z'\,\left(1\;-\;e^{-\phi(z')}\right)\;\approx\;1-e^{-\phi_1}-e^{-\phi_2}$$

where $\phi_1 \;=\; \frac{\hat{z}^2}{2}$ and $\phi_2 \;=\; \hat{z}\;-\;\ln\hat{z}$

- Denoting $e^{-\phi}$ by $S(r, Y; b)$ we obtain the following equation for S from BK equation:

$$\boxed{-\frac{\partial S(r,Y;b)}{\partial Y} \;=\; \bar{\alpha}_S\,\int\frac{d^2r'\,r^2}{(r-r')^2\,r'^2}\Big[\,S\,(r,Y;b) \;-\; S\left(r',Y;b-\frac{1}{2}\,(r-r')\right)S\left(r-r',Y;b-\frac{1}{2}r'\right)\,\Big]}$$

First Solution (LT) $r' \sim r \gg 1/Q_s$

In this region $S \ll 1 \quad \Longrightarrow \quad S^2 \approx 0$

$$\frac{\partial S(r,Y;b)}{\partial Y} = -\,\bar{\alpha}_S \int_{1/Q_s^2}^{r^2} \frac{dr'^2\, r^2}{|r^2 - r'^2|\, r'^2}\, S\,(r, Y; b)$$

$$= -\,\bar{\alpha}_S\, \ln(Q_s^2\, r^2)\; S\,(r, Y; b)$$

If $z = \ln(Q_s^2(Y;b)\, r^2)$

$$\Longrightarrow \frac{\chi(1-\gamma_{cr})}{1-\gamma_{cr}} \frac{dS(z)}{dz} = -z S(z)$$

$$\Longrightarrow S(z) = \exp\left(-\frac{1-\gamma_{cr}}{\chi(1-\gamma_{cr})}\frac{z^2}{2}\right)$$

Second Solution (LK) $1/Q_s \approx |r - r'|$ *or* $r' \ll r$

In this region $S(r - r', Y, b) \approx 1$

$$-\frac{\partial S(r,Y)}{\partial Y} = \bar{\alpha}_S \int \frac{d^2 r' r^2}{(r-r')^2 r'^2} \left(S\,(r, Y) - 2S\,(r', Y)_{|r-r'| \ll r} \right)$$

$$= \bar{\alpha}_S \int \frac{d^2 r'\; r^2}{(r-r')^2\; r'^2} \left(S\,(r, Y) + \{ S\,(r, Y) - 2S\,(r', Y) \}_{BFKL} \right)$$

If $z = \ln(Q_s^2(Y; b)\, r^2)$

$$\Longrightarrow \quad \frac{\chi(1-\gamma_{cr})}{1-\gamma_{cr}} \frac{dS(z)}{dz} = z S\,(z) + \frac{1}{1-\phi'_z} S\,(z)$$

where $\quad S\,(z) \;=\; e^{-\phi(z)}$

$$\boxed{\frac{\chi(1-\gamma_{cr})}{1-\gamma_{cr}}\,\frac{dS(z)}{d\,z} \;=\; z\,S\,(z)\;+\;\frac{1}{1-\phi'_z}\,S\,(z)}$$

$$\phi'_z{}^{(\pm)} = \frac{1}{2C}\left(z + C \pm \sqrt{(z+C)^2 + 4C(1-z)}\right)$$

where $C \;=\; \frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}$

At large z:

- $\phi'_z{}^{(+)} \longrightarrow \frac{z}{C}$

  $$S_1\,(z) \equiv e^{-\phi_1} = e^{-\int^z dz'\phi'_z{}^{(+)}(z')} = \exp\left(-\frac{z^2}{2C}\right)$$

- $\phi'_z{}^{(-)} \longrightarrow 1 - \frac{1}{z}$

  $$S_2\,(z) \equiv e^{-\phi_2} = e^{-\int^z dz'\phi'_z{}^{(-)}(z')} = z\exp\,(-z)$$

Numerical check of the new solution

Motivations:

- **The analytical consideration is correct only for large $z \gg 1$.**
- **Matching between negative z and large but positive z is still out of theoretical control.**
- **Singularities of the BFKL kernel give essential contributions, which are missing in simplified models.**

• BK equation in momentum representation (without b dependence):

$$\frac{\partial N(y,q)}{\partial y} = \bar{\alpha}_S \left(\frac{1}{\pi} \int \frac{d^2 q'}{(q-q')^2} \left[N(y,q') - \frac{q^2}{q'^2 + (q-q')^2} N(y,q) \right] - N^2(y,q) \right)$$

or after integration by angles:

$$\frac{\partial N(y,q)}{\partial y} = \bar{\alpha}_S \left\{ \int_0^\infty dq'^2 \left[\frac{N(y,q'^2)}{|q'^2 - q^2|} - \frac{q^2}{q'^2} \cdot \frac{N(y,q^2)}{|q'^2 - q^2|} + \frac{q^2}{q'^2} \cdot \frac{N(y,q^2)}{\sqrt{4q'^4 + q^4}} \right] - N^2(y,q^2) \right\}$$

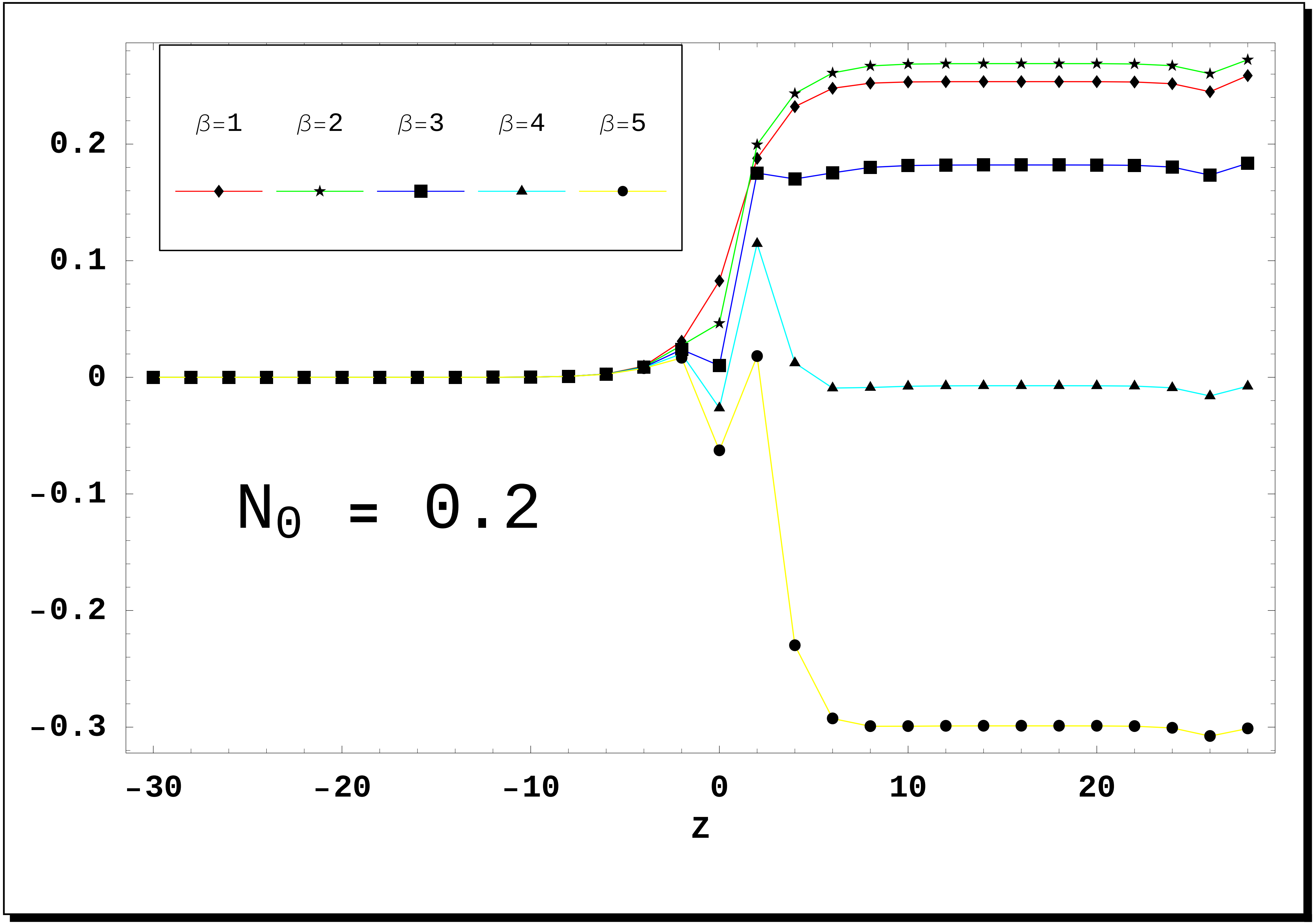
$$N(y,q) \longrightarrow N(z)$$

$$\frac{\partial N(z)}{\partial z} = \frac{1-\gamma_{cr}}{\chi(\gamma_{cr})} \left\{ \int_{-\infty}^{\infty} dz' \left[\frac{N(z') \cdot e^{-z'}}{|e^{-z'} - e^{-z}|} - \frac{N(z) \cdot e^{-z}}{|e^{-z'} - e^{-z}|} + \frac{N(z) \cdot e^{-z}}{\sqrt{(2e^{-z'})^2 + (e^{-z})^2}} \right] - N^2(z) \right\}$$

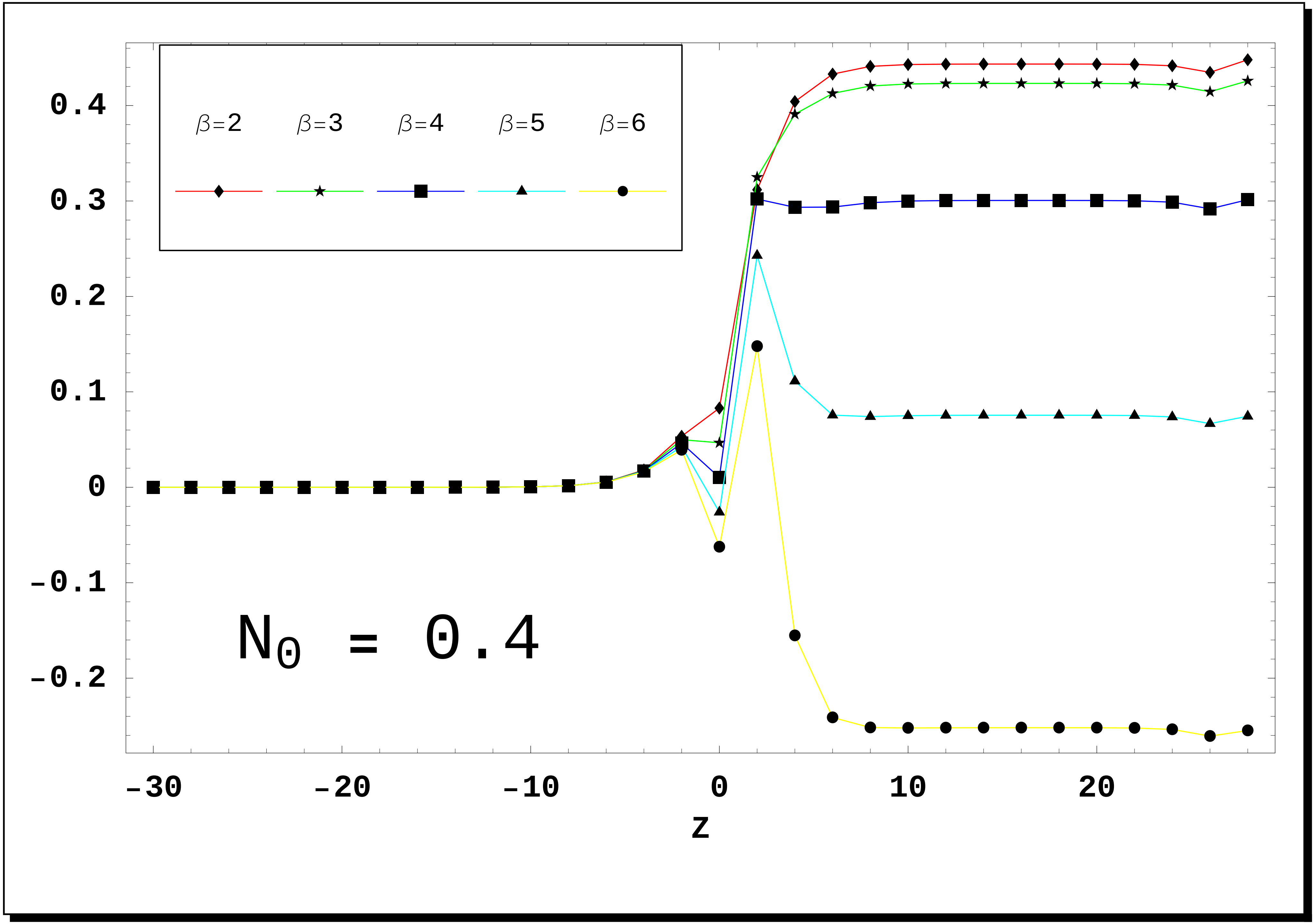
Solution to homogeneous equation:

$$0 = \frac{1-\gamma_{cr}}{\chi(\gamma_{cr})}\left\{\int_{-\infty}^{\infty} dz' \left[\frac{N(z')\cdot e^{-z'}}{|e^{-z'}-e^{-z}|} - \frac{N(z)\cdot e^{-z}}{|e^{-z'}-e^{-z}|} + \frac{N(z)\cdot e^{-z}}{\sqrt{(2e^{-z'})^2+(e^{-z})^2}}\right] - N^2(z)\right\}$$

$$N(z) = \begin{cases} N_0 e^{(1-\gamma_{cr})z} & z < 0 \ ; \\ N_0 + \frac{1}{2}\int_0^z \left(1 - e^{-\left(z' - \log\left[z' + \frac{1-2N_0(1-\gamma_{cr})}{\beta}\right]\right)}\right) dz' & z > 0; \end{cases}$$



β=1
β=2
β=3
β=4
β=5
0.2
0.1
0
-0.1
-0.2
-0.3
N0 = 0.2
-30
-20
-10
0
10
20
z



β=2
β=3
β=4
β=5
β=6
0.4
0.3
0.2
0.1
0
-0.1
-0.2
N0 = 0.4
-30
-20
-10
0
10
20
Z

Conclusions:

- We found that the BK homogeneous equation has a solution ;
- This solution gives the approaching to asymptotic behavior since we assume the geometrical scalling, which says that the only characteristic transverse momentum is Q_S ;
- This solution is determined by the singularities of the BFKL kernel ;
- We should be very careful approximating the full BK equation, by the simplified model of the BFKL kernel (diffusion approximation) ;