An introduction to BFKL dynamics

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1. Gluon reggeization
2. BFKL equation and its solution
3. Möbius invariance
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Gluon reggeization at high energies

Elastic scattering

$$A + B \to A' + B'$$

Mandelstam variables

$$s = (p_A + p_B)^2\,,\ t = (p_A - p_{A'})^2\,,\ u = (p_A - p_{B'})^2$$

Regge kinematics

$$s = 4E^2 \gg -t = |q|^2 \approx E^2\,\theta^2$$

QCD Born amplitude

$$M_{AB}^{A'B'}(s,t)|_{Born} = g\,T^c_{A'A}\,\delta_{\lambda_{A'}\lambda_A}\,\frac{2s}{t}\,g\,T^c_{B'B}\,\delta_{\lambda_{B'}\lambda_B}$$

Leading logarithmic approximation

$$M_{AB}^{A'B'}(s,t) = M_{AB}^{A'B'}(s,t)|_{Born}\,s^{\omega(t)},$$

$$\alpha_s \ln s \sim 1\,,\ \alpha_s = \frac{g^2}{4\pi} \ll 1$$

Gluon Regge trajectory in LLA

$$\omega(-|q|^2) = -\frac{\alpha_s N_c}{4\pi^2}\int d^2k\,\frac{|q|^2}{|k|^2|q-k|^2} \approx -\frac{\alpha_s N_c}{2\pi}\,\ln\frac{|q^2|}{\lambda^2}$$

Inelastic amplitudes

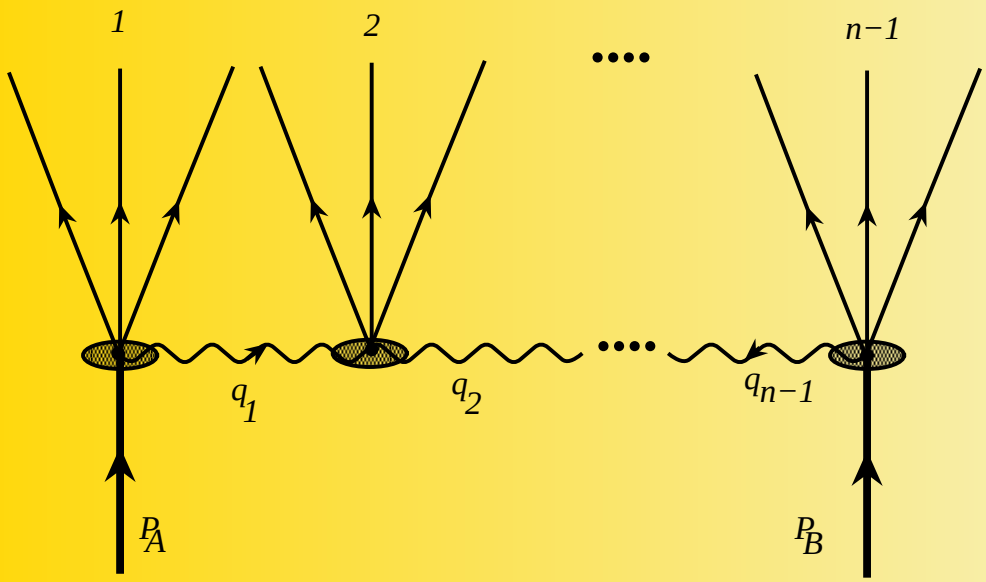

Figure 1: Quasi-multi-Regge kinematics

Gluon production amplitude in LLA

$$M_{2\to 1+n} \sim \frac{s_1^{\omega_1}}{|q_1|^2}\, g T^{d_1}_{c_2 c_1} C(q_2, q_1)\, \frac{s_2^{\omega_2}}{|q_2|^2} ... C^*(q_n, q_{n-1})\, \frac{s_n^{\omega_n}}{|q_n|^2}$$

Multi-Regge kinematics

$$s \gg s_r = (k_{r-1} + k_r)^2 \gg |q_r|^2\,,\ \ k_r = q_r - q_{r+1}$$

Reggeized gluon vertices

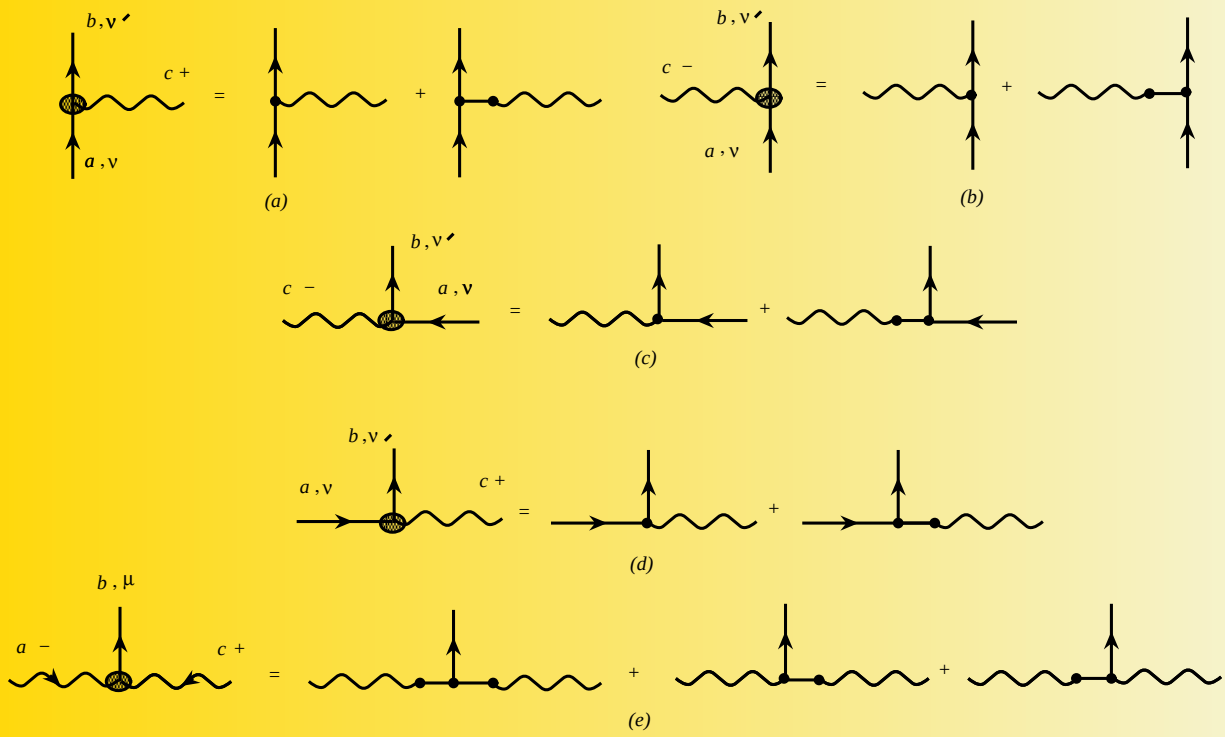

Figure 2: GGR and RRG effective vertices

$$\gamma^{\nu\nu'+}(P_A,q) = -g^{\nu\nu'} + \frac{P_{A'}^{\nu} P_B^{\nu'} + P_A^{\nu'} P_B^{\nu}}{P_A P_B} + q^2 \, \frac{P_B^{\nu} P_B^{\nu'}}{2(P_A P_B)^2} \, ,$$

$$C^{\mu} = (q_2+q_1)_{\perp}^{\mu} - P_A^{\mu}\left(\frac{kP_B}{P_A P_B} + \frac{q_1^2}{kP_A}\right) + P_B^{\mu}\left(\frac{kP_A}{P_A P_B} + \frac{q_2^2}{kP_B}\right) ,$$

$$C_{\mu} e^{\mu}(k) = C^*(q_2,q_1)e + C(q_2,q_1)e^* \, , \;\; C(q_2,q_1) = \frac{q_1 q_2^*}{k^*}$$

General relations

Dispersion representation

$$A(s,t) = \int_{th}^{\infty} \frac{ds'}{\pi} \frac{A_1(s',t)}{s'-s} + \int_{th}^{\infty} \frac{du'}{\pi} \frac{A_2(u',t)}{u'-u} + sub.term$$

Unitarity condition

$$A_1(s,t) = \sum_{n=1}^{\infty} \int d\tau_n \, M_{2\to 1+n} \, \tilde{M}^*_{2\to 1+n}$$

Phase space in the MR kinematics

$$d\tau_n = \frac{1}{s} \prod_{r=1}^{n} \frac{d^2 k_\perp^r}{16\pi^3} \prod_{l=1}^{n} \frac{ds_l}{s_l} \, 2\pi \, \delta(\ln s - \sum_{t=1}^{n} \ln s_t)$$

δ-function decomposition

$$\delta(\ln s - \sum_{t=1}^{n} \ln s_t) = \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} \, e^{\omega\left(\ln s - \sum_{t=1}^{n} \ln s_t\right)}$$

Watson-Sommerfeld representation

$$A^P(s,t) = \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \, \xi_\omega^P \, s^\omega \, f_\omega^P(t) \, , \; \xi_\omega^P = \frac{-e^{-\pi i \omega} + P}{\sin \pi\omega}$$

Gribov-Froissart formula

$$f_\omega^P(t) = \int_{th}^{\infty} \frac{d\,s}{s} \, s^{-\omega} \, A_1^P(s,t)$$

BFKL equation in LLA

Two-reggeon propagator

$$\int_{th}^{\infty} \frac{ds_l}{s_l}\, s_l^{-\omega+\omega_l+\tilde{\omega}_l} = \frac{1}{\omega - \omega(-|k_l|^2) - \omega(-|q-k_l|^2)}$$

Product of two RRG vertices

$$R(\vec{k}_1, \vec{k}_2; \vec{q}) = -C_\mu(k_1, k_2)\, C^\mu(q-k_1, q-k_2) =$$

$$\frac{k_1 k_2^* (q-k_1)^* (q-k_2) + k_1^* k_2 (q-k_1)(q-k_2)^*}{|k_1 - k_2|^2}$$

Colour structure decomposition

$$f_{aa'd}\, f_{bb'd}\, \Psi^{(T)}_{a'b'} = r^{(T)} \Psi^{(T)}_{ab}\,, \quad r^{(0)} = N_c\,, \quad r^{(N_c^2-1)} = \frac{N_c}{2}$$

Balitsky-Fadin-Kuraev-Lipatov equation (1975)

$$\left(\omega - \omega(-|k|^2) - \omega(-|q-k|^2)\right)\, \phi^{(T)}_\omega(\vec{k}, \vec{q} - \vec{k}) =$$

$$\Phi^{(T)}_{\vec{k},\vec{q}} + \frac{\alpha_s}{\pi}\, r^{(T)} \int \frac{d^2k'}{2\pi} R(\vec{k}, \vec{k}'; \vec{q})\, \frac{\phi^{(T)}_\omega(\vec{k}', \vec{q} - \vec{k}')}{|k'|^2\, |q-k'|^2}$$

Bootstrap solution

$$\phi^{(T)}_\omega(\vec{k}, \vec{q} - \vec{k}) = \frac{\Phi^{(T)}_{0,\vec{q}}}{\omega - \omega(-|q^2|)}\,, \quad T = N_c^2 - 1$$

t-channel unitarity constraints

Born production amplitude

$$M^{Born}_{2\to 2+n} = 2s\frac{\Gamma^{\mu'}_{\mu}(p_A,q_1)}{|q_1|^2}\,\Gamma^{\sigma_1}(q_2,q_1)\frac{1}{|q_2|^2}\cdots\frac{\Gamma^{\nu'}_{\nu}(p_B,q_{n+1})}{|q_{n+1}|^2}$$

Pole unitarity

$$\Im_{t_r} M^{Born}_{2\to 2+n} = \pi\delta(|q_r|^2)M^{Born}_{2\to r}\,M^{Born}_{2\to 2+n-r}$$

Relation between vertices

$$\lim_{|q|^2_r\to 0} C^{\sigma_r}(q_{r-1},q_r) = \frac{2q_r p_A}{p_B p_A}\,\gamma^{\sigma\sigma_r -}(-q_r,q_{r-1})\,p^{\sigma}_B + k^{\sigma_r}_{r-1}$$

Mellin transformation in LLA

$$M^{LLA}_{2\to 2}(s,t) = M^{Born}_{2\to 2}(s,t)\int_{-i\infty}^{i\infty}\frac{d\omega}{2\pi i}\,s^{\omega}\,f_{\omega}(t)\,,$$

$$f_{\omega}(t) = \frac{1}{\omega-\omega(t)}$$

Two-particle t-channel unitarity

$$\Im_t\, f_{\omega}(t) = \frac{\alpha_s\,N_c}{4}\,|f_{\omega}(t)|^2$$

Bootstrap and s-channel unitarity

Sudakov variables

$$k_r = \beta_r\, p_A + \alpha_r\, p_B + k_r^{\perp}\,,\ \ (k_r^{\mu})^2 = s\alpha_r\beta_r - |k_r|^2 = 0$$

Multi-Regge kinematics

$$1 \gg \beta_1 \gg \beta_2 \gg ...\beta_n \gg \frac{|k|^2}{s}\,,$$

$$\frac{|k|^2}{s} \ll \alpha_1 \ll \alpha_2 \ll ...\alpha_n \ll 1\,,\ \ |k_r| \sim |k|\,,$$

$$s_r \approx s\alpha_r\beta_{r-1}\,,\ \ \prod_{r=1}^{n+1} s_r = s\prod_{r=1}^{n} |k_r|^2 \sim s$$

Dispersion relations in LLA

$$\sum_{r=1}^{n} \Im_{s_{0r}}\, M^{LLA}_{2\to 2+n} = \pi\omega(t_1) M^{LLA}_{2\to 2+n}\,,\ \ s_{0r} = 2k_r p_A$$

Generalized bootstrap equations

$$\pi\omega(t_1) M^{LLA}_{2\to 2+n} = \sum_{r=1}^{n}\sum_{k=2}^{\infty}\int d\tau_k\, M^{LLA}_{4+n\to k}\, \tilde{M}^{LLA*}_{2\to k}$$

Colourless particle scattering in QCD

Impact factor representation

$$A = s \int \frac{d^2k}{2\pi} \frac{d^2k'}{2\pi} \frac{\Omega(\vec{k},\, \vec{q}-\vec{k})}{|k|^2\, |q-k|^2}\, f(\vec{k},\, \vec{k'},\, \vec{q};\, s)\, \frac{\Omega(\vec{k'},\, \vec{q}-\vec{k'})}{|k'|^2\, |q-k'|^2}$$

Impact factors

$$\Omega(\vec{k},\, \vec{q}-\vec{k}) = \int_{-\infty}^{\infty} \frac{d\,(s\alpha)}{(2\,\pi)^2 i}\, \Phi_{\mu\mu'}(k, q-k, p_1)\, \frac{p_B^{\mu}\, p_B^{\mu'}}{s^2}\,.$$

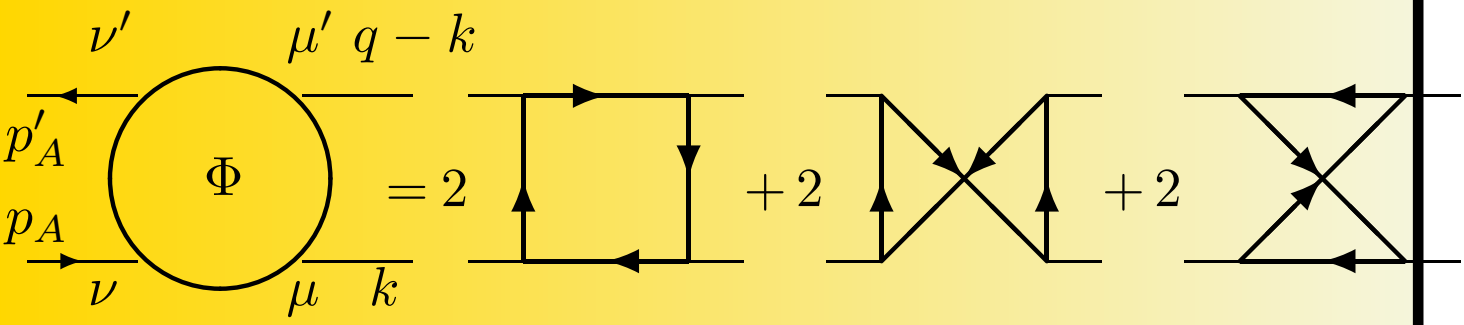

Figure 3: Photon-gluon scattering amplitude

Mellin representation

$$f(\vec{k},\, \vec{k'},\, \vec{q};\, s) = i \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \left(\frac{s}{s_0}\right)^{\omega} \phi_{\omega}^{BFKL}(\overrightarrow{k}, \overrightarrow{k}; \overrightarrow{q})$$

Solution of the BFKL equation for $q = 0$

Optical theorem

$$\sigma_t = \frac{\Im A(s,0)}{s}$$

Homogeneous BFKL equation

$$Ef(\overrightarrow{k},\overrightarrow{q}) = Hf(\overrightarrow{k},\overrightarrow{q}) = \ln\frac{|k|^2}{\lambda^2}\, f(\overrightarrow{k},\overrightarrow{q}) +$$

$$\ln\frac{|q-k|^2}{\lambda^2} f(\overrightarrow{k},\overrightarrow{q}) - \int \frac{d^2k'}{\pi}\, \frac{R_\lambda(\overrightarrow{k},\overrightarrow{k'},\overrightarrow{q})}{|k'|^2\,|q-k'|^2} f(\overrightarrow{k'},\overrightarrow{q})$$

BFKL Pomeron intercept

$$\sigma_t \sim s^{\Delta}\,,\ \Delta = -\frac{\alpha_s N_c}{2\pi}\, E_0 = \frac{4\alpha_s N_c}{\pi}$$

Complete set of eigenfunctions for $q = 0$

$$f_{m,\widetilde{m}}(\overrightarrow{k}) = |k|^{-1+2i\nu}\left(\frac{k}{k^*}\right)^{\frac{n}{2}}\,,$$

$$m = \frac{1}{2} + i\nu + \frac{n}{2}\,,\ \widetilde{m} = \frac{1}{2} + i\nu - \frac{n}{2}$$

Eigenvalues of H

$$E_{\nu n} = 4\Re\,\psi(\frac{1}{2} + i\nu + \frac{|n|}{2}) - 4\psi(1)\,,\ \psi(x) = \frac{d}{dx}\ln\Gamma(x)$$

Coordinate representation

Holomorphic coordinates and momenta

$$\rho_r = x_r + iy_r\,,\ \rho_r^* = x_r - iy_r\,,\ p_r = i\frac{\partial}{\partial \rho_r}\,,\ p_r^* = i\frac{\partial}{\partial \rho_r^*}$$

BFKL equation in the operator form

$$E\,f = \phi + H_{12}\,f\,,\ H_{12} = \ln|p_1|^2 + \ln|p_2|^2 - 4\psi(1) +$$

$$+\frac{1}{p_1 p_2^*}\,\ln|\rho_{12}|\,p_1 p_2^* + \frac{1}{p_1^* p_2}\,\ln|\rho_{12}|^2\,p_1^* p_2\,,\ \omega = -\frac{\alpha_s N_c}{2\pi}\,E$$

Inhomogenious term

$$\phi(\overrightarrow{\rho}_1, \overrightarrow{\rho}_2; \overrightarrow{\rho}'_1, \overrightarrow{\rho}'_1) = \pi^2\,\ln|\rho_{11'}|^2\,\ln|\rho_{22'}|^2$$

Colour transparency

$$\Omega(\vec{k},\,\vec{q}-\vec{k})_{|k|=0} = \Omega(\vec{k},\,\vec{q}-\vec{k})_{|k-q|=0} = 0$$

"Gauge equivalence" of solutions

$$\delta f(\overrightarrow{\rho}_1, \overrightarrow{\rho}_2) = \chi(\overrightarrow{\rho}_1) + \chi(\overrightarrow{\rho}_2)$$

Möbius space (J.B.,L.L.,G.V. (2004))

$$f(\overrightarrow{\rho}, \overrightarrow{\rho}) = 0$$

BFKL equation in the Möbius space

Möbius-invariant inhomogenious term

$$\phi(\overrightarrow{\rho}_1, \overrightarrow{\rho}_2; \overrightarrow{\rho}'_1, \overrightarrow{\rho}'_1) \to 2\pi^2 \ln \frac{|\rho_{11'}||\rho_{22'}|}{|\rho_{12'}||\rho_{21'}|} \ln \frac{|\rho_{11'}||\rho_{22'}|}{|\rho_{12}||\rho_{1'2'}|}$$

Holomorphic separability of H_{12} (L.L (1989))

$$H_{12} = h_{12} + h^*_{12}$$

Holomorphic hamiltonian

$$h_{12} = \ln(p_1 p_2) + \frac{1}{p_1} \ln(\rho_{12})\, p_1 + \frac{1}{p_2} \ln(\rho_{12})\, p_2 - 2\psi(1)$$

Other representations

$$h_{12} = \ln(\rho_{12}^2\, p_1) + \ln(\rho_{12}^2\, p_2) - 2\ln(\rho_{12}) - 2\psi(1) = -2\psi(1) +$$

$$\frac{1}{2}\left(\psi(\rho_{12}p_1) + \psi(1+\rho_{21}p_1) + \psi(\rho_{12}p_2) + \psi(1+\rho_{21}p_2)\right)$$

Möbius invariance (L.L. (1986))

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d}$$

Conformal weights

$$m = \frac{1}{2} + i\nu + \frac{n}{2}\,, \quad \widetilde{m} = \frac{1}{2} + i\nu - \frac{n}{2}$$

Solution of the BFKL equation

Eigenfunctions and eigenvalues of H_{12}

$$f_{m,\widetilde{m}}(\overrightarrow{\rho_1},\overrightarrow{\rho_2};\overrightarrow{\rho_0}) =< 0|\phi(\overrightarrow{\rho_1})\,\phi(\overrightarrow{\rho_2})\,O_{m,\widetilde{m}}(\overrightarrow{\rho_0})|0 >=$$

$$\left(\frac{\rho_{12}}{\rho_{10}\,\rho_{20}}\right)^m \left(\frac{\rho^*_{12}}{\rho^*_{10}\,\rho^*_{20}}\right)^{\widetilde{m}}\,,\ \omega(\nu,n) = -\frac{\alpha_s N_c}{2\pi}\,E_{\nu,n}$$

Solution of the equation (L.L. (1986))

$$f(\overrightarrow{\rho}_1,\overrightarrow{\rho}_2;\overrightarrow{\rho}'_1,\overrightarrow{\rho}'_1) = \sum_{n=-\infty}^{\infty}\int_{-i\infty}^{i\infty}\frac{G_{\nu n}(\overrightarrow{\rho}_1,\overrightarrow{\rho}_2;\overrightarrow{\rho}'_1,\overrightarrow{\rho}'_2)}{\omega-\omega(\nu,n)}$$

$$\frac{(\nu^2+n^2/4)\,d\nu}{[\nu^2+(n-1^2/4)]\,[\nu^2+(n+1^2/4)]} = f(x,x^*)$$

Pomeron Green function

$$G_{\nu n}(\overrightarrow{\rho}_1,\overrightarrow{\rho}_2;\overrightarrow{\rho}'_1,\overrightarrow{\rho}'_2) =$$

$$\int d^2\rho_0\, f^*_{m,\widetilde{m}}(\overrightarrow{\rho}'_1,\overrightarrow{\rho}'_2;\overrightarrow{\rho_0})\, f_{m,\widetilde{m}}(\overrightarrow{\rho_1},\overrightarrow{\rho_2};\overrightarrow{\rho_0}) = g_{\nu n}(x,x^*)$$

Anharmonic ratio

$$x = \frac{\rho_{12}\rho_{1'2'}}{\rho_{11'}\rho_{22'}}$$

Fields

Gluon fields

$$v_\mu(x) = -iT^a v^a_\mu(x)\,,\ [T^a, T^b] = i f_{abc} T^c\,,\ Tr\, T^a T^b = \frac{\delta^{ab}}{2}$$

Reggeized gluon and quark fields

$$A_\pm(x) = -iT^a A^a_\pm(x)\,,\ \psi(x)\,,\ \bar{\psi}(x)$$

Global gauge transformations

$$\delta v_\mu(x) = [v_\mu(x), \chi],\ \delta\psi(x) = -\chi\,\psi(x),\ \delta A(x) = [A(x), \chi]$$

Local gauge transformations

$$\delta v_\mu(x) = \frac{1}{g}[D_\mu, \chi(x)],\ \delta\psi(x) = -\chi(x)\,\psi(x),\ \delta A_\pm(x) = 0$$

Kinematical constraints

$$\partial_\mp A_\pm(x) = 0$$

Gluon fields in light-cone gauges

$$v^L_1 + iv^L_2 = -i\partial^* \phi^*\,,\ v^R_1 + iv^R_2 = -i\partial^* \phi\,,$$

$$v^L_\mu P^\mu_A = 0\,,\ v^R_\mu P^\mu_B = 0$$

Symmetry between two gauges

Multi-Regge effective action (L.L. (1991))

$$S = \int d^4x \, (L_{free} + L_{prod} + L_{scat})$$

Free lagrangian

$$L_{free} = -\frac{1}{2} \left((\partial_+ \partial \phi_a)(\partial_- \partial^* \phi_a^*) + 4(\partial A_a^+)(\partial^* A_a^-) \right)$$

Production contribution

$$L_{prod} = j_a \phi_a^* + j_a^* \phi_a \, , \; j_a = g(\partial^* A^+) T_a (\partial A^-)$$

Scattering contribution

$$L_{scat} = j_+^a A^{+a} + j_-^a A^{-a} \, , \; j_+^a = g(\partial^* \phi^*) T^a i \partial_+ (\partial \phi) + h.c.$$

Lagrangian in the gauge-invariant form

$$\frac{(F_{\mu\nu}^a)^2}{4} + \frac{g}{2} \left(-A_+^a (F_{-\sigma} T^a i \partial_-^{-1} F_{-\sigma}) - A_-^a (F_{+\sigma} T^a i \partial_+^{-1} F_{+\sigma} + \right.$$

$$\left. (\partial_-^{-1} F_{-\sigma}^a)(A_- T^a i \partial_\sigma A_+) + (\frac{i}{\partial_+ \partial_-} F_{+-}^a) \partial_\sigma A_+ T^a \partial_\sigma A_- + \ldots \right)$$

$$F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$$

Steinman relation is absent

Effective gauge invariant Lagrangian (L.L. (1995)

Effective action local in rapidity y

$$S = \int d^4x \left(L_0 + L_{ind}\right) \ , \ |y - y_0| < \eta \ , \ y = \frac{1}{2} \ln \frac{\epsilon_k + |k|}{\epsilon_k - |k|}$$

QCD Lagrangian

$$L_0 = i\bar{\psi}\hat{D}\psi + \frac{1}{2} Tr\, G^2_{\mu\nu}, \ D_\mu = \partial_\mu + g v_\mu, \ G_{\mu\nu} = \frac{1}{g}[D_\mu, D_\nu]$$

Induced contribution

$$L_{ind} = Tr\left(L^k_{ind} + L^{GR}_{ind}\right), \ L^k_{ind} = -\partial_\mu A^a_+ \partial_\mu A^a_-$$

Gluon-Reggeon couplings

$$L^{GR}_{ind} = -\frac{1}{g}\partial_+ \, P \exp\left(-\frac{1}{2}\int_{-\infty}^{x^+} v_+(x')d(x')^+\right) \partial^2_\sigma A_- -$$

$$\frac{1}{g}\partial_- \, P \exp\left(-\frac{1}{2}\int_{-\infty}^{x^-} v_-(x')d(x')^+\right) \partial^2_\sigma A_+ =$$

$$Tr\left(v_+ - g v_+ \frac{1}{\partial_+} v_+ + g^2 v_+ \frac{1}{\partial_+} v_+ \frac{1}{\partial_+} v_+ - \ldots\right) \partial^2_\sigma A_-$$

$$+ \left(v_- - g v_- \frac{1}{\partial_-} v_- + g^2 v_- \frac{1}{\partial_-} v_- \frac{1}{\partial_-} v_- - \ldots\right) \partial^2_+ A_+$$

$$\partial_\mp A_\pm(x) = 0$$

Feynman rules in the momentum space

Momentum conservation for induced vertices

$$k_0^+ + k_1^+ + ... + k_m^+ = 0$$

Examples of the vertices

$$\Delta^{\nu_0 \nu_1 +}_{a_0 a_1 c} = \vec{q}_\perp^{\,2}\, T^c_{a_1 a_0}\, (n^+)^{\nu_1} \frac{1}{k_1^+} (n^+)^{\nu_0}, \;\; \Delta^{\nu_0 \nu_1 \nu_2 +}_{a_0 a_1 a_2 c} =$$

$$\vec{q}_\perp^{\,2}\, (n^+)^{\nu_0} (n^+)^{\nu_1} (n^+)^{\nu_2} \left(\frac{T^a_{a_2 a_0}\, T^c_{a_1 a}}{k_1^+ k_2^+} + \frac{T^a_{a_2 a_1}\, T^c_{a_0 a}}{k_0^+ k_2^+} \right)$$

Factorization

$$\Delta^{\nu_0 \nu_1 ... \nu_r +}_{a_0 a_1 ... a_r c} = -\vec{q}_\perp^{\,2} \prod_{s=0}^{r} (n^+)^{\nu_s}\, 2\, Tr\, \left(T^c G_{a_0 a_1 ... a_r} \right) .$$

General representation for $G_{a_0 a_1 ... a_r}$

$$\sum_{\{i_0, i_1, ..., i_r\}} \frac{T^{a_{i_0}} T^{a_{i_1}} T^{a_{i_2}} ... T^{a_{i_r}}}{k_{i_0}^+ (k_{i_0}^+ + k_{i_1}^+) ... (k_{i_0}^+ + k_{i_1}^+ + ... + k_{i_{r-1}}^+)} ,$$

Recurrent relations (Ward identities)

$$k_r^+\, \Delta^{\nu_0 \nu_1 ... \nu_r +}_{a_0 a_1 ... a_r} (k_0^+, ..., k_r^+) =$$

$$\sum_{i=0}^{r-1} i f_{a a_m a_i} \Delta^{\nu_0 ... \nu_r +}_{... a_{i-1} a a_{i+1} ... a_r} (k_0^+, ..., k_{i-1}^+, k_i^+ + k_r^+, k_{i+1}^+, ...)$$

Composite states of several reggeons

Equation for the n-gluon state (B.,K.,P. (1980))

$$E\Psi(\vec{\rho}_1,...,\vec{\rho}_n) = \sum_{k<l} \frac{T_k^a T_l^a}{(-N_c)} H_{k,l}\, \Psi(\vec{\rho}_1,...,\vec{\rho}_n)$$

Multi-colour anzatz

$$\Psi(\vec{\rho}_1,...,\vec{\rho}_n) = \sum_{(i_1,...,i_n)} Tr\,\left(T_{c_{i_1}} T_{c_{i_2}} ... T_{c_{i_n}}\right) f(\vec{\rho}_{i_1},...,\vec{\rho}_{i_n})$$

Scrödinger equation for $N_c \to \infty$

$$E f(\vec{\rho}_1,...,\vec{\rho}_n) = H f(\vec{\rho}_1,...,\vec{\rho}_n)\,,\;\; H = \frac{1}{2}\sum_{k=1}^{n} H_{k,k+1}$$

Holomorphic separability of H

$$H = \frac{h+h^*}{2}\,,\;\; h = \sum_{r=1}^{n} h_{r,r+1}$$

Holomorphic factorization of f (L.L. (1989))

$$f(\vec{\rho}_1,\vec{\rho}_2,...,\vec{\rho}_n) = \sum_{r,s} a_{r,s}\, f_r(\rho_1,...,\rho_n)\, \tilde{f}_s(\rho_1^*,...,\rho_n^*)$$

$$h f_r = \epsilon f_r\,,\;\; h^* \tilde{f}_s = \tilde{\epsilon} \tilde{f}_s\,,\;\; \omega = -\frac{\alpha_s N_c}{2\pi}\frac{1}{2}(\epsilon + \tilde{\epsilon})$$

Wave function normalization

New representation for the pair hamiltonian

$$h_{12} = \rho_{12}\,\ln(p_1 p_2)\,\rho_{12}^{-1} + 2\ln(\rho_{12}) - 2\psi(1)$$

Two forms of the holomorphic hamiltonian

$$h = \sum_{r=1}^{n} \left(p_r^{-1}\ln(\rho_{r,r-1}\rho_{r,r+1})p_r + 2\ln p_r - 2\psi(1)\right) =$$

$$\sum_{r=1}^{n} \left(\rho_{r,r+1}\,\ln(p_r p_{r+1})\,\rho_{r,r+1}^{-1} + 2\ln(\rho_{r,r+1}) - 2\psi(1)\right)$$

Two expressions for the transposed hamiltonian

$$h^t = \prod_{r=1}^{n} p_r\, h \left(\prod_{r=1}^{n} p_r\right)^{-1} = \left(\prod_{r=1}^{n} \rho_{r,r+1}\right)^{-1} h \prod_{r=1}^{n} \rho_{r,r+1}$$

Two normalization conditions

$$||f||_1^2 = \int f^* \prod_{r=1}^{n} d^2\rho_r\,|p_r|^2\, f,\ \ ||f||_2^2 = \int \prod_{r=1}^{n} \frac{d^2\rho_r}{|\rho_{r,r+1}|^2}|f|^2$$

First integral of motion (L.L. (1993))

$$A = \rho_{1,2}\rho_{2,3}...\rho_{n,1}\,p_1 p_2 ... p_n\ ,\ \ [A,h] = 0$$

Integrals of motion and integrability

New representation for A

$$A = Tr\ (L_1 L_2 ... L_n)\ , L_k = \begin{pmatrix} \rho_k\, p_k & p_k \\ -\rho_k^2\, p_k & -\rho_k\, p_k \end{pmatrix}$$

Transfer matrix (L.L.(1993))

$$T(u) = Tr\, L_1(u) ... L_n(u) = \sum_{r=0}^{n} q_r u^{n-r}\ ,\ L_k(u) = u\, I + L_k$$

Integrals of motion

$$q_r = \sum_{k_1 < k_2 < ... < k_r} \rho_{k_1 k_2} ... \rho_{k_{r-1} k_r}\, p_{k_1} p_{k_2} ... p_{k_r}\ ,\ \ [q_r, h] = 0$$

Monodromy matrix

$$t(u) = L_1(u) L_2(u) ... L_n(u)$$

Yang-Baxter equation (L.L.(1993))

$$t_{r_1'}^{s_1}(u)\, t_{r_2'}^{s_2}(v)\, l_{r_1 r_2}^{r_1' r_2'}(v-u) = l_{s_1' s_2'}^{s_1 s_2}(v-u)\, t_{r_2}^{s_2'}(v)\, t_{r_1}^{s_1'}(u)\ ,$$

L-operator for the Heisenberg model

$$l_{s_1' s_2'}^{s_1 s_2} = w \delta_{s_1'}^{s_1} \delta_{s_2'}^{s_2} + i \delta_{s_2'}^{s_1} \delta_{s_1'}^{s_2}$$

Duality symmetry (L.L. (1999))

Cyclic symmetry at large N_c

$$\rho_i \to \rho_{i+1}\,,\ p_i \to p_{i+1}$$

Duality transformations (L.L. (1999))

$$D:\ p_r \to \rho_{r,r+1} \to p_{r+1}\,,\ \sum_{r=1}^{n} p_r = 0$$

Symmetry of integrals of motion and hamiltonian

$$(D\,q_r)^t = q_r\,,\ (D\,h)^t = h$$

Wave function for $q = 0$

$$\psi_{m,\widetilde{m}}(\vec{\rho}_{12},...,\vec{\rho}_{n1}) = \int d^2\rho_0\, f_{m,\widetilde{m}}(\vec{\rho}_1,...,\vec{\rho}_n;\vec{\rho}_0)\,,$$

$$-\sum_{r=1}^{n} \rho_{r,r+1}^2 \partial_r \partial_{r+1} f_{m,\widetilde{m}} = m(m-1) f_{m,\widetilde{m}}\,,\ A\,f = \lambda_m\, f$$

Duality transformation

$$\psi_{m,\widetilde{m}} = \frac{|\lambda_m|}{2^{-n}} \int \prod_{k=1}^{n-1} \frac{d^2\rho'_{k,k+1}}{2\pi} \prod_{k=1}^{n} \frac{e^{\vec{\rho}_{k,k+1}\vec{\rho'}_k}}{|\rho'_{k,k+1}|^2} \psi^*_{m,\widetilde{m}}(\vec{\rho'}_{12},...)$$

Odderon wave function

$$f_{m,\widetilde{m}}(\overrightarrow{\rho_1},\overrightarrow{\rho_2},\overrightarrow{\rho_3};\overrightarrow{\rho_0}) = z^m z^{*\,\widetilde{m}} \varphi_{m,\widetilde{m}}(\overrightarrow{x})\,,$$

$$z = \frac{\rho_{23}}{\rho_{20}\,\rho_{30}}\,,\;\; x = \frac{\rho_{12}\,\rho_{30}}{\rho_{10}\,\rho_{32}}$$

Normalization condition

$$\left\|\varphi_{m,\widetilde{m}}\right\|_1^2 = \int \frac{d^2x}{|x(1-x)|^2}\,\left|\varphi_{m,\widetilde{m}}(\overrightarrow{x})\right|^2$$

Integral of motion (L.L. (1993))

$$-ix(1-x)\left(x(1-x)\partial^2 + (2-m)\left((1-2x)\partial - 1 + m\right)\right)\partial$$

Holomorphic factorization (Y.,W. (1997))

$$\varphi_{m,\widetilde{m}}(\overrightarrow{x}) = \varphi_f^{(m)}(x)\,\varphi_f^{(\widetilde{m})}(x^*) + c_2\,\varphi_r^{(m)}(x)\,\varphi_r^{(\widetilde{m})}(x^*) +$$

$$c_1\left(\varphi_s^{(m)}(x)\,\varphi_r^{(\widetilde{m})}(x^*) + \varphi_r^{(m)}(x)\,\varphi_s^{(\widetilde{m})}(x^*)\right) + (\lambda \to -\lambda)$$

Eigenvalue equation

$$A\,\varphi_m = a_{1-m}\,a_m\,\varphi_m = \lambda\,\varphi_m\,,\;\; a_m = x(1-x)\,p^{1+m}$$

Duality relations (L.L. (1999))

$$a_m\varphi^{(m)} = \lambda^m\varphi^{(1-m)}\,,\;\; |c_1| = |\lambda|\,,\;\; \Im\frac{c_2}{c_1} = \Im\frac{m+\widetilde{m}}{m\,\widetilde{m}}$$

Odderon hamiltonian

Holomorphic hamiltonian

$$\sum_{k=1}^{n}\left(\ln\left(\frac{\rho_{k+2,0}\;\rho_{k,k+1}^2}{\rho_{k+1,0}\;\rho_{k+1,k+2}}\,\partial_k\,\frac{\rho_{k-2,0}\;\rho_{k,k-1}^2}{\rho_{k-1,0}\;\rho_{k-1,k-2}}\,\partial_k\right)+2\gamma\right)$$

Odderon case (L.L. (1993))

$$h=6\gamma+\ln\left(x^2\partial\right)+\ln\left((1-x)^2\partial\right)+\ln\left(x^2\partial+\frac{m\,x^2}{1-x}\right)+$$

$$\ln\left(\partial+\frac{m}{1-x}\right)+\ln\left((1-x)^2(\partial-\frac{m}{x})\right)+\ln\left(\partial-\frac{m}{x}\right)$$

Algebraic relations

$$\ln(x^{-1}\,P)=-\ln\,x+\frac{1}{2}\psi(-P)+\frac{1}{2}\psi(1+P)\,,\;\;P=x\partial$$

Normal form (L.L. (1999))

$$\frac{h}{2}=3\gamma-\ln x+\psi(1-P)+\psi(-P)+\psi(m-P)+\sum_{k=1}^{\infty}x^k f_k(P),$$

$$f_k(P)=-\frac{2}{k}+\frac{1}{2}\left(\frac{1}{P+k-m}+\frac{1}{P+k}\right)+\sum_{t=0}^{k}\frac{c_t(k)}{P+t}\,,$$

$$c_t(k)=\frac{(-1)^{k-t}\;\Gamma(m+t)\;\left((t-k)\,(m+t)+m\,k/2\right)}{k\,\Gamma(m-k+t+1)\,\Gamma(t+1)\,\Gamma(k-t+1)}$$

Odderon intercepts

Relation with the integral of motion

$$\frac{h}{2} = \log(B) + 3\gamma + \sum_{r=1}^{\infty} \frac{c_r}{B^{2r}}\,, \;\; B = iA$$

Redefinition of the wave function

$$\varphi_m(P) = \Gamma(-P)\,\Gamma(1-P)\,\Gamma(m-P)\,\exp(i\pi P)\;\Phi_m(P)$$

Expansion at large $\lambda = -i\mu$

$$\Phi_m(P) = \sum_{n=0}^{\infty} \mu^{-n}\Phi_m^n(P)\,,\; \Phi_m^0(P) = 1$$

Duality recurrent relation

$$\Phi_m^n(P) = \sum_{k=1}^{P} \Omega_m^{n-1} - \frac{1}{2}\sum_{k=1}^{m} \Omega_m^{n-1},\; \Omega_m^{n-1} = (k-1)(k-1-m)$$

$$\big((k-m)\,\Phi_m^{n-1}(k-1) + (k-2)\,\Phi_{1-m}^{n-1}(k-1-m)\big)$$

Odderon energy (L.L. (1999))

$$\frac{\varepsilon}{2} = \ln\mu + 3\gamma + \frac{\Phi_m'(1)}{\Phi_m(1)} + \frac{m(1-m)}{2\mu} =$$

$$\ln\mu + 3\gamma + \left(\frac{3}{448} + \frac{13}{120}(m-1/2)^2 - \frac{1}{12}(m-1/2)^4\right)\frac{1}{\mu^2} +$$

Analytic Bethe anzatz

Monodromy matrix

$$\widetilde{t}(u) = \widetilde{L}_n(u)...\widetilde{L}_1(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

L-operator

$$\widetilde{L}_k(u) = \begin{pmatrix} u + p_k\rho_{k0} & -p_k\rho_{k0}^2 \\ p_k\rho_{k0}^2 & u - p_k\rho_{k0} \end{pmatrix}$$

Pseudo-vacuum state (L.F.,G.K. (1995))

$$\Psi^{(0)}(\overrightarrow{\rho_1},\, \overrightarrow{\rho_2},\, \ldots,\overrightarrow{\rho_n};\overrightarrow{\rho_0}) = \prod_{k=1}^{n} \frac{1}{|\rho_{k0}|^4}\,,\ C(u)\,\Psi^{(0)} = 0$$

Baxter equation

$$\Lambda^{(n)}(\lambda;\ \vec{\mu})\,Q\,(\lambda;\,m,\vec{\mu}) = (\lambda+i)^n\ Q\,(\lambda+i;\,m,\vec{\mu}) +$$

$$+(\lambda-i)^n\ Q\,(\lambda-i;\,m,\vec{\mu})\,,\ \Lambda^{(n)}(\lambda;\ \vec{\mu}) = \sum_{k=0}^{n}(-i)^k\,\mu_k\,\lambda^{n-k}$$

Sklyanin anzatz

$$f(\rho_1,...\rho_n) = \prod_{r=1}^{n-1} Q(\widehat{\lambda}_r;\,m,\vec{\mu})\Psi^{(0)}, B(u) = -P\prod_{r=1}^{n-1}(u-\widehat{\lambda}_r)$$

Baxter-Sklyanin representation (H.V.,L.L.)

Unitary transformation

$$\Phi(\vec{P},\vec{\lambda}_1,...,\vec{\lambda}_{n-1}) = \int \prod d^2\rho_r U_{\vec{\lambda}_1,...,\vec{\lambda_{n-1}};\vec{P}}(\vec{\rho}_1,...,\vec{\rho}_n)\Psi$$

Equation for the kernel

$$B(u)U_{\vec{\lambda}_1,...,\vec{\lambda}_{n-1}} = -P\prod_{r=1}^{n-1}(u-\lambda_r)U_{\vec{\lambda}_1,...,\vec{\lambda}_{n-1}}$$

Quantization conditions

$$\lambda_r = \sigma + i\frac{N}{2}\,,\ \ \lambda_r^* = \sigma - i\frac{N}{2}$$

Pomeron wave function

$$\frac{\Phi_{m,\widetilde{m}}(\vec{P},\vec{\lambda})}{|\lambda|^2 P^m P^{*\widetilde{m}}} = Q(\lambda^*,\widetilde{m})Q(\lambda,m) + (-1)^n\,(\lambda\to-\lambda)$$

Pomeron Baxter function

$$Q(\lambda,m) = -\frac{\pi^2 m(1-m)}{\sin(\pi m)}F_{32}(-i\lambda+1,2-m,1+m;2;1)$$

Meromorphic properties

$$r_l(m) = \lim_{\lambda\to il}(\lambda-il)Q(\lambda,m) = -\frac{\sin(\pi m)}{i\pi}Q(-il,m)$$

Hamiltonian in BS representation (H.V.,L.L.)

Property of the BFKL hamiltonian

$$\lim_{|p|\to 0} \frac{1}{p^{-i\lambda^*}(p^*)^{-i\lambda}} H_{12} p^{-i\lambda^*}(p^*)^{-i\lambda} = 4\gamma$$

$$-\ln|p|^2 + \psi(1+i\lambda^*) + \psi(1-i\lambda^*) + \psi(1+i\lambda) + \psi(1-i\lambda)$$

H_{12} in the BS representation

$$\lim_{\lambda,\lambda^*\to i} H_{12}\Phi = i\left(\frac{\partial}{\partial\lambda} + \frac{\partial}{\partial\lambda^*} + \frac{1}{\lambda - i} + \frac{1}{\lambda^* - i} - 2i\right)\Phi$$

Pomeron energy

$$i \lim_{\lambda,\lambda^*\to i}\left(\frac{\partial}{\partial\lambda}\ln\left((\lambda - i)\lambda^2 Q(\lambda, m)\right) + (\lambda \to \lambda^*, m \to \widetilde{m})\right)$$

General expression for energy

$$i \lim_{\lambda,\lambda^*\to i} \frac{\partial}{\partial\lambda}\frac{\partial}{\partial\lambda^*}\ln\left[(\lambda - i)^{n-1}(\lambda^* - i)^{n-1}|\lambda|^{2n}\, Q_{m,\widetilde{m},\vec{\mu}}\right]$$

Holomorphic factorization

$$Q_{m,\widetilde{m},\vec{\mu}}\left(\overrightarrow{\lambda}\right) = \sum_{t,l} C_{t,l}\, Q^{(t)}(\lambda;\, m, \vec{\mu})\, Q^{(l)}\left(\lambda^*;\, \widetilde{m}, \vec{\mu^s}\right)$$

Meromorphic anzatz (H.V.,L.L.)

Baxter equation

$$\Omega(x,\vec{\mu})\,Q(x,\vec{\mu}) = (x{+}1)^n\,Q(x{+}1,\vec{\mu}){+}(x{-}1)^n\,Q(x{-}1,\vec{\mu}),$$

$$\Omega(x,\vec{\mu}) = \sum_{k=0}^{n}(-1)^k\,\mu_k\,x^{n-k}$$

Auxiliary functions

$$f_r(x,\vec{\mu}) = \sum_{l=0}^{\infty}\left[\frac{\widetilde{a}_l(\vec{\mu})}{(x-l)^r} + \frac{\widetilde{b}_l(\vec{\mu})}{(x-l)^{r-1}} + \ldots + \frac{\widetilde{g}_l(\vec{\mu})}{x-l}\right]$$

Recurrent relations

$$\widetilde{a}_l(\vec{\mu}) = \widetilde{A}_l(a_{l-1},a_{l-2},\ldots)\,,\ \widetilde{b}_l(\vec{\mu}) = \widetilde{B}_l(a_r,b_{l-1},b_{l-2}\ldots),\ldots$$

$$\widetilde{a}_0 = 1, \widetilde{b}_0 = \widetilde{c}_0 = \ldots = \widetilde{g}_0 = 0$$

Minimal set of solutions

$$Q^{(t)}(x,\vec{\mu}) = \sum_{r=1}^{t} C_r^{(t)}(\vec{\mu})\,f_r(x,\vec{\mu}){+}$$

$$\beta^{(t)}(\vec{\mu})\sum_{r=1}^{n-1-t} C_r^{(n-1-t)}(\vec{\mu^s})\,f_r(-x,\vec{\mu^s})\,,$$

Solution of the Baxter equation (H.V.,L.L.)

Asymptotic behaviour

$$\lim_{x\to\infty} x^k\, Q^{(t)}(x,\vec{\mu}) = 0 \quad , \quad k = 1, 2, \ldots, n-2\,,$$

$$C_t^{(t)}(\vec{\mu}) = C_{n-1-t}^{(n-1-t)}(\vec{\mu}) = 1$$

Linear relations

$$\left[\delta^{(r)}(\vec{\mu}) + \pi\, \cot(\pi x)\right] Q^{(r)}(x,\vec{\mu}) =$$

$$Q^{(r+1)}(x,\vec{\mu}) + \alpha^{(r)}(\vec{\mu})\, Q^{(r-1)}(x,\vec{\mu})$$

Holomorphic energies

$$\epsilon = \frac{b_1}{a_1} + n = b_0 - \frac{\mu_{n-1}}{\mu_n}$$

Quantization conditions

$$\epsilon^{(0)} = \epsilon^{(1)} = \ldots = \epsilon^{(n)} \;\; \textit{or} \;\; \delta^{(r)}(\vec{\mu}) = 0$$

Total energy

$$E_{m,\widetilde{m}} = \epsilon_m(\vec{\mu}) + \epsilon_{\widetilde{m}}(\vec{\mu}^{s*}) = \epsilon_m(\vec{\mu}^{s}) + \epsilon_{\widetilde{m}}(\vec{\mu}^{*})$$

Odderon in the BS representation (H.V.,L.L.)

New variables for the Odderon

$$t = \ln\left[\frac{p_1\,(p_1+p_2)}{(p_2+p_3)\,p_3}\right]\,,\;\; z = \frac{p_1\,p_3}{(p_2+p_3)(p_1+p_2)}$$

Transformation to the BS representation

$$\Psi_{m,\widetilde{m}} = \int \frac{d^2p_1 d^2p_3}{|p_1|^2|1-p_1-p_3|^2|p_3|^2} U^*_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{t},\overrightarrow{z})\Psi_{m,\widetilde{m}}$$

Factorization of the kernel

$$U_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{t},\overrightarrow{z}) = C_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}\;\; e^{it\frac{\lambda_1^*+\lambda_2^*}{2}}\;\; e^{it^*\frac{\lambda_1+\lambda_2}{2}}\;\; U^{\overrightarrow{z}}_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}$$

Explicit expression

$$U^{\overrightarrow{z}}_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}/K_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}} = \chi_{\lambda_1\lambda_2}(z^*)\chi_{\lambda_1^*\lambda_2^*}(z) - \chi_{\lambda_2\lambda_1}(z^*)\chi_{\lambda_2^*\lambda_1^*}(z)$$

$$\chi_{\lambda_1\lambda_2}(z) = a_{\lambda_1\lambda_2}\;\; z^{i\frac{\lambda_1-\lambda_2}{2}}\;\; F(-i\lambda_2\,,\,i\lambda_1\,;\,1+i(\lambda_1-\lambda_2)\,;\,z)\,,$$

$$a_{\lambda_1\lambda_2} = \frac{\Gamma(i\lambda_1)\Gamma(-i\lambda_2)}{\Gamma(1+i(\lambda_1-\lambda_2))}\,,$$

$$K_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}} = i\,|\lambda_2|^2\;\; \frac{\sinh(\pi\lambda_2)\;\sinh(\pi\lambda_1)}{\sinh(\pi(\lambda_1-\lambda_2))}$$

Baxter functions for Odderon (H.V.,L.L.)

Baxter equation

$$\left[2x^3 + m(m-1)x + \mu\right] Q\,(x;\, m, \mu) =$$

$$(x+1)^3\, Q\,(x+1;\, m, \mu) + (x-1)^3\, Q\,(x-1;\, m, \mu)$$

Auxiliary functions

$$f_2\,(x;\, m,\, \mu) = \sum_{l=0}^{\infty} \left[\frac{a_l(m,\, \mu)}{(x-l)^2} + \frac{b_l(m,\, \mu)}{x-l} \right] ,$$

$$f_1\,(x;\, m,\, \mu) = \sum_{l=0}^{\infty} \frac{a_l(m, \mu)}{x-l} \,,\ a_0 = 1\,,\ b_0 = 0$$

Recurrent relations

$$(r+1)^3\, a_{r+1} = \left[2\, r^3 + m(m-1)\, r - \mu\right]\, a_r - (r-1)^3\, a_{r-1}\,,$$

$$(r+1)^3\, b_{r+1} = \left[2\, r^3 + m(m-1)\, r - \mu\right]\, b_r - (r-1)^3\, b_{r-1}$$

$$-3\,(r+1)^2\, a_{r+1} + \left[6\, r^2 + m(m-1)\right]\, a_r - 3\,(r-1)^2\, a_{r-1}$$

Independent solutions

$$Q^{(2)}\,(x;\, m, \mu) = f_2\,(x;\, m,\, \mu) + B(m, \mu)\, f_1\,(x;\, m,\, \mu)\ ,$$

$$Q^{(1)}\,(x;\, m, \mu) = f_1\,(x;\, m,\, \mu) + C(m, \mu)\, f_1\,(-x;\, m,\, -\mu)\ ,$$

$$Q^{(0)}\,(x;\, m, \mu) = Q^{(2)}\,(-x;\, m, -\mu)\ ,\ \lim_{x\to\infty} x\, Q^{(r)}(x) = 0$$

Properties of Odderon (H.V., L.L.)

Solution for $\mu = 0$

$$Q(\lambda; m, 0) = -\frac{Q(\lambda, m)}{i\pi\lambda}\,, \quad \lim_{\lambda\to 0} Q(\lambda, m) = \frac{i\pi}{\lambda} + Q_0(m)\,,$$

$$Q_0(m) = \pi\left(\frac{\pi}{\sin(\pi m)} + \psi(m) + \psi(1-m) - 2\psi(1)\right)$$

Energies for $\mu = 0\,,\ \nu = 0$

$$E_{|n=0} = 2\pi - 8\ln 2 > 0\,,\ E_{|n=1} = 0$$

Eigenfunction of A for $\mu = 0\,,\ n = 2k+1$

$$\varphi = 1 + (-x)^m(-x^*)^{\widetilde{m}} + (x-1)^m(x^*-1)^{\widetilde{m}}\,,\ \|\varphi\|_1^2 < \infty$$

BLV solution with $\mu = 0\,,\ n = 2k+1$

$$f_{m,\widetilde{m}} = \sum_{r=1}^{3} \delta^2(\rho_{r,r+1})|\rho_{r+1,r+2}|^2 f_{m,\widetilde{m}}(\vec{\rho}_{r+1}, \vec{\rho}_{r+2}; \vec{\rho}_0)$$

Linear relation

$$[\pi\cot\pi x + \delta(m,\mu)]\,Q^{(1)}(x) = Q^{(2)}(x) - C(m,\mu)Q^{(2)}(-x)$$

Quantization of μ

$$\delta(m,\mu) = 0 \to \mu_1 = 0.20525\,,\ E_1 = 0.49434\,, \ldots$$

Pomeron at a non-zero temperature (H.V.,L.L.)

t-channel invariants

$$t = 4E^2\,,\;\; s = -2\vec{p}^{\,2}(1-\cos\theta)$$

Constraints at a finite temperature

$$A(\tau + \frac{1}{T}) = A(\tau)\,,\;\; E_l = 2\pi i T l$$

Constraints in the s-channel

$$A(y+2\pi) = A(y)\,,\;\; q_{2l} = l\,,\;\; 2\pi T = 1$$

Gluon Regge trajectory and Green function

$$\omega(-\vec{q}^{\,2}) = -\frac{g^2}{8\pi^2}\, N_c\, \Omega(-\vec{q}^{\,2})\,,\;\; \Omega(-\vec{q}^{\,2}) = \Omega(q) + \Omega(q^*)\,,$$

$$\Omega(q) = \frac{\pi T}{\lambda} + \frac{1}{2}\,[\psi(1+iq) + \psi(1-iq) - 2\psi(1)]\,,$$

$$G(\vec{\rho}_{\,12}) = -\frac{\pi\, T}{\lambda} + \ln\left(2\sinh\frac{\rho_{12}}{2}\right) + \ln\left(2\sinh\frac{\rho^*_{12}}{2}\right)$$

BFKL hamiltonian

$$H_{12} = h_{12} + h^*_{12}\,,\;\; h_{12} = -2\psi(1) +$$

$$\sum_{r=1}^{2}\left[\frac{1}{2}\psi(1+ip_r) + \frac{1}{2}\psi(1-ip_r) + \frac{1}{p_r}\,\ln\left(2\sinh\frac{\rho_{12}}{2}\right)\, p_r\right]$$

Pomeron wave function at finite T (H.V., L.L.)

Small-T expansion

$$h_{12} = h_{12}^0 + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} \sum_{r=1,2} \left[\frac{(-1)^{k+1}}{p_r^{2k}} + \frac{1}{p_r} \frac{\rho^{2k}}{(2k)!} \, p_r \right]$$

Integral of motion

$$A = 4 \sinh^2 \left(\frac{\rho}{2} \right) p_1 \, p_2 \, , \;\; \rho = \rho_{12} \, , \;\; [A, h_{12}] = 0$$

Eigenvalue equation

$$\left[\frac{Q^2}{4} + \frac{\partial^2}{\partial \rho^2} \right] \Psi(\rho, Q) = \frac{m(m-1)}{4 \sinh^2 \frac{\rho}{2}} \;\; \Psi(\rho, Q) \, , \; t = -4 \, |Q|^2$$

Holomorphic solution

$$\Psi^{(m)}(\rho, \, Q) = e^{\frac{i}{2} Q \, \rho} \;\; (e^{\rho} - 1)^m \;\; F \left(iQ + m, m; 2m; 1 - e^{\rho} \right)$$

Pomeron wave function

$$\Psi^{(m, \widetilde{m})}(\overrightarrow{\rho}, \, \overrightarrow{Q}) =$$

$$\frac{\Gamma(m)\Gamma(m + iQ)\Gamma(\widetilde{m})\Gamma(\widetilde{m} - iQ^*)}{\Gamma(2m)\Gamma(iQ)\Gamma(2\widetilde{m})\Gamma(-iQ^*)} \Psi^{(m)}(\rho, Q) \Psi^{(\widetilde{m})}(\rho^*, Q^*)$$

$$- \left(m \to 1 - m \, , \;\; \widetilde{m} \to 1 - \widetilde{m} \right)$$

Integrability at finite temperature (H.V., L.L.)

Conformal transformation

$$\rho_r = \ln\, \rho'_r$$

A and h_{12} in new coordinates

$$A = -(\rho'_{12})^2 \frac{\partial}{\partial \rho'_1}\, \frac{\partial}{\partial \rho'_2}\,,$$

$$h_{12} = \ln(p'_1\, p'_2) + \frac{1}{p'_1}\, \log(\rho'_{12})\, p'_1 + \frac{1}{p'_2}\, \log(\rho'_{12})\, p'_2 - 2\psi(1)$$

Operator identity

$$\frac{1}{2}\left(\psi\left(-z\frac{\partial}{\partial z}\right) + \psi\left(1 + z\frac{\partial}{\partial z}\right)\right) = \ln\, z + \ln\, \frac{\partial}{\partial z}$$

Pomeron wave function

$$\Psi^{(m,\widetilde{m})} = \left(\frac{\sinh\frac{\rho_{12}}{2}}{2\sinh\frac{\rho_{10}}{2}\,\sinh\frac{\rho_{20}}{2}}\right)^m \left(\frac{\sinh\frac{\rho^*_{12}}{2}}{2\sinh\frac{\rho^*_{10}}{2}\,\sinh\frac{\rho^*_{20}}{2}}\right)^{\widetilde{m}}$$

Integrable Heisenberg model

$$h = \sum_{r=1}^{n} h(\vec{M}_r \vec{M}_{r+1})\,,$$

$$M^z_k = \partial_k\,,\ M^+_k = e^{-\rho_k}\partial_k\,,\ M^-_k = -e^{\rho_k}\partial_k$$

Regge trajectories (H.V.,L.L.)

Pomeron wave function at small ρ

$$f(\vec{\rho},\vec{Q}) = \rho^m\ (\rho^*)^{\tilde{m}} + e^{i\,\delta_{m,\tilde{m}}(\vec{Q})}\ \rho^{1-m}\ (\rho^*)^{1-\tilde{m}}$$

Relative phase

$$e^{i\,\delta_{m,\tilde{m}}(\vec{Q})} = e^{i\,\delta^0_{m,\tilde{m}}(\vec{Q})}\ e^{i\,\delta^T_{m,\tilde{m}}(\vec{Q})}$$

Zero-temperature contribution

$$e^{i\,\delta^0_{m,\tilde{m}}(\vec{Q})} = \frac{\Gamma(m)}{\Gamma(1-m)}\frac{\Gamma(2-2\,m)}{\Gamma(2\,m)}\frac{\Gamma(\tilde{m})}{\Gamma(1-\tilde{m})}\frac{\Gamma(2-2\,\tilde{m})}{\Gamma(2\,\tilde{m})}$$

Temperature correction

$$e^{i\,\delta^T_{m,\tilde{m}}(\vec{Q})} = |Q|^{-4\,i\,\nu}\ \frac{\Gamma(\frac{1}{2}+i\,\nu+i\,Q)}{\Gamma(\frac{1}{2}-i\,\nu+i\,Q)}\frac{\Gamma(\frac{1}{2}+i\,\nu-i\,Q^*)}{\Gamma(\frac{1}{2}-i\,\nu-i\,Q^*)}$$

Pomeron trajectories

$$\omega_k(Q) = \alpha_s(Q)\ \chi(\nu_k(Q))\,,\ \alpha_s(Q) = \frac{4\,\pi}{9\,\log\frac{4\,Q^2}{\Lambda^2}}\,,$$

$$\frac{4}{9\,\alpha_s(Q)}\left[\frac{1}{\chi(\nu_k)}\int_0^{\nu_k} dx\ \chi(x) - \nu_k\right] - \frac{1}{2\,\pi}\ \delta^T_{\nu_k}(\vec{Q}) = k + \frac{1}{4},$$

$$\chi(\nu) = -\frac{6}{\pi}\mathrm{Re}\left[\gamma + \psi\left(\frac{1}{2}+i\,\nu\right)\right]$$

Anomalous dimensions

Parton distributions

$$f_a(x,Q^2) \;=\; \int_{k_\perp^2<Q^2} dk_\perp^2\, \varphi_a(x,k_\perp^2)\;,\;\; a=g,q,s$$

Mellin transformations

$$f_a(j,Q^2) = \int_0^1 d\,x\; x^{j-1}\, f_a(x,Q^2)$$

DGLAP equation and anomalous dimensions

$$\frac{d}{d\ln Q^2} f_a(j,Q^2) = \sum_b \gamma_{ab}(j) f_b(j,Q^2)$$

Light-cone components

$$O^a = \tilde{n}^{\mu_1}...\tilde{n}^{\mu_j}\, O^a_{\mu_1,...,\mu_j}\;,\;\; \tilde{\mathcal{O}}^a = \tilde{n}^{\mu_1}...\tilde{n}^{\mu_j}\, \tilde{\mathcal{O}}^a_{\mu_1,...,\mu_j}$$

Mixed projections

$$\tilde{n}^{\mu_1}...\tilde{n}^{\mu_{1+\omega}}\, O^a_{\mu_1,...,\mu_{1+\omega},\sigma_1,...,\sigma_{|n|}}\, l_\perp^{\sigma_1}...l_\perp^{\sigma_{|n|}}$$

Eigenvalues of the BFKL equation

$$\omega = j-1 = \omega(\gamma\,,|n|)\,,\;\; m=\gamma+\frac{n}{2}\,,\; \tilde{m}=\gamma-\frac{n}{2}$$

BFKL equation in $N = 4$ SUSY (A.K., L.L.)

Eigenvalue of the kernel

$$\omega = 4\,\hat{a}\;\chi(n,\gamma) + 4\;\hat{a}^2\,\Delta(n,\gamma)\,,\;\; \hat{a} = g^2 N_c/(16\pi^2)$$

Non-analyticity in QCD

$$\Delta_{QCD}(n,\gamma) = c_0\delta_{n,0} + c_2\delta_{n,0} + ...$$

Hermitial separability in $N = 4$ SUSY

$$\Delta(n,\gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M) + \rho(M^*)}{2\hat{a}/\omega}\,,\; M = \gamma + \frac{|n|}{2}\,,$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2)\,,\; \beta'(z) = \frac{1}{4}\left[\Psi'\left(\frac{z+1}{2}\right) - \Psi'\left(\frac{z}{2}\right)\right]$$

Property of maximal complexity

$$\phi(M) = 3\zeta(3) + \Psi''(M) - 2\Phi(M) + 2\beta'(M)\Big(\Psi(1) - \Psi(M)\Big)$$

$$\Phi(M) = \sum_{k=0}^{\infty} \frac{\beta'(k+1)}{k+M} +$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+M}\left(\Psi'(k+1) \;-\; \frac{\Psi(k+1) - \Psi(1)}{k+M}\right)$$

Relation between BFKL and DGLAP equations

Operator expansion of the Green function

$$< \phi(\overrightarrow{\rho_1})\, \phi(\overrightarrow{\rho_2})\, \phi(\overrightarrow{\rho_{1'}})\, \phi(\overrightarrow{\rho_{2'}}) >$$

$$\sim < \phi(\overrightarrow{\rho_1})\, \phi(\overrightarrow{\rho_2}) \sum_n |\rho_{1'2'}|^{2\Gamma_\omega(n)} \left(\frac{\rho_{1'2'}}{\rho^*_{1'2'}}\right)^{\frac{|n|}{2}} O_{|n|,\nu_\omega}(\overrightarrow{\rho_{1'}}) >$$

Perturbative expansion

$$\Gamma_\omega = 1 + \frac{|n|}{2} - \gamma_n(\omega)\,,\ \gamma_n(\omega)|_{\omega\to 0} = \frac{g^2 N_c}{4\pi^2\omega} + O(g^4)$$

Hypothesis about $\gamma(j)$ (A.K., L.L. (2000))

$$\lim_{j\to -r} \gamma(j) = \lim_{|n|\to -r-1} \gamma_n(j+r)$$

Predictions (A.K., L.L. (2003))

$$\gamma(j)|_{j\to -r} \;=\; \frac{g^2 N_c}{4\pi^2}\frac{1}{j+r} + \left(\frac{g^2 N_c}{4\pi^2}\right)^2 \frac{1+(-1)^r}{2(j+r)^3} + \ldots$$

Independent calculation (L.L. (1997))

$$\gamma(j) \;=\; \frac{g^2 N_c}{16\pi^2}\gamma^{LLA}(j),\quad \gamma^{LLA}(j) \;=\; 4\Big(\Psi(1) - \Psi(j-1)\Big)$$

Integrable Heisenberg spin model (L.L. (1997))

One loop anomalous dimensions

Wilson twist-2 operators

$$\mathcal{O}^g_{\mu_1,\ldots,\mu_j} = \hat{S} G^a_{\rho\mu_1} D_{\mu_2} D_{\mu_3} \ldots D_{\mu_{j-1}} G^a_{\rho\mu_j} \,,$$

$$\tilde{\mathcal{O}}^g_{\mu_1,\ldots,\mu_j} = \hat{S} G^a_{\rho\mu_1} D_{\mu_2} D_{\mu_3} \ldots D_{\mu_{j-1}} \tilde{G}^a_{\rho\mu_j} \,,$$

$$\mathcal{O}^q_{\mu_1,\ldots,\mu_j} = \hat{S} \bar{\Psi}^a \gamma_{\mu_1} D_{\mu_2} \ldots D_{\mu_j} \Psi^a \,,$$

$$\tilde{\mathcal{O}}^q_{\mu_1,\ldots,\mu_j} = \hat{S} \bar{\Psi}^a \gamma_5 \gamma_{\mu_1} D_{\mu_2} \ldots D_{\mu_j} \Psi^a \,,$$

$$\mathcal{O}^\varphi_{\mu_1,\ldots,\mu_j} = \hat{S} \bar{\Phi}^a D_{\mu_1} D_{\mu_2} \ldots D_{\mu_j} \Phi^a$$

Anomalous dimension matrix (L.L. (1999))

$$\gamma_{gg} = -\frac{8}{j-1} + \frac{8}{j} - \frac{8}{j+1} + \frac{8}{j+2} + 8S_1(j)\,, \quad \gamma_{q\varphi} = -\frac{16}{j}\,,$$

$$\gamma_{gq} = -\frac{8}{j-1} + \frac{8}{j} - \frac{4}{j+1}\,, \quad \gamma_{g\varphi} = -\frac{8}{j-1} + \frac{8}{j}\,,$$

$$\gamma_{qg} = -\frac{16}{j} + \frac{32}{j+1} - \frac{32}{j+2}\,, \quad \gamma_{qq} = -\frac{16}{j} + \frac{16}{j+1} + 8S_1(j),$$

$$\gamma_{\varphi g} = -\frac{24}{j+1} + \frac{24}{j+2}\,, \quad \gamma_{\varphi q} = -\frac{12}{j+1}\,, \quad \gamma_{\varphi\varphi} = 8S_1(j),$$

$$\widetilde{\gamma}_{gg} = -\frac{16}{j} + \frac{16}{j+1} + 8S_1(j)\,, \quad \widetilde{\gamma}_{gq} = -\frac{8}{j} + \frac{4}{j+1}\,,$$

$$\widetilde{\gamma}_{qg} = \frac{16}{j} - \frac{32}{j+1}\,, \quad \widetilde{\gamma}_{qq} = \frac{8}{j} - \frac{8}{j+1} + 8S_1(j)$$

Two-loop universal anomalous dimension

Diagonalization in the Born Approximation

$$\begin{vmatrix} 8S_1(j-2) & 0 & 0 \\ 0 & 8S_1(j) & 0 \\ 0 & 0 & 8S_1(j+2) \end{vmatrix},$$

$$\begin{vmatrix} 8S_1(j-1) & 0 \\ 0 & 8S_1(j+1) \end{vmatrix}$$

Universal anomalous dimension

$$U\gamma U^+ = \gamma_{uni}(j)\,,\ \gamma^{(0)}_{uni}(j) = -4S_1(j-2)\,,\ S_r(j) = \sum_{i=1}^{j}\frac{1}{i^r}$$

Most complicated functions (A.K.,L.L. (2000))

$$\gamma_{uni}(j) = \hat{a}\gamma^{(0)}_{uni}(j) + \hat{a}^2\gamma^{(1)}_{uni}(j) + \hat{a}^3\gamma^{(2)}_{uni}(j) + \ldots$$

Two-loop dimension (A.K.,L.L.,V.V (2003))

$$\gamma^{(1)}_{uni}(j+2)/8 = 2S_1(j)\left(S_2(j)+S_{-2}(j)\right)-2S_{-2,1}(j)+S_3(j)$$

$$+S_{-3}(j),\ S_{-r}(j) = \sum_{i=1}^{j}\frac{(-1)^i}{i^r},\ S_{-2,1} = \sum_{m=1}^{j}\frac{(-1)^m}{m^2}S_1(m)$$

Three-loop anomalous dimension

QCD dimension matrix (S.M., J.V., A. V. (2004)

$$\gamma_{QCD}^{ik}(j) = ..., \; i,k = g,q$$

N=4 universal dimension (A.K.,L.L.,A.O.,V.V.)

$$\gamma_{uni}^{N=4}(j) = \hat{a}\gamma_{uni}^{(0)}(j) + \hat{a}^2\gamma_{uni}^{(1)}(j) + \hat{a}^3\gamma_{uni}^{(2)}(j) + ...$$

Most complicated functions

$$\gamma_{uni}^{(2)}(j+2) = 24\,S_{-2,1,1,1} - 12\,(S_{-3,1,1} + S_{-2,1,2} + S_{-2,2,1})$$

$$+6\,(S_{-4,1} + S_{-3,2} + S_{-2,3}) - 3\,S_{-5} - 2\,S_3\,S_{-2} - S_5$$

$$-2\,S_1^2\,(3\,S_{-3} + S_3 - 2\,S_{-2,1}) - S_2\,(S_{-3} + S_3 - 2\,S_{-2,1})$$

Harmonic sums

$$S_a(j) = \sum_{m=1}^{j} \frac{1}{m^a}\,, \;\; S_{a,b,c,\cdots}(j) = \sum_{m=1}^{j} \frac{1}{m^a}\,S_{b,c,\cdots}(m)\,,$$

$$S_{-a}(j) = \sum_{m=1}^{j} \frac{(-1)^m}{m^a}, \; S_{-a,b,\cdots}(j) = \sum_{m=1}^{j} \frac{(-1)^m}{m^a}\,S_{b,\cdots}(m),$$

$$\overline{S}_{-a,b,c\cdots}(j) = (-1)^j S_{-a,b,\cdots}(j) + S_{-a,b,\cdots}(\infty)\Big(1-(-1)^j\Big)$$

Comparison with other approaches (A.K.,L.L.,A.O.,V.V.)

Singularities at $j = 1 + \omega \to 0$

$$\gamma_{uni}^{N=4}(j) = \hat{a}\frac{4}{\omega} - 32\zeta_3\,\hat{a}^2 + 32\zeta_3\,\hat{a}^3\frac{1}{\omega} + ...$$

DL resummation at $j + 2r = \omega \to 0$

$$\gamma_{uni} = 4\,\frac{\hat{a}}{\omega} + \frac{\gamma_{uni}^2}{\omega}$$

Anomalous dimensions at large j

$$\gamma_{uni} = a(z)\,\ln j\,,\;\; z = \frac{\alpha N_c}{\pi} = 4\hat{a}$$

Perturbative results

$$a = -z + \frac{\pi^2}{12}\,z^2 - \frac{11}{720}\pi^4 z^3 + ...$$

Polyakov AdS/CFT prediction

$$\lim_{z\to\infty} a = -z^{1/2} + \frac{3\ln 2}{4\pi} + ...$$

Resummation

$$\widetilde{a} = -z + \frac{\pi^2}{12}\,\widetilde{a}^2 = -z + \frac{\pi^2}{12}\,z^2 - \frac{1}{72}\pi^4 z^3 + ...$$

Results

1. Reggeization of gluons and quarks in QCD
2. Pomeron as a composite state of two reggeons
3. Odderon as a composite state of three reggeons
4. Möbius invariance of the BFKL equation
5. Holomorphic separability of BFKL Hamiltonian
6. Duality symmetry of BKP equations at $N_c \to \infty$
7. Integrability of the BFKL dynamics at $N_c \to \infty$
8. Effective action for reggeized gluon interactions
9. s- and t- chanel unitarity
10. Next-to-leading corrections to the BFKL equation
11. Remarkable properties of high energy dynamics in N=4 SUSY