June 2005

Workshop on Low x Physics
Sinaia

"Revisiting the Collinear Limit at Small x"

1. Collinear behaviour of the NLL Green's function
2. Renormalization-Group improved BFKL kernel
3. Bessel method in γ space
4. Bessel method in $\mathbf{k}_t$ space
5. Present and future work

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1. Collinear behaviour of the NLL Green's function

$\overline{\text{MS}}$ <u>NLL BFKL kernel</u> [Fadin, Lipatov]:

$$\int d^2\vec{q}_2\, \mathcal{K}\left(\vec{q}_1, \vec{q}_2\right) f\left(q_2^2\right) =$$

$$\int \frac{d^2\vec{q}_2}{\left|q_1^2 - q_2^2\right|} \left\{ \left[\bar{\alpha}_s + \bar{\alpha}_s^2 \left(\mathcal{S} - \frac{\beta_0}{4N_c} \ln \left(\frac{\left|q_1^2 - q_2^2\right|^2}{\max\left(q_1^2, q_2^2\right) \mu^2} \right) \right) \right] \right.$$

$$\left. \times \left(f\left(q_2^2\right) - 2 \frac{\min\left(q_1^2, q_2^2\right)}{\left(q_1^2 + q_2^2\right)} f\left(q_1^2\right) \right) - \frac{\bar{\alpha}_s^2}{4} \left(\mathcal{T}\left(q_1^2, q_2^2\right) + \ln^2 \left(\frac{q_1^2}{q_2^2} \right) \right) f\left(q_2^2\right) \right\},$$

An alternative is the GB scheme:

$$\bar{\alpha}_s^{\text{GB}} = \bar{\alpha}_s \left(1 + \mathcal{S}\, \bar{\alpha}_s\right)$$

where $\mathcal{S} = \left(4 - \pi^2 + 5\beta_0/N_c\right)/12$ common in soft gluon resummations.

Usual notation:

$$\int d^2\vec{q}_2\, \mathcal{K}\left(\vec{q}_1, \vec{q}_2\right) \left(\frac{\bar{\alpha}_s\left(q_2^2\right)}{\bar{\alpha}_s\left(q_1^2\right)} \right)^{-\frac{1}{2}} \left(\frac{q_2^2}{q_1^2} \right)^{\gamma - 1} = \bar{\alpha}_s\left(q_1^2\right) \chi_0\left(\gamma\right) + \bar{\alpha}_s^2 \chi_1\left(\gamma\right)$$

1. Collinear behaviour of the NLL Green's function

$$\chi_0(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$$

$$\chi_1(\gamma) = \mathcal{S}\chi_0(\gamma) + \frac{1}{4}\left(\psi''(\gamma) + \psi''(1-\gamma)\right) - \frac{1}{4}\left(\phi(\gamma) + \phi(1-\gamma)\right)$$

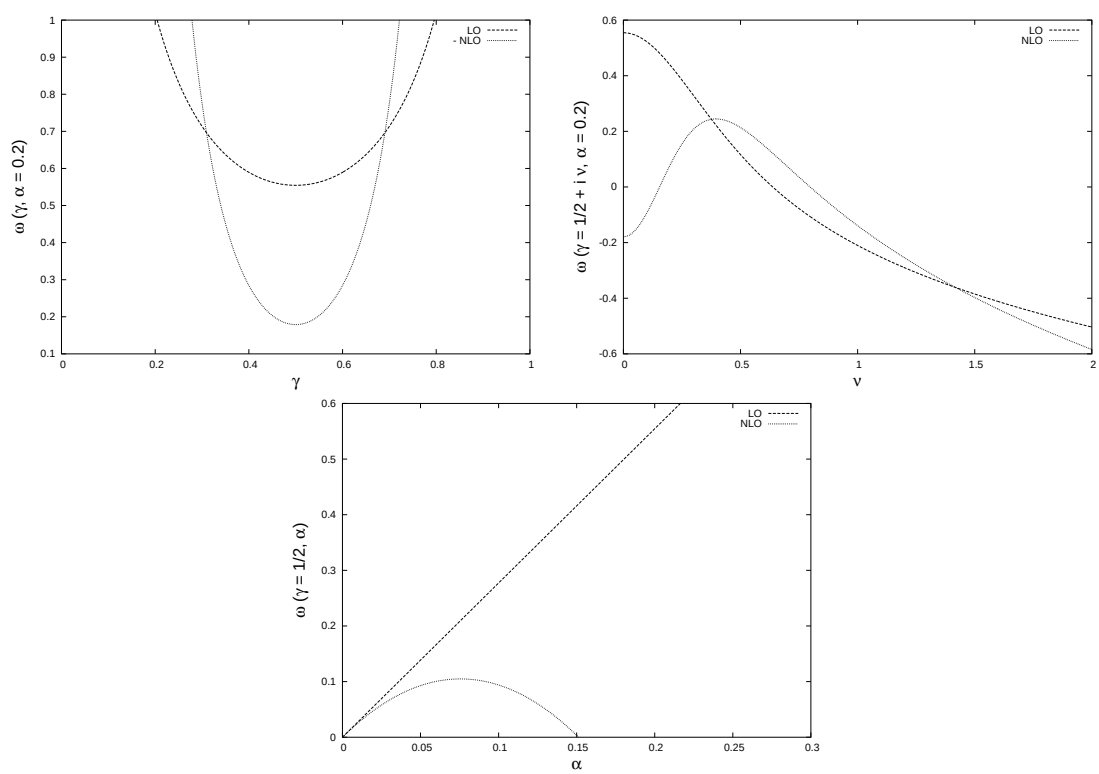
$$-\frac{\pi^2\cos(\pi\gamma)}{4\sin^2(\pi\gamma)(1-2\gamma)}\left(3 + \left(1 + \frac{n_f}{N_c^3}\right)\frac{(2+3\gamma(1-\gamma))}{(3-2\gamma)(1+2\gamma)}\right) + \frac{3}{2}\zeta_3 - \frac{\beta_0}{8N_c}\chi_0^2(\gamma)$$

Collinear/anticollinear behaviour, fixed coupling:

$$\chi_0(\gamma) \simeq \frac{1}{\gamma} + \{\gamma \to 1-\gamma\}$$

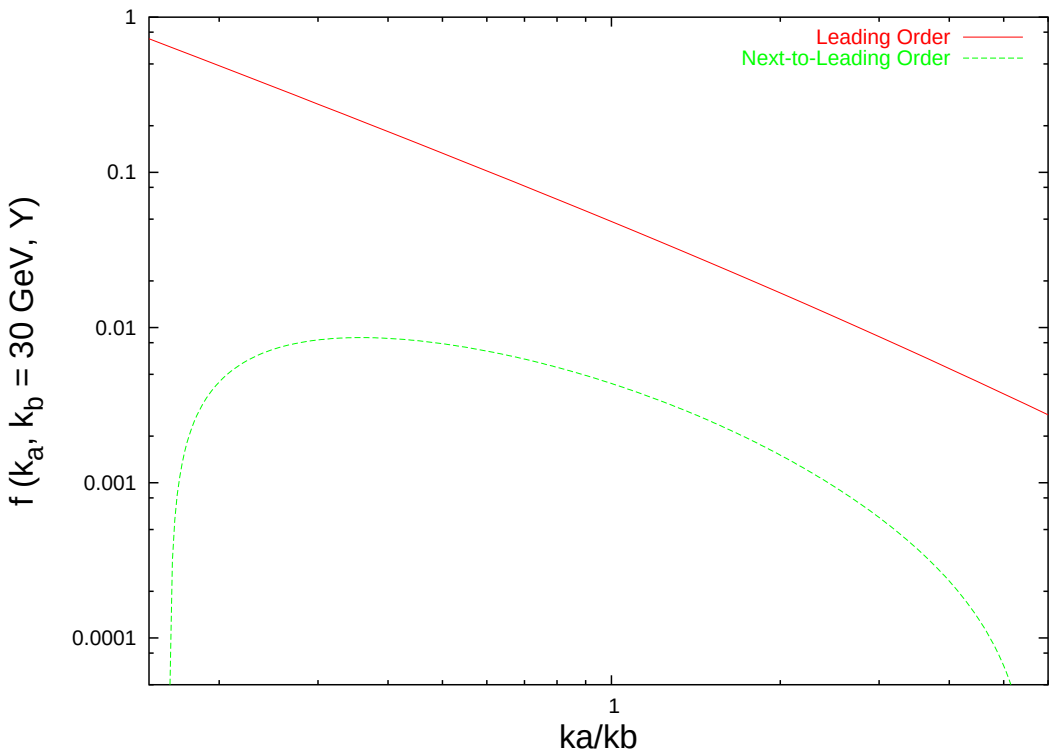
$$\chi_1(\gamma) \simeq \frac{\mathrm{a}}{\gamma} + \frac{\mathrm{b}}{\gamma^2} - \frac{1}{2\gamma^3} + \{\gamma \to 1-\gamma\}$$

$$\mathrm{a} = \frac{5}{12}\frac{\beta_0}{N_c} - \frac{13}{36}\frac{n_f}{N_c^3} - \frac{55}{36}, \quad \mathrm{b} = -\frac{1}{8}\frac{\beta_0}{N_c} - \frac{n_f}{6N_c^3} - \frac{11}{12}$$



1. Collinear behaviour of the NLL Green's function

This produces oscillations in $\mathbf{k}_t$-space [Ross]



generating a negative Green's function in asymmetric configurations.

LL eigenfunctions [Salam]

$$f\left(\vec{k}_a,\vec{k}_b,Y\right) = \frac{1}{\pi|\vec{k}_a||\vec{k}_b|}\int\frac{d\omega}{2\pi i}e^{\omega Y}\int\frac{d\gamma}{2\pi i}\left(\frac{\vec{k}_a^2}{\vec{k}_b^2}\right)^{\gamma-\frac{1}{2}}\frac{1}{\omega-\omega(\gamma)}$$

$$\omega(\gamma) = \bar{\alpha}_s\left(2\Psi(1)-\Psi(\gamma)-\Psi(1-\gamma)\right)$$

$$\begin{aligned} f\left(\vec{k}_a,\vec{k}_b,s\right) &= \frac{1}{\pi|\vec{k}_a||\vec{k}_b|}\int\frac{d\omega}{2\pi i}\left(\frac{s}{k_a k_b}\right)^{\omega}\int\frac{d\gamma}{2\pi i}\left(\frac{\vec{k}_a^2}{\vec{k}_b^2}\right)^{\gamma-\frac{1}{2}}\frac{1}{\omega-\omega(\gamma)} \\ &= \frac{1}{\pi|\vec{k}_a||\vec{k}_b|}\int\frac{d\omega}{2\pi i}\left(\frac{s}{k_a^2}\right)^{\omega}\int\frac{d\gamma}{2\pi i}\left(\frac{\vec{k}_a^2}{\vec{k}_b^2}\right)^{\gamma-\frac{1}{2}+\frac{\omega}{2}}\frac{1}{\omega-\omega(\gamma)} \\ &= \frac{1}{\pi|\vec{k}_a||\vec{k}_b|}\int\frac{d\omega}{2\pi i}\left(\frac{s}{k_a^2}\right)^{\omega}\int\frac{d\gamma}{2\pi i}\left(\frac{\vec{k}_a^2}{\vec{k}_b^2}\right)^{\gamma-\frac{1}{2}}\frac{1}{\omega-\omega(\gamma-\frac{\omega}{2})} \end{aligned}$$

In collinear limit $\gamma\sim 0$ $\omega(\gamma)\sim\frac{\bar{\alpha}_s}{\gamma}$

$$\omega\sim\frac{\bar{\alpha}_s}{\gamma-\frac{\omega}{2}}\longrightarrow\omega\sim\sum_{n=0}^{\infty}\frac{(2n)!}{2^n n!(n+1)!}\frac{(\bar{\alpha}_s)^{n+1}}{\gamma^{2n+1}}$$

$$\omega\sim\frac{\bar{\alpha}_s}{\gamma}+\frac{\bar{\alpha}_s^2}{2\gamma^3}+\sum_{n=2}^{\infty}\frac{(2n)!}{2^n n!(n+1)!}\frac{(\bar{\alpha}_s)^{n+1}}{\gamma^{2n+1}}$$

Double logs not allowed by DGLAP. Second one cancelled by NLL kernel. The rest are numerically large.

2. Renormalization-Group improved BFKL kernel

Solution [Salam, Ciafaloni, Colferai]:

Redefine the original kernel:

$$\omega^{\text{new}}(\gamma) \quad = \quad \omega\left(\gamma + \frac{\omega}{2}\right)$$

$$\begin{aligned}
f\left(\vec{k}_a, \vec{k}_b, s\right) &= \frac{1}{\pi|\vec{k}_a||\vec{k}_b|}\int\frac{d\omega}{2\pi i}\left(\frac{s}{k_a k_b}\right)^{\omega}\int\frac{d\gamma}{2\pi i}\left(\frac{\vec{k}_a^2}{\vec{k}_b^2}\right)^{\gamma-\frac{1}{2}}\frac{1}{\omega-\omega^{\text{new}}(\gamma)} \\
&= \frac{1}{\pi|\vec{k}_a||\vec{k}_b|}\int\frac{d\omega}{2\pi i}\left(\frac{s}{k_a^2}\right)^{\omega}\int\frac{d\gamma}{2\pi i}\left(\frac{\vec{k}_a^2}{\vec{k}_b^2}\right)^{\gamma-\frac{1}{2}+\frac{\omega}{2}}\frac{1}{\omega-\omega(\gamma+\frac{\omega}{2})} \\
&= \frac{1}{\pi|\vec{k}_a||\vec{k}_b|}\int\frac{d\omega}{2\pi i}\left(\frac{s}{k_a^2}\right)^{\omega}\int\frac{d\gamma}{2\pi i}\left(\frac{\vec{k}_a^2}{\vec{k}_b^2}\right)^{\gamma-\frac{1}{2}}\frac{1}{\omega-\omega(\gamma)}
\end{aligned}$$

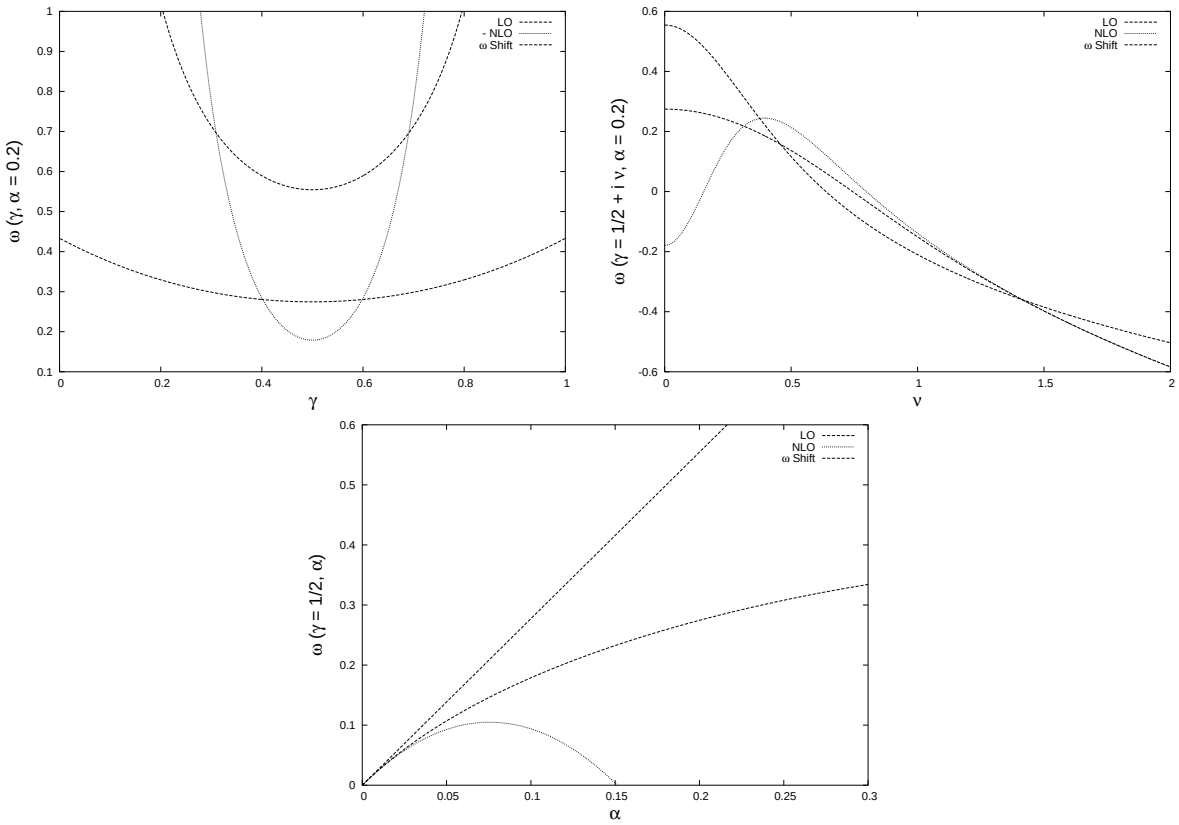
Collinear limit free from unphysical double logs.

In general need to match NLL accuracy [Salam]

$$\begin{aligned}
\omega \quad &= \\
&\bar{\alpha}_s\left(1+\left(\mathrm{a}+\frac{\pi^2}{6}\right)\bar{\alpha}_s\right)\left(2\psi(1)-\psi\left(\gamma+\frac{\omega}{2}-\mathrm{b}\,\bar{\alpha}_s\right)-\psi\left(1-\gamma+\frac{\omega}{2}-\mathrm{b}\,\bar{\alpha}_s\right)\right) \\
&\quad +\bar{\alpha}_s^2\left(\chi_1(\gamma)+\left(\frac{1}{2}\chi_0(\gamma)-\mathrm{b}\right)\left(\psi'(\gamma)+\psi'(1-\gamma)\right)-\left(\mathrm{a}+\frac{\pi^2}{6}\right)\chi_0(\gamma)\right)
\end{aligned}$$

2. Renormalization-Group improved BFKL kernel

The kernel is stable again



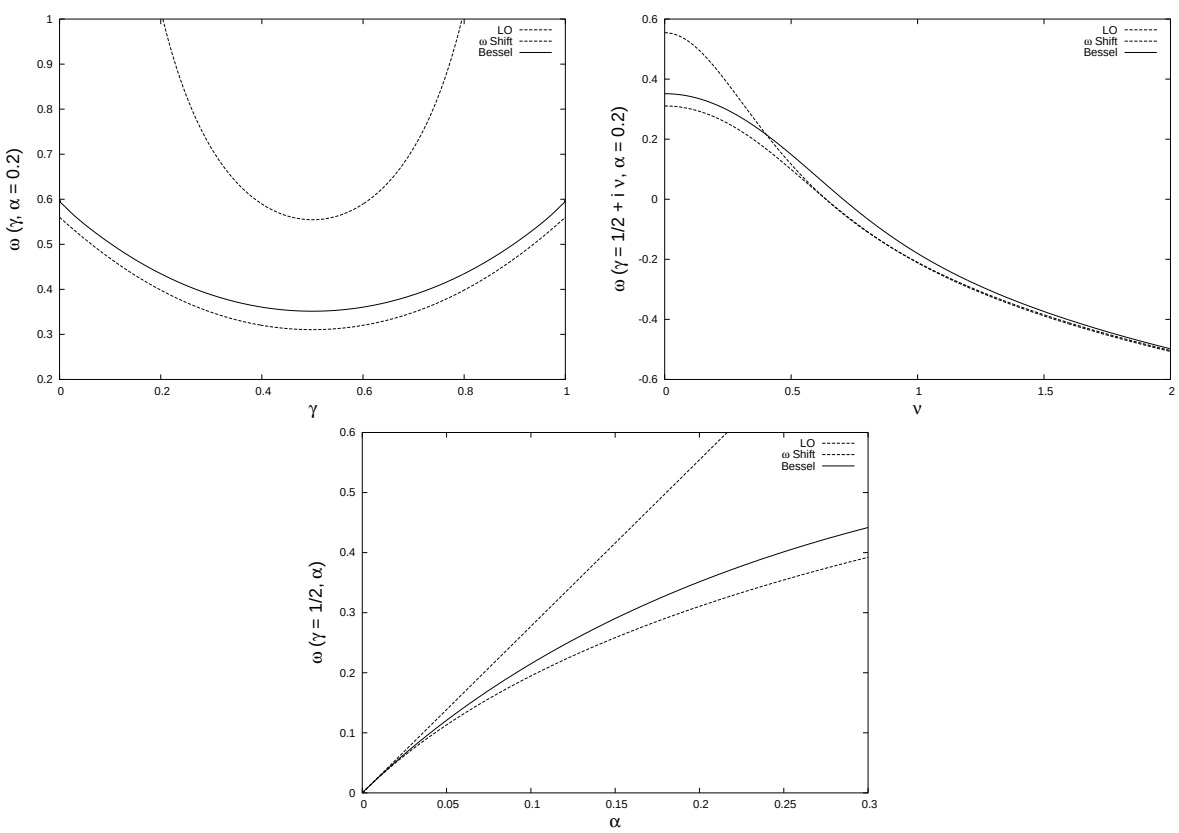
3. Bessel method in γ space

<u>Main idea</u>: [Sabio Vera, NPB 2005]

$$\omega = \bar{\alpha}_s \left(2\psi(1) - \psi\left(\gamma + \frac{\omega}{2}\right) - \psi\left(1 - \gamma + \frac{\omega}{2}\right)\right)$$

can be approximated very well by

$$\omega = \int_0^1 \frac{dx}{1-x} \left\{ \left(x^{\gamma-1} + x^{-\gamma}\right) \sqrt{\frac{2\bar{\alpha}_s}{\ln^2 x}} J_1\left(\sqrt{2\bar{\alpha}_s \ln^2 x}\right) - 2\bar{\alpha}_s \right\}$$



<u>Derivation</u> for RG-improved NLL corrections:

Take a general shift

$$\omega = \bar{\alpha}_s\,(1+\mathsf{A}\,\bar{\alpha}_s)\left(2\psi(1)-\psi\left(\gamma+\frac{\omega}{2}+\mathsf{B}\,\bar{\alpha}_s\right)-\psi\left(1-\gamma+\frac{\omega}{2}+\mathsf{B}\,\bar{\alpha}_s\right)\right)$$

Use this representation

$$\chi_0(\gamma) \quad = \quad \sum_{m=0}^{\infty}\left(\frac{1}{\gamma+m}+\frac{1}{1-\gamma+m}-\frac{2}{m+1}\right)$$

$$\omega = \bar{\alpha}_s\,(1+\mathsf{A}\bar{\alpha}_s)\sum_{m=0}^{\infty}\left(\frac{1}{\gamma+m+\frac{\omega}{2}+\mathsf{B}\,\bar{\alpha}_s}+\frac{1}{1-\gamma+m+\frac{\omega}{2}+\mathsf{B}\,\bar{\alpha}_s}-\frac{2}{m+1}\right)$$

Close to $\gamma \simeq -m$ a good approximation is

$$\omega \quad \simeq \quad \frac{\bar{\alpha}_s\,(1+\mathsf{A}\bar{\alpha}_s)}{\gamma+m+\frac{\omega}{2}+\mathsf{B}\,\bar{\alpha}_s}$$

with the straightforward solution

$$\omega \quad = \quad -(\gamma+m+\mathsf{B}\,\bar{\alpha}_s)+|\gamma+m+\mathsf{B}\,\bar{\alpha}_s|\left(1+\frac{2\bar{\alpha}_s\,(1+\mathsf{A}\bar{\alpha}_s)}{(\gamma+m+\mathsf{B}\,\bar{\alpha}_s)^2}\right)^{\frac{1}{2}}$$

Perform this at <u>all poles</u> and put it together ...

3. Bessel method in γ space

$$\omega = \sum_{m=0}^{\infty} \left\{ -(1+2m+2\,\mathsf{B}\,\bar{\alpha}_s) + |\gamma+m+\mathsf{B}\,\bar{\alpha}_s| \left(1+\frac{2\bar{\alpha}_s\,(1+\mathsf{A}\bar{\alpha}_s)}{(\gamma+m+\mathsf{B}\,\bar{\alpha}_s)^2}\right)^{\frac{1}{2}} \right.$$
$$\left. + |1-\gamma+m+\mathsf{B}\,\bar{\alpha}_s| \left(1+\frac{2\bar{\alpha}_s\,(1+\mathsf{A}\bar{\alpha}_s)}{(1-\gamma+m+\mathsf{B}\,\bar{\alpha}_s)^2}\right)^{\frac{1}{2}} - \frac{2\bar{\alpha}_s\,(1+\mathsf{A}\bar{\alpha}_s)}{m+1} \right\}$$

All orders expansion of the square roots

$$\omega = \sum_{m=0}^{\infty} \left\{ -\frac{2\bar{\alpha}_s\,(1+\mathsf{A}\bar{\alpha}_s)}{m+1} \right.$$
$$\left. + \left(\sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^n n!(n+1)!} \frac{\left(\bar{\alpha}_s + \mathsf{A}\bar{\alpha}_s^2\right)^{n+1}}{(\gamma+m+\mathsf{B}\,\bar{\alpha}_s)^{2n+1}} + \{\gamma \to 1-\gamma\} \right) \right\}$$

At $\gamma = 0, 1$ it generates

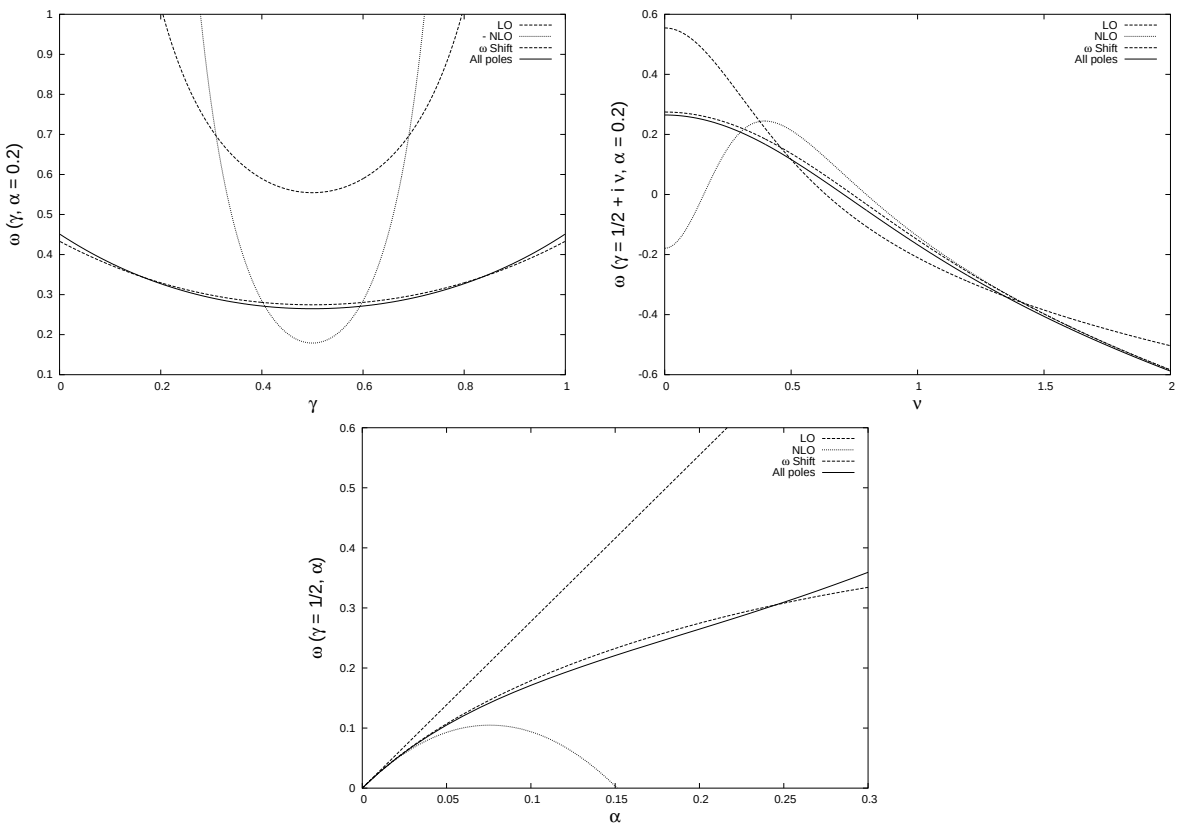
$$\omega \simeq \frac{\bar{\alpha}_s}{\gamma} + \bar{\alpha}_s^2 \left(\frac{\mathsf{A}}{\gamma} - \frac{\mathsf{B}}{\gamma^2} - \frac{1}{2\gamma^3} \right) + \{\gamma \to 1-\gamma\}$$

To match original NLL: $\mathsf{A} = \mathsf{a}$ and $\mathsf{B} = -\mathsf{b}$.

Finally ...

3. Bessel method in γ space

$$
\begin{aligned}
\omega &= \bar{\alpha}_s \chi_0(\gamma) + \bar{\alpha}_s^2 \chi_1(\gamma) \\
&+ \left\{ \sum_{m=0}^{\infty} \left[\left(\sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^n n!(n+1)!} \frac{\left(\bar{\alpha}_s + \mathrm{a}\,\bar{\alpha}_s^2\right)^{n+1}}{(\gamma + m - \mathrm{b}\,\bar{\alpha}_s)^{2n+1}} \right) \right.\right. \\
&\left.\left. -\frac{\bar{\alpha}_s}{\gamma + m} - \bar{\alpha}_s^2 \left(\frac{\mathrm{a}}{\gamma + m} + \frac{\mathrm{b}}{(\gamma + m)^2} - \frac{1}{2(\gamma + m)^3} \right) \right] + \{\gamma \to 1 - \gamma\} \right\}.
\end{aligned}
$$



Introduce the auxiliary function

$$\begin{aligned}
\Omega(\gamma) &\equiv \int_0^\infty \frac{dq^2}{|k^2-q^2|}\left(\left(\frac{q^2}{k^2}\right)^{\gamma-1-\mathsf{b}\bar{\alpha}_s\frac{|k-q|}{k-q}} - 2\frac{\min\left(k^2,q^2\right)}{k^2+q^2}\right) \\
&= \int_0^1 \frac{dx}{1-x}\left(x^{\gamma-1-\bar{\alpha}_s\mathsf{b}} + x^{-\gamma-\bar{\alpha}_s\mathsf{b}} - 2\right) \\
&= 2\psi(1) - \psi(\gamma-\bar{\alpha}_s\mathsf{b}) - \psi(1-\gamma-\bar{\alpha}_s\mathsf{b}) \\
&= \sum_{m=0}^\infty\left(\frac{1}{\gamma+m-\mathsf{b}\,\bar{\alpha}_s} + \frac{1}{1-\gamma+m-\mathsf{b}\,\bar{\alpha}_s} - \frac{2}{m+1}\right)
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^\infty \frac{(-1)^n\left(\bar{\alpha}_s+\mathsf{a}\,\bar{\alpha}_s^2\right)^{n+1}}{2^n n!\,(n+1)!}\frac{d^{2n}}{d\gamma^{2n}}\Omega(\gamma) \quad = \\
&\int_0^\infty \frac{dq^2}{|k^2-q^2|}\left(\frac{q^2}{k^2}\right)^{\gamma-1-\mathsf{b}\bar{\alpha}_s\frac{|k-q|}{k-q}}\sum_{n=1}^\infty \frac{(-1)^n\left(\bar{\alpha}_s+\mathsf{a}\,\bar{\alpha}_s^2\right)^{n+1}}{2^n n!\,(n+1)!}\ln^{2n}\left(\frac{q^2}{k^2}\right) \\
&\quad = \sum_{m=0}^\infty\sum_{n=1}^\infty \frac{(-1)^n(2n)!}{2^n n!\,(n+1)!}\frac{\left(\bar{\alpha}_s+\mathsf{a}\,\bar{\alpha}_s^2\right)^{n+1}}{(\gamma+m-\mathsf{b}\,\bar{\alpha}_s)^{2n+1}} + \{\gamma\to 1-\gamma\}\,.
\end{aligned}$$

$$\begin{aligned}
&\int_0^\infty \frac{dq^2}{|k^2-q^2|}\Bigg(-2\left(\bar{\alpha}_s+\mathsf{a}\,\bar{\alpha}_s^2\right)\frac{\min(k^2,q^2)}{k^2+q^2} \\
&\quad + \left(\frac{q^2}{k^2}\right)^{\gamma-1-\mathsf{b}\bar{\alpha}_s\frac{|k-q|}{k-q}}\sqrt{\frac{2\left(\bar{\alpha}_s+\mathsf{a}\,\bar{\alpha}_s^2\right)}{\ln^2\left(\frac{q^2}{k^2}\right)}}J_1\left(\sqrt{2\left(\bar{\alpha}_s+\mathsf{a}\,\bar{\alpha}_s^2\right)\ln^2\left(\frac{q^2}{k^2}\right)}\right)\Bigg)
\end{aligned}$$

It generates just a Bessel function in $\mathbf{k}_t$ space:

$$\int_0^\infty \frac{dq^2}{|k^2-q^2|}\left(-2\left(\bar{\alpha}_s+\mathsf{a}\,\bar{\alpha}_s^2\right)\frac{\min(k^2,q^2)}{k^2+q^2}\right.$$
$$\left.+\left(\frac{q^2}{k^2}\right)^{\gamma-1-\mathsf{b}\bar{\alpha}_s\frac{|k-q|}{k-q}}\sqrt{\frac{2\left(\bar{\alpha}_s+\mathsf{a}\,\bar{\alpha}_s^2\right)}{\ln^2\left(\frac{q^2}{k^2}\right)}}J_1\left(\sqrt{2\left(\bar{\alpha}_s+\mathsf{a}\,\bar{\alpha}_s^2\right)\ln^2\left(\frac{q^2}{k^2}\right)}\right)\right)$$

$$\left(\frac{q^2}{k^2}\right)^{-\mathsf{b}\bar{\alpha}_s\frac{|k-q|}{k-q}}\sqrt{\frac{2\left(\bar{\alpha}_s+\mathsf{a}\,\bar{\alpha}_s^2\right)}{\ln^2\left(\frac{q^2}{k^2}\right)}}J_1\left(\sqrt{2\left(\bar{\alpha}_s+\mathsf{a}\,\bar{\alpha}_s^2\right)\ln^2\left(\frac{q^2}{k^2}\right)}\right)$$
$$\simeq\bar{\alpha}_s+\mathsf{a}\,\bar{\alpha}_s^2-\frac{\bar{\alpha}_s^2}{4}\ln^2\left(\frac{q^2}{k^2}\right)-\mathsf{b}\,\bar{\alpha}_s^2\frac{|k-q|}{k-q}\ln\left(\frac{q^2}{k^2}\right)+\ldots$$

Finally ...

Prescription:

Remove the double log

$$-\frac{\bar{\alpha}_s^2}{4}\frac{1}{(\vec{q}-\vec{k})^2}\ln^2\left(\frac{q^2}{k^2}\right)$$

and introduce the all-orders resummation:

$$\frac{1}{(\vec{q}-\vec{k})^2}\left\{\left(\frac{q^2}{k^2}\right)^{-\mathrm{b}\bar{\alpha}_s\frac{|k-q|}{k-q}}\sqrt{\frac{2\left(\bar{\alpha}_s+\mathrm{a}\,\bar{\alpha}_s^2\right)}{\ln^2\left(\frac{q^2}{k^2}\right)}}J_1\left(\sqrt{2\left(\bar{\alpha}_s+\mathrm{a}\,\bar{\alpha}_s^2\right)\ln^2\left(\frac{q^2}{k^2}\right)}\right)\right.$$
$$\left.-\bar{\alpha}_s-\mathrm{a}\,\bar{\alpha}_s^2+\mathrm{b}\,\bar{\alpha}_s^2\frac{|k-q|}{k-q}\ln\left(\frac{q^2}{k^2}\right)\right\}$$

For symmetric configurations $q^2\sim k^2$:

$$J_1\left(\sqrt{2\bar{\alpha}_s\ln^2\left(\frac{q^2}{k^2}\right)}\right)\quad\simeq\quad\sqrt{\frac{\bar{\alpha}_s}{2}\ln^2\left(\frac{q^2}{k^2}\right)}$$

For asymmetric configurations $q^2 <<,>> k^2$:

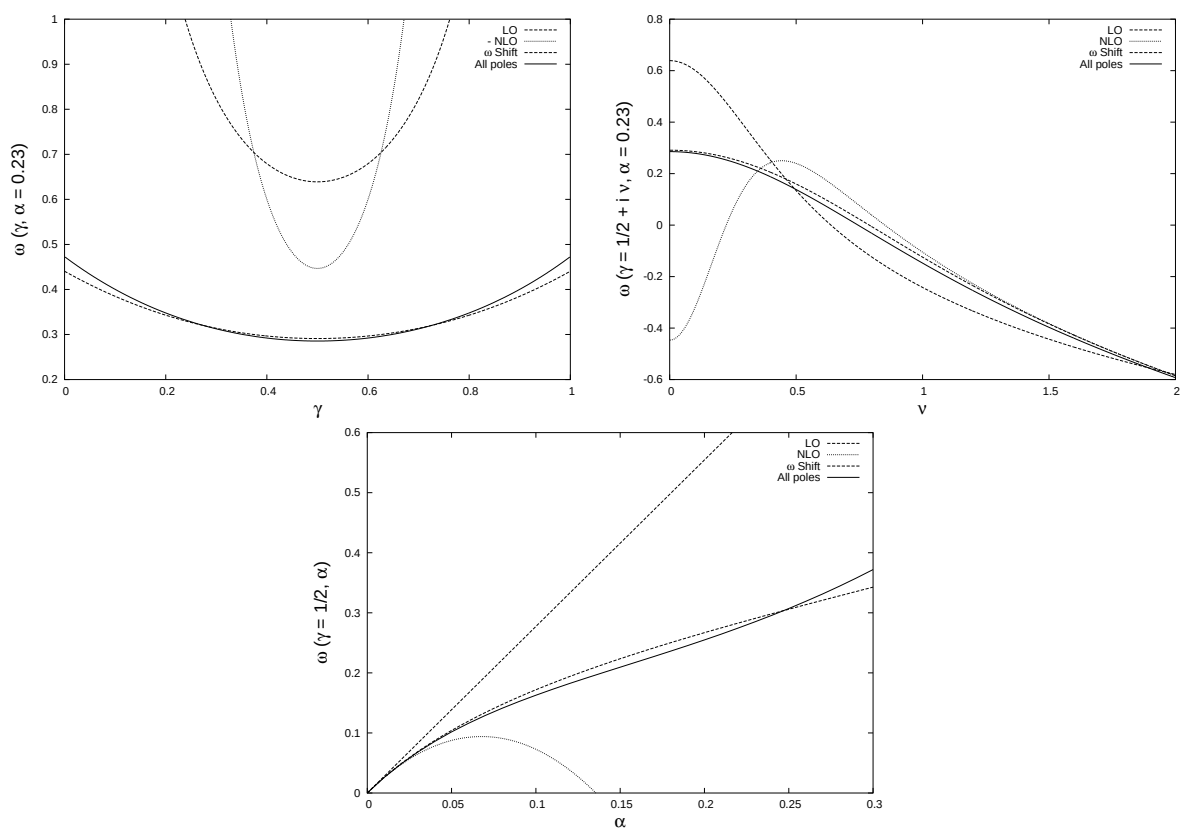
$$J_1\left(\sqrt{2\bar{\alpha}_s\ln^2\left(\frac{q^2}{k^2}\right)}\right)\simeq\left(\frac{2}{\pi^2\bar{\alpha}_s\ln^2\left(\frac{q^2}{k^2}\right)}\right)^{\frac{1}{4}}\cos\left(\sqrt{2\bar{\alpha}_s\ln^2\left(\frac{q^2}{k^2}\right)}-\frac{3\pi}{4}\right)$$

In GB scheme

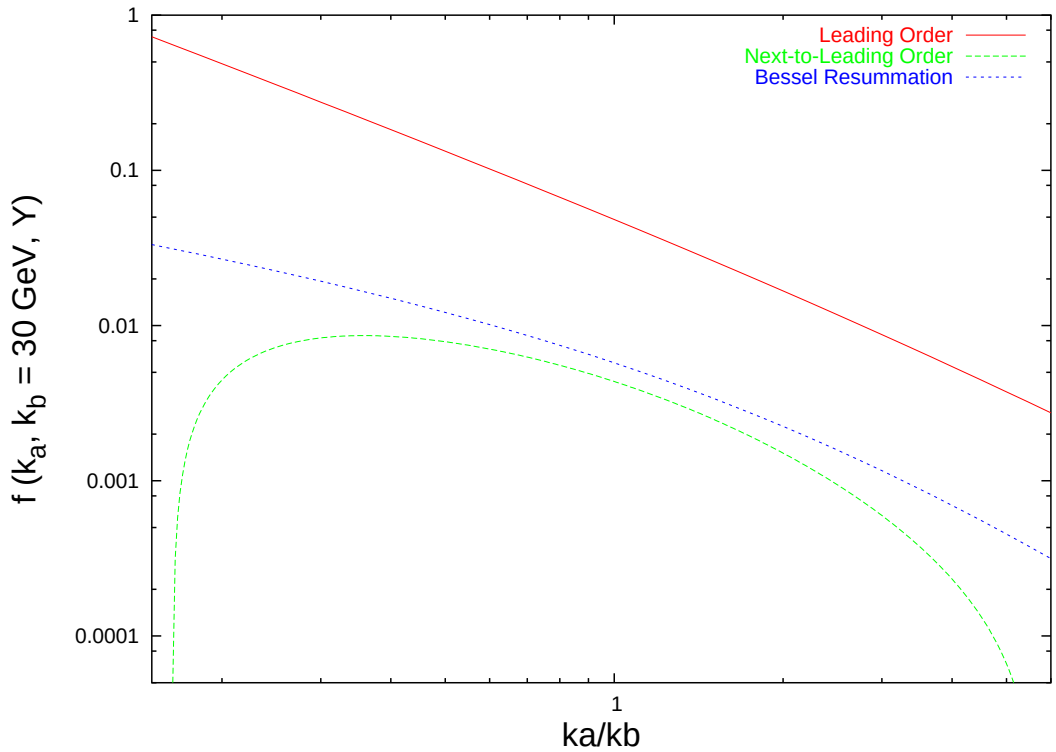
$$\Lambda_{\mathrm{GB}} = \Lambda_{\overline{\mathrm{MS}}} \exp\left(\mathcal{S}\frac{2N_c}{\beta_0}\right)$$

only the coefficient a changes, b does not

$$\mathrm{a} = \frac{\pi^2}{12} - \frac{13}{36}\frac{n_f}{N_c^3} - \frac{67}{36}$$



No oscillations, good collinear behaviour



All-poles Bessel resummation suitable for iterative approach.

5. Present and future work

• Transverse and longitudinal spaces have been decoupled for RG improved kernels.

• Suitable representation for iterative method [Andersen, SV]

• Ready for full NLO phenomenology in cross sections [Chachamis, Papa ...]

• Preparing for final states (jet multiplicities at NLO) [Bartels, Schwennsen, SV]

• Behaviour of the NLO nonforward case? [Andersen, Fadin, SV]

• Target: BFKL driven exclusive observables at the LHC.