The QCD Evolution Equations at High Energy and Large-Nc

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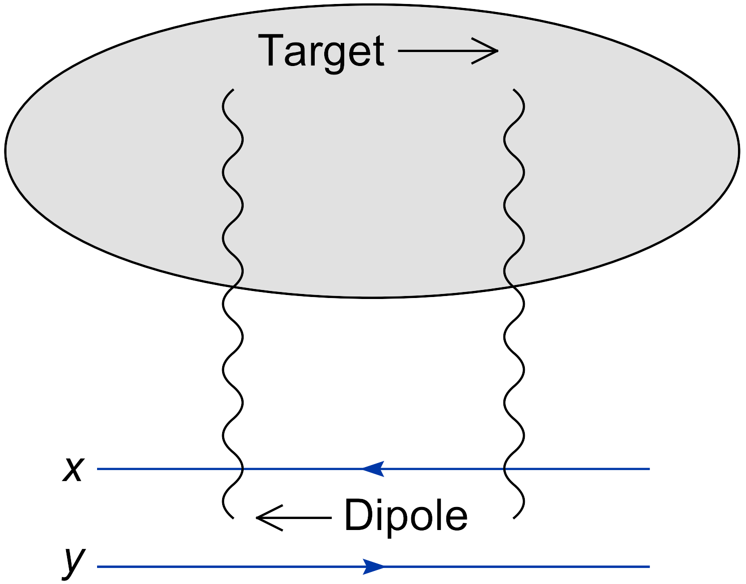
Based on : E. Iancu, D.N.T., Nucl. Phys. **A 756** (2005) 419, Phys. Lett. **B 610** (2005) 253
J.-P. Blaizot, E. Iancu, K. Itakura, D.N.T., Phys. Lett. **B 615** (2005) 221
Y. Hatta, E. Iancu, L. McLerran, A. Stasto, D.N.T., hep-ph/0504182

Outline

- Approaches to High Energy QCD ($s \to \infty$, $\Lambda^2_{\rm QCD} \ll Q^2 \ll s$)
  - Color Dipole Picture
  - Color Glass Condensate (CGC) - JIMWLK equation
  - Effective Action, Pomeron Vertices, Reggeized Gluons,...
  - QCD - Statistical Physics correspondence
- Outline:
  - The BFKL Pomeron and the BFKL Equation
  - Pomeron Mergings and Saturation
  - Pomeron Splittings and Fluctuations
  - Pomeron Loops
  - Evolution Equations at High Energy
  - Duality, Hamiltonian Approach
  - Conclusion, Perspectives

The BFKL Pomeron

- Probe gluon distribution of generic hadron with small color dipole
  Dipole size : $\boldsymbol{r}^2 = (\boldsymbol{x}-\boldsymbol{y})^2 \ll \Lambda_{\rm QCD}^{-2}$, Gluon momentum : $Q^2 \sim 4/\boldsymbol{r}^2$



Lowest order in pQCD

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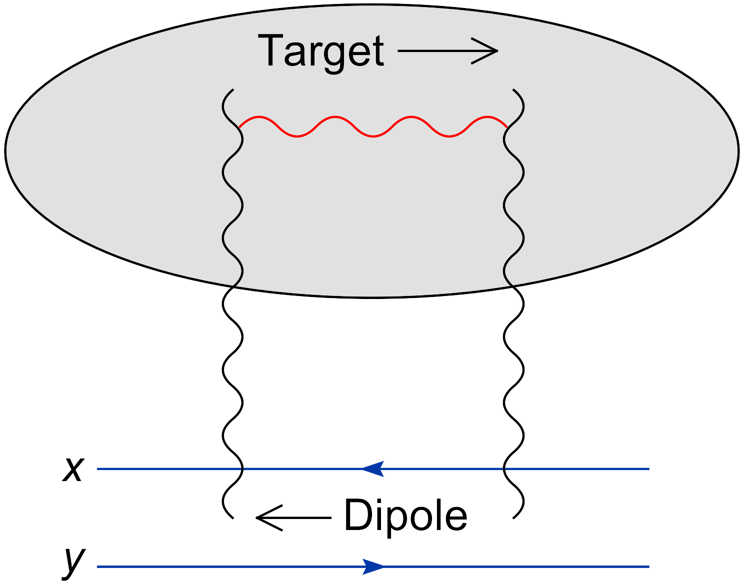

One soft gluon: $\alpha_s Y$

- $Y = \ln(1/x) = \ln(p^+/k^+)$

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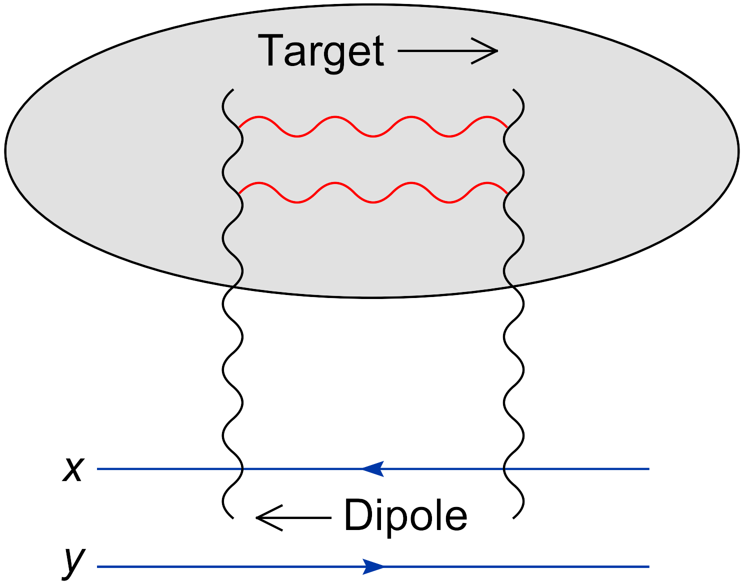

Two soft gluons: $(\alpha_s Y)^2$

- $Y = \ln(1/x) = \ln(p^+/k^+)$

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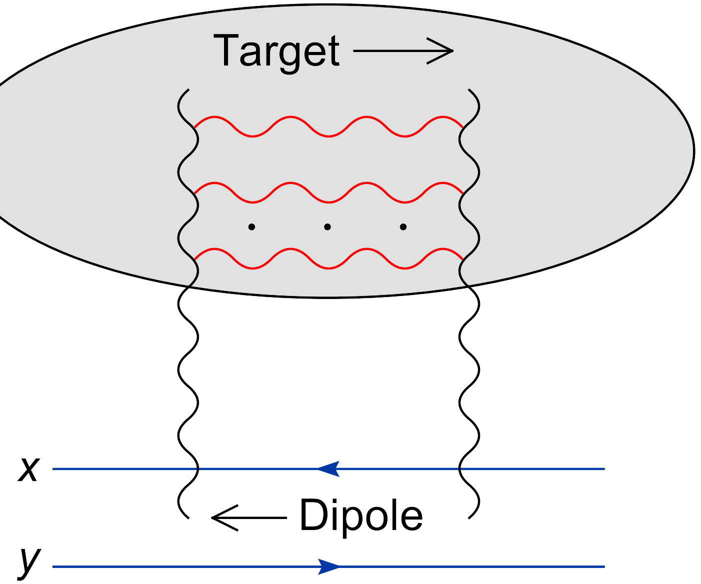
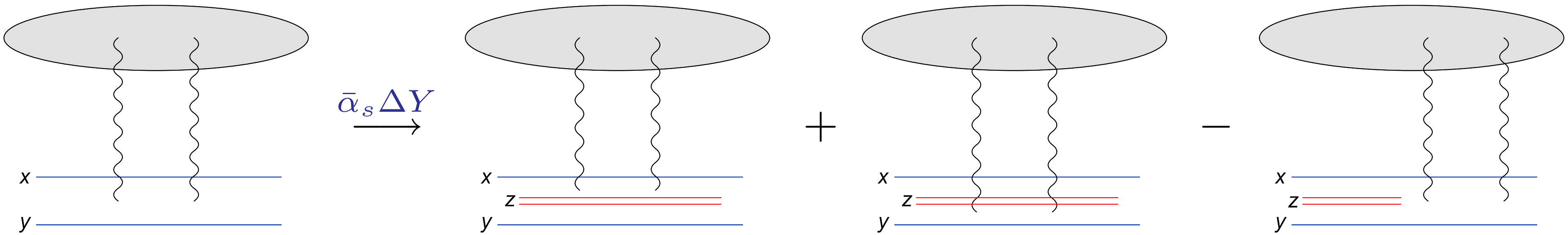


n soft gluons: $(\alpha_s Y)^n$

- $Y = \ln(1/x) = \ln(p^+/k^+)$
- Resum all $(\alpha_s Y)^n$ terms when $\alpha_s Y \gtrsim 1$
  Gluon ladder : BFKL Pomeron

The BFKL Equation (1/2)



- Diagram resummation $\leftrightarrow$ Evolution equation for scattering amplitude T
- View soft gluon emission in projectile

  At large-N_c: Gluon $\longrightarrow$ Quark-Antiquark pair

  Either daughter dipole can scatter off target
- BFKL Equation (coordinate space)

$$\frac{\mathrm{d}T_{\boldsymbol{xy}}}{\mathrm{d}Y} = \frac{\bar{\alpha}_s}{2\pi} \int \mathrm{d}^2\boldsymbol{z}\; \underbrace{\mathcal{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})}_{\frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2}}\; [T_{\boldsymbol{xz}} + T_{\boldsymbol{zy}} - T_{\boldsymbol{xy}}] \equiv \mathcal{K} \otimes T_{\boldsymbol{xy}}$$

  - Kernel : dipole splitting differential probability

The BFKL Equation (2/2)

- Linear evolution $\rightsquigarrow$ Solve eigenvalue problem
  High energy, fixed $\boldsymbol{r}^2$ one eigenvalue dominates

$$T \sim \alpha_s \varphi \sim \alpha_s^2\, n \sim \alpha_s^2 \exp[\omega_{\mathbb{P}} Y]$$

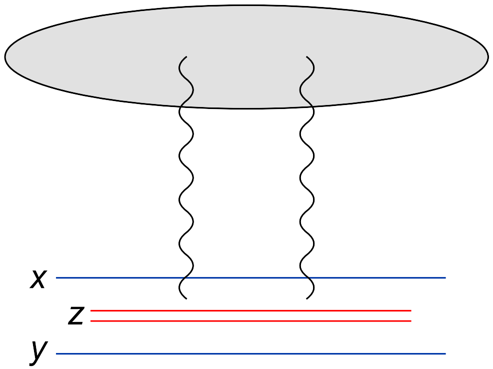
  Exponential increase of gluon distribution φ, dipole density n in target

- Unitarity violation: Amplitude must satisfy $T(\boldsymbol{r}, \boldsymbol{b}) \leq 1$
  Maximal allowed gluon density is $\varphi \sim a^\dagger a \sim \mathcal{A}^2 \lesssim 1/g^2 \sim 1/\alpha_s$

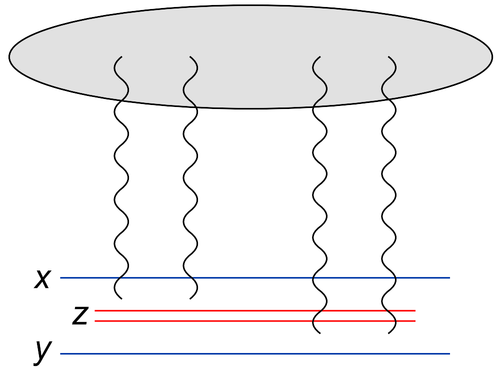
- Sensitivity to non-perturbative physics:
  Non-local in transverse coordinates ($\sim$ momentum) kernel $\rightsquigarrow$
  Random-walk in $\ln \boldsymbol{r}^2$ $\rightsquigarrow$ Diffusion to infrared: $\boldsymbol{r}^2 \gtrsim 1/\Lambda_{\mathrm{QCD}}^2$
  BFKL evolution is not self-consistent

- Next to leading BFKL: resum $\alpha_s(\alpha_s Y)^n$ terms
  Will not save from difficulties $\rightsquigarrow$ Simply adds a correction to $\omega_{\mathbb{P}}$

Saturation, Unitarity, CGC (1/2)



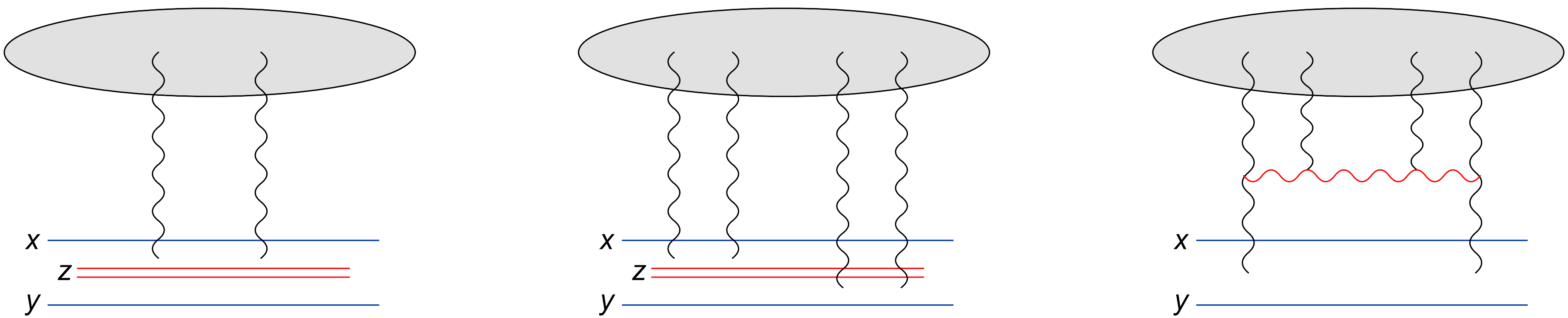
$\bar{\alpha}_s \Delta Y \mathcal{O}(\alpha_s \varphi)$



$\bar{\alpha}_s \Delta Y \mathcal{O}(\alpha_s^2 \varphi^2)$

- Second diagram small in perturbation theory $\varphi \ll 1/\alpha_s$
  Equally important at high density $\varphi \sim 1/\alpha_s \rightsquigarrow$
  $T^{(2)} \sim T$: Allow both dipoles to scatter

Saturation, Unitarity, CGC (1/2)



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  Equally important at high density $\varphi \sim 1/\alpha_s \rightsquigarrow$
  $T^{(2)} \sim T$: Allow both dipoles to scatter
- Third diagram (equiv to second): target evolution
  Merging of two pomerons
  Gluon recombination

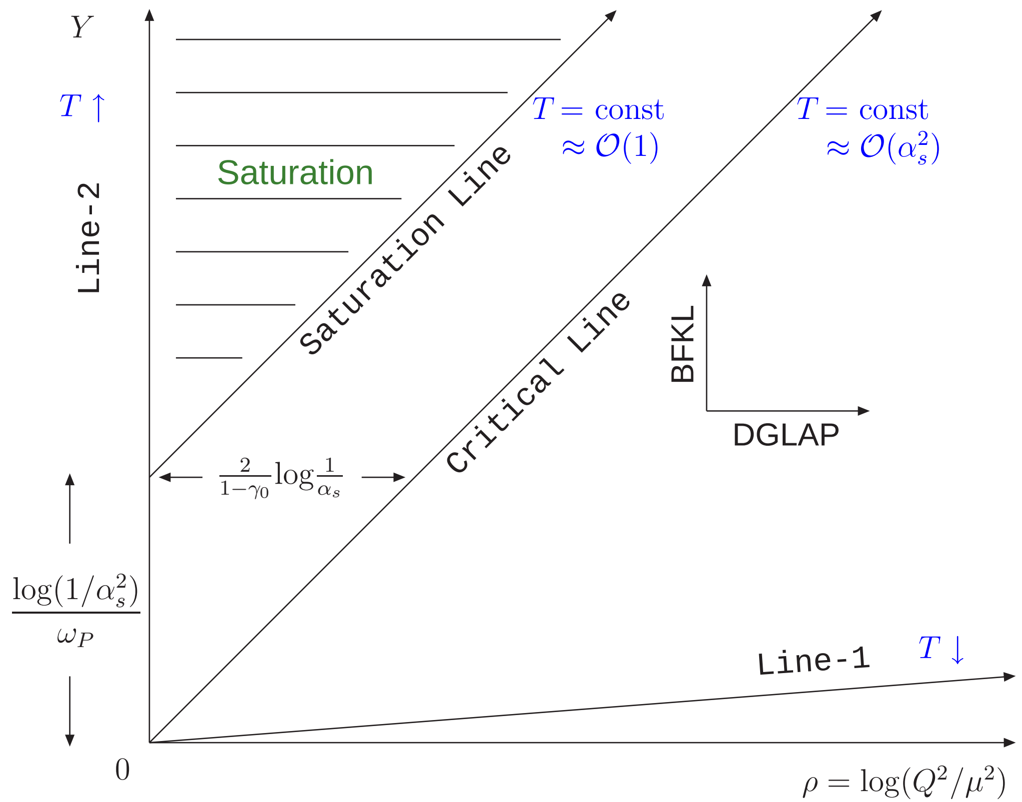
Saturation, Unitarity, CGC (2/2)

- Resum $\alpha_s Y$ terms in presence of background field $\rightsquigarrow$
  Functional RGE of CGC = JIMWLK Equation
  "Projection" on dipole scattering gives First Balitsky Equation

$$\frac{\mathrm{d}T_{\boldsymbol{xy}}}{\mathrm{d}Y} = \mathcal{K}_{\mathrm{BFKL}} \otimes T_{\boldsymbol{xy}} - \frac{\bar{\alpha}_s}{2\pi} \int \mathrm{d}^2\boldsymbol{z} \mathcal{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) T^{(2)}_{\boldsymbol{xz};\boldsymbol{zy}}$$

- "Mean field" approximation: $T^{(2)}(\boldsymbol{xz};\boldsymbol{zy}) \simeq T(\boldsymbol{x},\boldsymbol{z})T(\boldsymbol{z},\boldsymbol{y})$
  - $T = 1 \rightsquigarrow$ stable fixed point
- Pathologies are cured
  - Amplitude satisfies unitarity bound
  - Non-linear term cuts diffusion to the infrared
- Saturation line $Q_s^2 \approx \Lambda^2 \exp(\lambda_s Y)$ where $T(r \sim 2/Q_s, Y) = \mathcal{O}(1)$
  Justifies weak coupling approximation: $\alpha_s(Q_s) \ll 1$

The Logarithmic Plane



- Increase momentum, increase rapidity so that $T = \mathrm{const}$
- `Line-1`: DGLAP $\gamma \to 0$
- `Line-2`: Hard Pomeron Intercept $\gamma = 1/2$
- `Saturation-Line`: $0 < \gamma_s < 1/2$

Deficiencies of Balitsky-JIMWLK

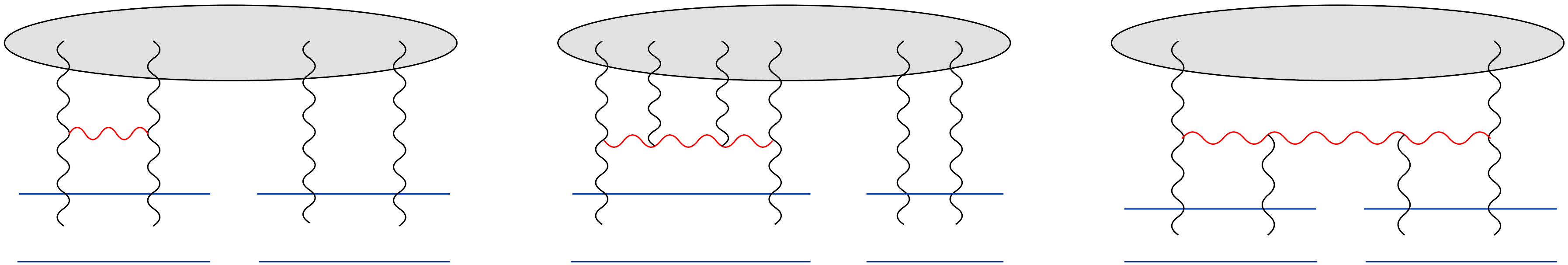
- Extreme sensitivity to the UV:
  Reconstructing solution in two (or more) steps by completeness
  Contributions from momenta up to $\ln(Q^2/Q_s^2) \lesssim \sqrt{\bar{\alpha}_s D Y}$
  Embarrassing: Some orders of magnitude in Q^2
- Violation of Unitarity (!)
  $1 \geq T \sim \dfrac{1}{\alpha_s^2} T_{\rm a} T_{\rm b} \qquad \text{and for} \quad T_{\rm a} < \alpha_s^2 \quad \text{then} \quad T_{\rm b} > 1$
- Absence of Pomeron splittings:

$$\frac{\mathrm{d}T^{(\kappa)}}{\mathrm{d}Y} = \kappa \bar{\alpha}_s T^{(\kappa)} - \kappa \bar{\alpha}_s T^{(\kappa+1)} + \mathcal{O}(1/N_c^2)$$

  Two ladders merge, but how could we have them in the first place?
  - Nucleus target $\rightsquigarrow$ Many sources $\rightsquigarrow$ Many BFKL pomerons
    No more dynamics needed - Initial condition to be lost at high energy
  - Pomeron Splittings

The Missing Diagram(s)

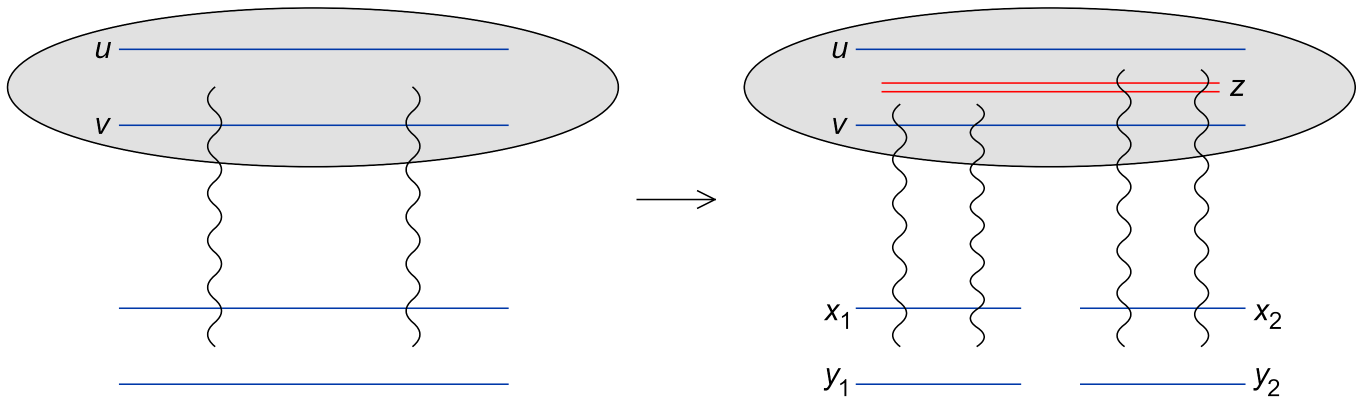
- Diagrammatic illustration of splitting



$\bar{\alpha}_s \Delta Y \mathcal{O}(\alpha_s^2 \varphi^2)$ $\bar{\alpha}_s \Delta Y \mathcal{O}(\alpha_s^3 \varphi^3)$ $\bar{\alpha}_s \Delta Y \mathcal{O}(\alpha_s^3 \varphi)$

- Third diagram not included in JIMWLK
- Important when $\varphi \sim \alpha_s \Rightarrow T \sim \alpha_s^2$, where JIMWLK has problems
- Low density region $\rightsquigarrow$ fluctuations
- “Measure” fluctuations: probe with two dipoles
- First Balitsky equation remains unchanged

Pomeron Splittings



- Large-N_c: Target dipole splits into two
  Measure BOTH child dipoles $\rightsquigarrow$
  Evolution equation for dipole-pair scattering at large-N_c

$$\left.\frac{\mathrm{d}T^{(2)}_{\boldsymbol{x}_1\boldsymbol{y}_1;\boldsymbol{x}_2\boldsymbol{y}_2}}{\mathrm{d}Y}\right|_{\mathrm{split}} = \left(\frac{\alpha_s}{2\pi}\right)^2 \frac{\bar{\alpha}_s}{2\pi} \int_{\boldsymbol{uvz}} \mathcal{M}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})\, \underbrace{\mathcal{A}_0(\boldsymbol{x}_1,\boldsymbol{y}_1|\boldsymbol{u},\boldsymbol{z})}_{\mathrm{Dip-Dip\ Scatt}} \mathcal{A}_0(\boldsymbol{x}_2,\boldsymbol{y}_2|\boldsymbol{z},\boldsymbol{v})\, \underbrace{\nabla^2_{\boldsymbol{u}}\nabla^2_{\boldsymbol{v}}T_{\boldsymbol{uv}}}_{\sim\mathrm{Dip-density}}$$

- Low density fluctuations are the seed for higher-point correlations
- Equivalent to Bartels' $1 \longrightarrow 2$ Vertex

The Large-Nc Equations

- At large-N_c, only $1 \to 2$ process $\rightsquigarrow T^{(\kappa)} \to T^{(\kappa+1)}$
- Structure of Equations (adding large-N_c Balitsky)

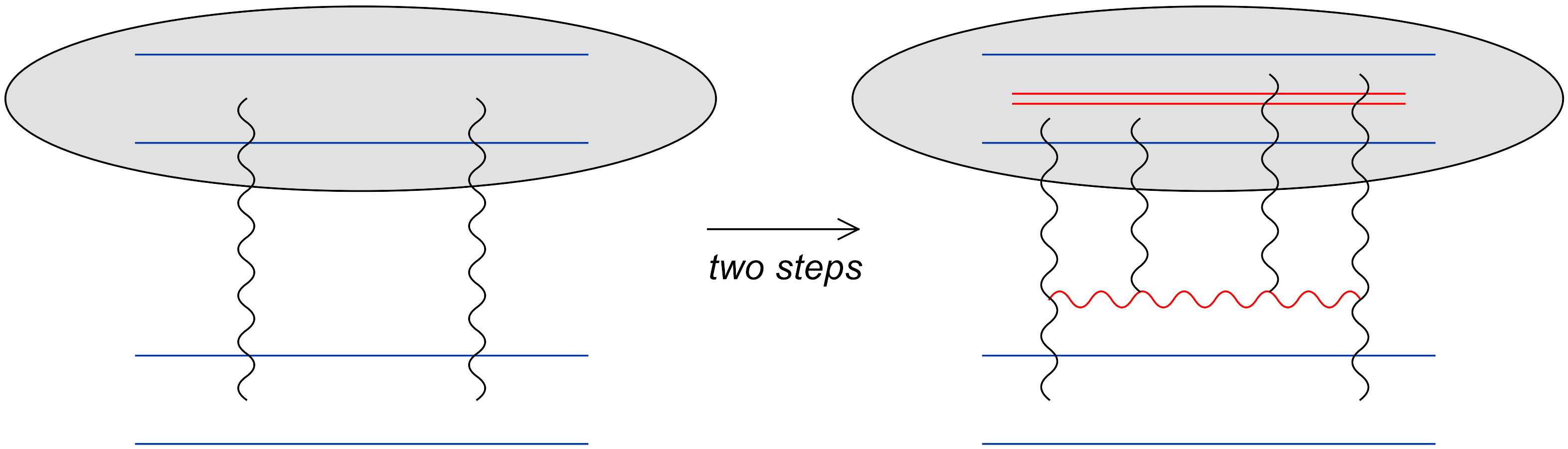
$$\frac{\mathrm{d}T^{(\kappa)}}{\mathrm{d}Y} = \underbrace{\kappa\bar{\alpha}_s T^{(\kappa)}}_{\text{BFKL}} - \underbrace{\kappa\bar{\alpha}_s T^{(\kappa+1)}}_{\text{merging}} + \underbrace{\kappa(\kappa-1)\bar{\alpha}_s \alpha_s^2 T^{(\kappa-1)}}_{\text{splitting}}$$

Not consistent with factorization $T^2 \neq \mathrm{c}TT$...

- Can be summarized in a Langevin Equation
  Certain approximations $\rightsquigarrow$ stochastic-FKPP equation

$$\frac{\mathrm{d}T}{\mathrm{d}Y} = \bar{\alpha}_s T - \bar{\alpha}_s T^2 + \sqrt{2\bar{\alpha}_s \alpha_s^2 T}\,\nu \quad \text{with} \quad \langle \nu(Y)\nu(Y')\rangle = \delta(Y-Y')$$

Pomeron Loops - Duality



- Splittings + Mergings $\longrightarrow$ Loops
- Simplest loop, Pomeron Loops will be built through evolution

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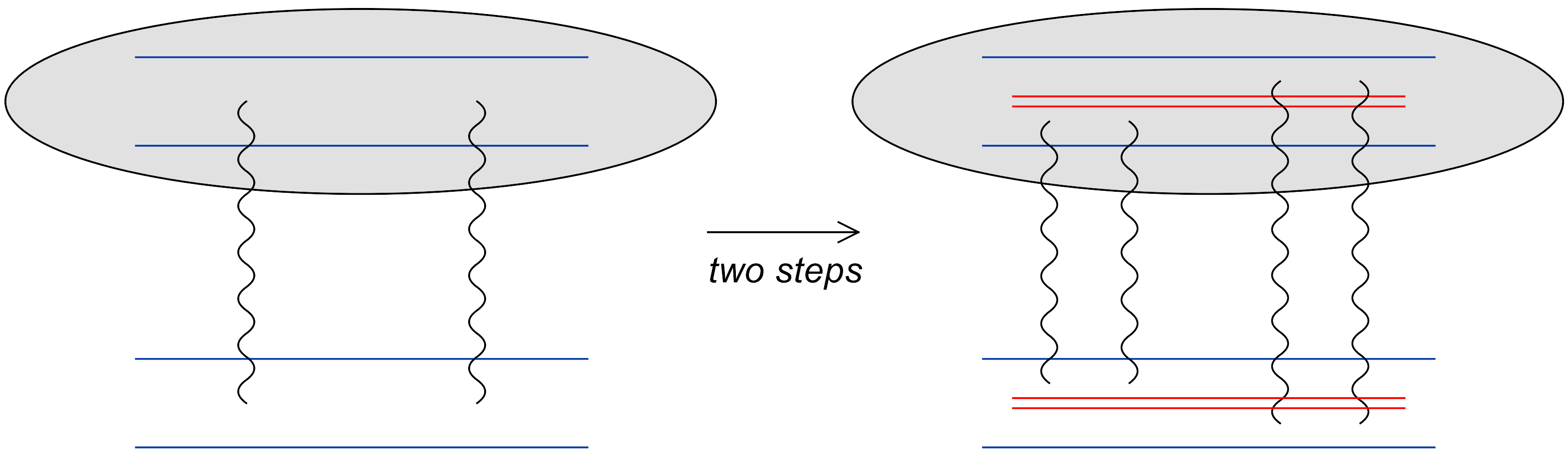

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- Simplest loop, Pomeron Loops will be built through evolution
- Merging in target $\leftrightarrow$ splitting in projectile
  Pomeron Loop manifestly symmetric
- Effective theory with both splittings and mergings is self-dual
  High density $\leftrightarrow$ Low density or Saturation $\leftrightarrow$ Fluctuation Duality

Hamiltonian Approach - Duality

- Pomeron Effective Theory

$$H_{\mathbb{P}} = H_{\rm BFKL} + H_{1\to 2} + H_{2\to 1}\ , \qquad \frac{{\rm d}T^{(\kappa)}}{{\rm d}Y} = H_{\mathbb{P}} \otimes T^{(\kappa)}$$

- BFKL + 3$\mathbb{P}$-Vertices
  - $H_{\rm BFKL} + H_{1\to 2} \leftrightarrow$ Color Dipole Picture
  - $H_{\rm BFKL} + H_{2\to 1} \leftrightarrow$ Balitsky-JIMWLK at large-N_c

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- Lorentz (boost) invariance in dipole-dipole scattering requires

$$\text{Duality Condition :} \quad H_{\mathbb{P}}\left[\alpha, \frac{\delta}{\mathrm{i}\,\delta\alpha}\right] = H_{\mathbb{P}}\left[\frac{\delta}{\mathrm{i}\,\delta\rho}, \rho\right]$$

Color charge density ρ, emitted field α : $\nabla^2\alpha = -\rho$

- Indeed: H_{BFKL} = self-dual and $H_{1\to 2} \leftrightarrow H_{2\to 1}$

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- Indeed: $H_{\rm BFKL}$ = self-dual and $H_{1\to 2} \leftrightarrow H_{2\to 1}$
- Scattering amplitude: $T(\boldsymbol{x},\boldsymbol{y}) = \dfrac{g^2}{4N_c}\left[\alpha^a(\boldsymbol{x}) - \alpha^a(\boldsymbol{y})\right]^2$

Conclusion-Perspectives

- Evolution equations at large-N_c
- Pomeron Loops:
  - Basic “building block” to reach unitarity
  - NO diffusion to IR, NO diffusion to UV (good for numerics)
  - Free of divergencies
  - More important than Next to Leading-BFKL corrections
- Self-dual effective theory
- Go beyond multicolor limit at large or low density (done)
- Arbitrary density (semi-done)
- Phenomenology may change even at qualitative level