Solution for the BFKL Pomeron Calculus in zero transverse dimensions

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M. Kozlov and E. Levin, arXiv:hep-ph/0604039

Outline

1. **Motivations: Pomeron Loops $\Longleftrightarrow$ Markov Chain**
   - Evolution Equation with Pomeron Loops
   - Markov's chain (zero transverse dimensions)
   - From Markov's chain to Partial Deferential Eq.
2. **Evolution equation and its general properties**
   - Sturm-Liouville problem
   - General properties of evolution equation
3. **The exact solution to the problem**
   - Solution
   - Comparison of exact solution with approximate methods
4. **Summary**

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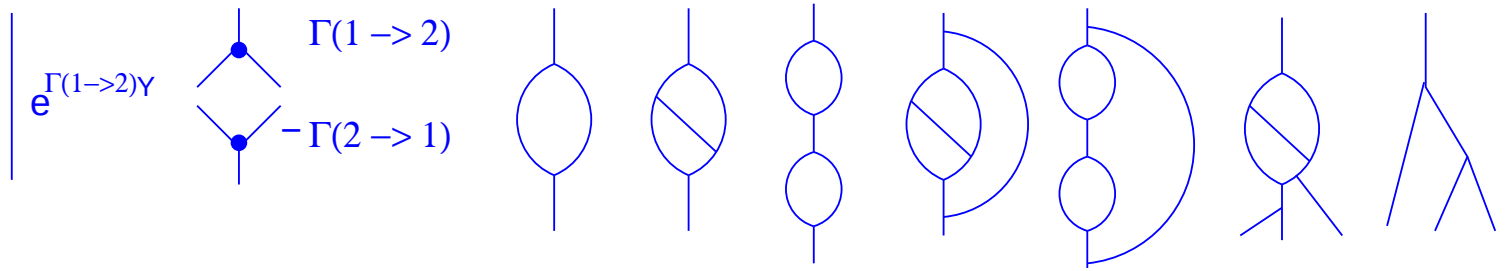
Motivations

Problem [Mueller, Patel (1994), Iancu, Mueller (2003)]

For correct interpretation of the scattering amplitude at high energy, it is mandatory to take into account the rare fluctuations in the target.

Solution

- Taking into account Pomeron Loops in evolution, which leads to functional evolution equations;
- In zero transverse dimensions (Toy Model) functional equations $\Longleftrightarrow$ Markov's chain;



Pomeron and vertices of Pomeron interactions enhanced diagrams mean field approximation

Motivations

Markov's chain (zero transverse dimensions)

$$\frac{dP_n(Y)}{dY} = -\Gamma(1\to 2)\, n\, P_n(Y) + \Gamma(1\to 2)\,(n-1)\,P_{n-1}(Y) - \Gamma(2\to 1)\frac{n(n-1)}{2}\, P_n(Y) + \Gamma(2\to 1)\frac{n(n-1)}{2}\, P_{n+1}(Y)$$

Boundary Conditions:

$$\sum_{n=0}^{\infty} P_n(Y) = 1$$

Solution

- Equation belong to general class of differential-difference equations with polynomial coefficients.
- In spite the big practical importance (Population Grows, Network Communications, Queuing theory, ...), no general method for solution of such equations has been developed yet.

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From Markov's chain to Partial Deferential Eq.

Two Equivalent Possibilities:

- Langevin Equation (E.Iancu, D.N.Triantafyllopoulos - 2004);
- Fokker-Planck Equation (E.Levin, M.Lublinsky - 2004);

Generating Functional (A.Mueller 1995)

$$Z(Y;u) \;=\; \sum_{n=0}^{\infty} P_n(Y)\, u^n$$

Equation for Generating Functional (Fokker-Planck Equation)

$$\frac{\partial Z}{\partial Y} \;=\; u\,(1-u)\left(-\,\kappa\,\frac{\partial Z}{\partial u} \;+\; \frac{\partial^2 Z}{\partial u^2}\right)$$

where $Y \;=\; \Gamma(2\to 1)\,y$ and $\kappa \;=\; \frac{2\cdot\Gamma(1\to 2)}{\Gamma(2\to 1)} \;=\; \frac{1}{\alpha_s^2} \;\gg\; 1$

Initial and Boundary Conditions

$$Z(Y=0;u) \;=\; u \quad \text{and} \quad Z(Y;u=1) \;=\; \sum_{n=0}^{\infty} P_n(Y) \;=\; 1$$

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Sturm-Liouville problem

Second-order partial differential equation of parabolic type:

$$\frac{\partial Z}{\partial Y} = a(u)\frac{\partial^2 Z}{\partial u^2} + b(u)\frac{\partial Z}{\partial u}$$

where $a(u) = u(1-u)$ and $b(u) = -\kappa u(1-u)$

Separation of variables:

$$Z(Y,u) = \sum_{n=1}^{\infty} A_n\, Z_n(u)\, \psi_n(Y) = \sum_{n=1}^{\infty} A_n\, Z_n(u)\, e^{-\lambda_n Y}$$

u dependence:

$$a(u)\frac{d^2 Z_n(u)}{du^2} + b(u)\frac{dZ_n(u)}{du} + \lambda_n Z_n(u) = 0$$

Sturm-Liouville Problem

- $p(u) = exp\left[\int \frac{b(u)}{a(u)} du\right] = e^{-\kappa u}$
- $s(u) = -\frac{1}{a(u)} exp\left[\int \frac{b(u)}{a(u)} du\right] = \frac{1}{u(1-u)} e^{-\kappa u}$

Sturm-Liouville Eq.

$$s(u)\,\frac{\partial Z(Y,u)}{\partial Y} = \frac{\partial}{\partial u}\left(p(u)\,\frac{\partial Z(Y,u)}{\partial u}\right)$$

Without solving explicitly equation, we can say a lot about its solution !

General properties of evolution equation

Eigenvalues

1. Exist the infinite set of the eigenvalues $\omega_n = -\lambda_n$
2. All λ_n are Real
3. $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$ with $\lambda_n \to +\infty$
4. The multiplicity of each eigenvalue is equal to 1
5. There can exist only a finite number of negative λ_n, but in our case there are no negative eigenvalues λ_n and $\lambda_1 = 0$ is the least eigenvalue

Asymptotic behavior @ large n: $n \longrightarrow \infty$:

$$\lambda_n = \frac{\pi^2 n^2}{\pi^2} + O(1)$$

General properties of evolution equation

Eigenfunction

1. Eigenfunction $Z_n(u)$ has exactly $n-1$ zeroes in the interval $[0,1]$
2. Eigenfunctions $Z_n(u)$ and $Z_m(u)$ are orthogonal with weight $s(u)$: $\int_0^1 du\, s(u)\, Z_n(u)\, Z_m(u) = 0 \quad \text{for} \quad n \neq m$
3. An arbitrary function $F(u)$, that satisfies the boundary conditions can be expanded into absolute and uniformly convergent series in eigenfunctions:

$$F(u) \;=\; \sum_{n=1}^{\infty} F_n\, Z_n(u)\,; \quad \text{where} \quad F_n \;=\; \frac{1}{\|Z_n\|} \int\limits_u^1 du\, F(u)\, Z_n(u)$$

Asymptotic behaviour @ large n: $n \longrightarrow \infty$:

$$\frac{Z_n(u)}{\|Z_n\|} \;=\; \left(\frac{4}{\pi^2}\, u\,(1-u)\right)^{1/4} e^{\frac{\kappa}{2}\, u}\, \sin\left(-n\left(\frac{\pi}{2} + \arcsin(1-2\,u)\right)\right) + O\left(\frac{1}{n}\right)$$

Stationary Solution

$Y = \infty$ (Stationary Solution)

$$\frac{\partial Z}{\partial Y}\Big|_{Y=\infty} = 0 \qquad \Longleftrightarrow \qquad u\,(1-u)\left(-\,\kappa\,\frac{\partial Z}{\partial u} + \frac{\partial^2 Z}{\partial u^2}\right) = 0$$

Solution, which satisfies boundary conditions:
$Z(Y, u=0) = 0, \quad Z(Y, u=1) = 1$ is

$$Z_{asymp}(Y=\infty, u) = \frac{1-e^{\kappa\, u}}{1-e^{\kappa}}$$

Initial and Final stages of evolution:

The general solution must "*travel*" between two boundaries:

$$Z(Y=0, u) = u \qquad \longrightarrow \qquad Z_{asymp}(Y=\infty, u) = \frac{1-e^{\kappa\, u}}{1-e^{\kappa}}$$

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Constructing Hermitian Problem

Kolmogorov Backward $\longrightarrow$ Kolmogorov Forward (FP) equation:

$$\frac{\partial Z}{\partial y} = u\,(1-u)\left(-\,\kappa\,\frac{\partial Z}{\partial u} + \frac{\partial^2 Z}{\partial u^2}\right) \qquad Z(y,u) = u(1-u)\cdot\widetilde{Z}(y,u)$$

$$\frac{\partial \widetilde{Z}}{\partial y} = -\,\frac{\partial}{\partial u}\left(\kappa\, u(1-u)\cdot\widetilde{Z}(y,u)\right) + \frac{\partial^2}{\partial u^2}\left(u(1-u)\cdot\widetilde{Z}(y,u)\right)$$

Constructing Hermitian Hamiltonian

$$\hat{L}_{FP} \equiv \frac{\partial}{\partial u}a(u) + \frac{\partial^2}{\partial u^2}b(u) = \frac{\partial}{\partial u}b(u)e^{-\Phi(u)}\frac{\partial}{\partial u}e^{\Phi(u)}$$

Potential: $\Phi(u) \equiv \ln[b(u)] - \int\limits_{u_1}^{u}\frac{a(u')}{b(u')}\,du' = \ln[u(1-u)] - \kappa$

$$\hat{L}^{\dagger}_{FP} \neq \hat{L}_{FP} \quad \text{BUT} \quad \left(e^{\Phi(u)}\hat{L}_{FP}\right)^{\dagger} = e^{\Phi(u)}\hat{L}_{FP}$$

Contracting Hermitian Operator

Eigenfunction:

$$\hat{L}_H \;=\; e^{\Phi(u)/2}\,\hat{L}_{FP}\,e^{-\Phi(u)/2}$$

$$\hat{L}_H\,\phi_n(u) \;=\; \lambda_n\,\phi_n(u) \quad \xrightarrow[\psi_n(u)=e^{\Phi(u)/2}\phi_n(u)]{} \quad \hat{L}_{FP}\,\psi_n(u) \;=\; \lambda_n\,\psi_n(u)$$

- Orthogonality: $\int_{u_1}^{u_2} e^{\Phi(u)}\,\phi_n(u)\cdot\phi_m(u)\,du = \int_{u_1}^{u_2}\psi_n(u)\cdot\psi_m(u)\,du = \delta_{nm}$
- Completeness: $e^{\Phi(u)}\sum_n \phi_n(u)\cdot\phi_n(u') = \sum_n \psi_n(u)\cdot\psi_n(u') = \delta(u-u')$

In our particular case:

$$Z_n(u) \;=\; \frac{1-v^2}{4}\,e^{\frac{\kappa}{4}(1-v)}\,\mathcal{G}_n(v) \qquad \text{where} \qquad v = 1-2u$$

$$(1-v^2)\,\frac{d^2}{dv^2}\mathcal{G}_n(v) - 4\,v\,\frac{d}{dv}\mathcal{G}_n(v) + \left\{-\omega - 2 - \frac{\kappa^2}{16}(1-v^2)\right\}\mathcal{G}_n(v) \;=\; 0$$

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In our particular case:

$$Z_n(u) \;=\; \frac{1-\upsilon^2}{4}\,e^{\frac{\kappa}{4}(1-\upsilon)}\,\mathcal{G}_n(\upsilon) \qquad \text{where} \qquad \upsilon = 1-2u$$

$$(1-\upsilon^2)\,\frac{d^2}{d\upsilon^2}\mathcal{G}_n(\upsilon) - 4\,\upsilon\,\frac{d}{d\upsilon}\mathcal{G}_n(\upsilon) + \left\{-\omega - 2 - \frac{\kappa^2}{16}(1-\upsilon^2)\right\}\mathcal{G}_n(\upsilon) \;=\; 0$$

Prolate Angular Spheroidal Eq.

Sturm-Liouville problem $\Longleftrightarrow$ Prolate Angular Spheroidal Eq.

$$S_{n,m} \;=\; (1-\upsilon^2)^{\frac{m}{2}}\;\mathcal{G}_n(\upsilon)$$

$$\frac{d}{d\upsilon}\left((1-\upsilon^2)\,\frac{dS_{n,m}(c,\upsilon)}{d\upsilon}\right) + \left(\lambda_{n,m} - c^2\,\upsilon^2 - \frac{m^2}{1-\upsilon^2}\right) S_{n,m}(c,\upsilon) \;=\; 0$$

Equation for prolate spheroidal angular function with the fixed order parameter $m = 1$ and arbitrary degree parameter n: $S_{n,\,m=1}(c \;=\; i\,\frac{\kappa}{4},\upsilon)$

- Le-Wei Li, Xiao-Kang Kang and Mook-Seng Leong, Jon Wiley & Son Inc. 2002.
- P. Falloon, http://www.physics.uwa.edu.au/ falloon/spheroidal/spheroidal.html;

Spectrum

$$\omega_n \;=\; -\frac{\kappa^2}{16} \;-\; 2 \;+\; m(m+1) \;-\; \lambda_{n,m} \;=\; -\frac{\kappa^2}{16} \;-\; \lambda_{n,1}$$

Green's function and General Solution

Green's function for differential Eq. of parabolic type

$$s(v)\frac{\partial Z}{\partial Y} = \frac{\partial}{\partial v}\left[p(v)\frac{\partial Z}{\partial v}\right] - q(v)Z + \Phi(Y,v)$$

$$Z(Y,v) = \int_0^Y \int_{v_1}^{v_2} \Phi(\tau,\xi)\cdot G(Y-\tau,v,\xi)\,d\tau\,d\xi + \int_{v_1}^{v_2} S(\xi)\cdot Z(0,\xi)\cdot G(Y,v,\xi)\,d\xi$$

$$+p(v_1)\int_0^Y Z(Y,v_1)\cdot \Lambda_1(y-\tau,v)\,d\tau + p(v_2)\int_0^Y Z(Y,v_2)\cdot \Lambda_2(Y-\tau,v)\,d\tau$$

$$G(Y,v,\xi) = \sum_{n=1}^{\infty} \frac{Z_n(v)\,Z_n(\xi)}{||Z_n||^2}\exp(-\lambda_n Y), \qquad ||Z_n||^2 = \int_{v_1}^{v_2} s(v)\cdot Z_n^2(v)\,dv$$

Last transformation (For Initial and Boundary Conditions)

- $Z(Y,u)$:
  $Z(Y=0,u) = u, \quad Z(Y,u=0) = 0, \quad Z(Y,u=1) = 1$
- $\widehat{Z}(Y;u) = Z(Y;u) - Z(Y=0,u)$:
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Green's function and General Solution

$$Z(Y,u) = \int_0^Y \int_{v_1}^{v_2} \Phi(\tau,\xi)\cdot G(Y-\tau,v,\xi)\,d\tau\,d\xi$$

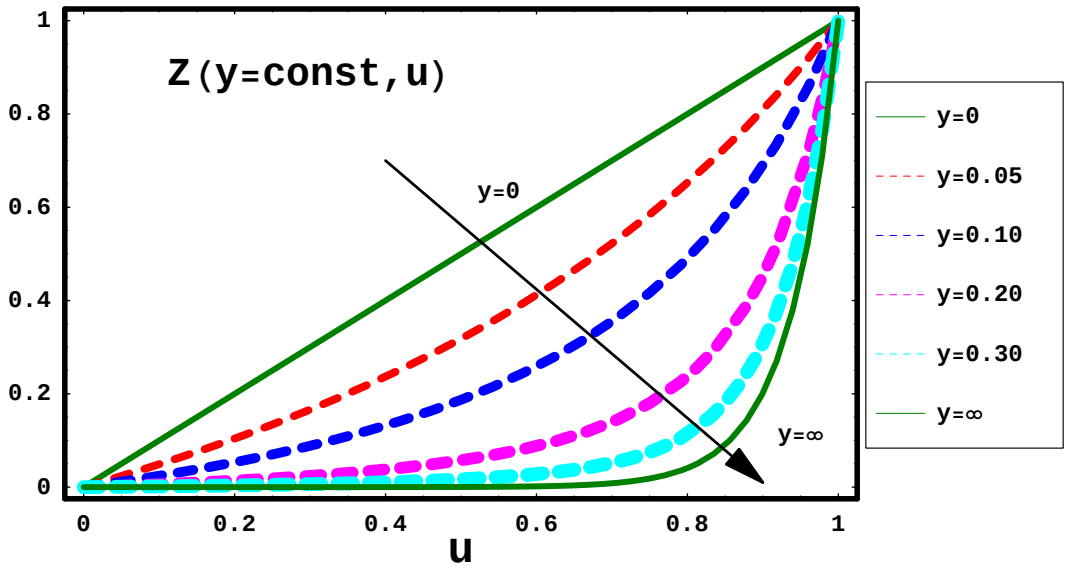
$$= u + \frac{u(1-u)}{\sqrt{1-(1-2u)^2}} e^{\frac{\kappa}{2}u} \sum_{n=1}^{\infty} \left(\frac{1-e^{-\lambda_{n,1}Y}}{\lambda_{n,1}}\right) S_{n,1}\left(i\frac{\kappa}{4}, 1-2u\right) \cdot$$

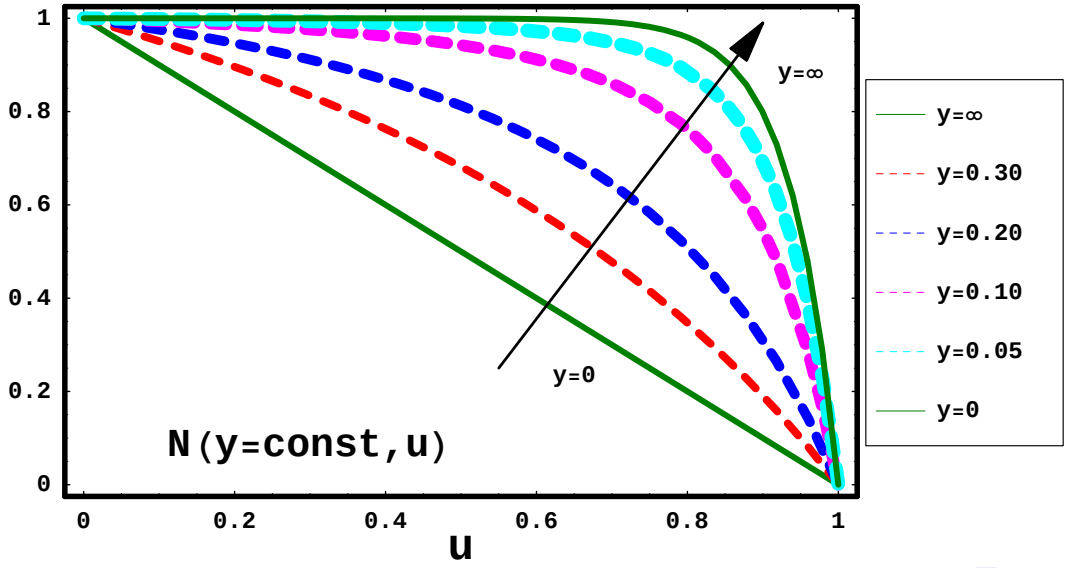
$$\cdot \frac{\int_{-1}^{1} d\xi\, S_{n,1}\left(i\frac{\kappa}{4},\xi\right)\left(-\kappa\sqrt{1-\xi^2}\,e^{-\frac{\kappa}{4}(1-\xi)}\right)}{\|S_{n,1}(i\frac{\kappa}{4},1-2u)\|^2}$$

General properties of given solution

- We found the exact analytical solution in terms of appropriate eigenfunctions (Prolate Angular Spheroidal Functions)
- Series is fast converged with n (practically, we need to take into account only about 10 first terms)

General Solution - u dependence





Comparison of exact solution with approximate methods

Aproximate methods

- Semi-classical Approach
- Mean field approximation
- Mueller-Patel-Salam-Iancu (MPSI) approximation
- Exact solution at $u \longrightarrow 1$
- Perturbation Method

Solution

All of these methods have big difference from exact solution for the BFKL Pomeron Calculus in zero transverse dimensions at energy region: $Y < 10$.

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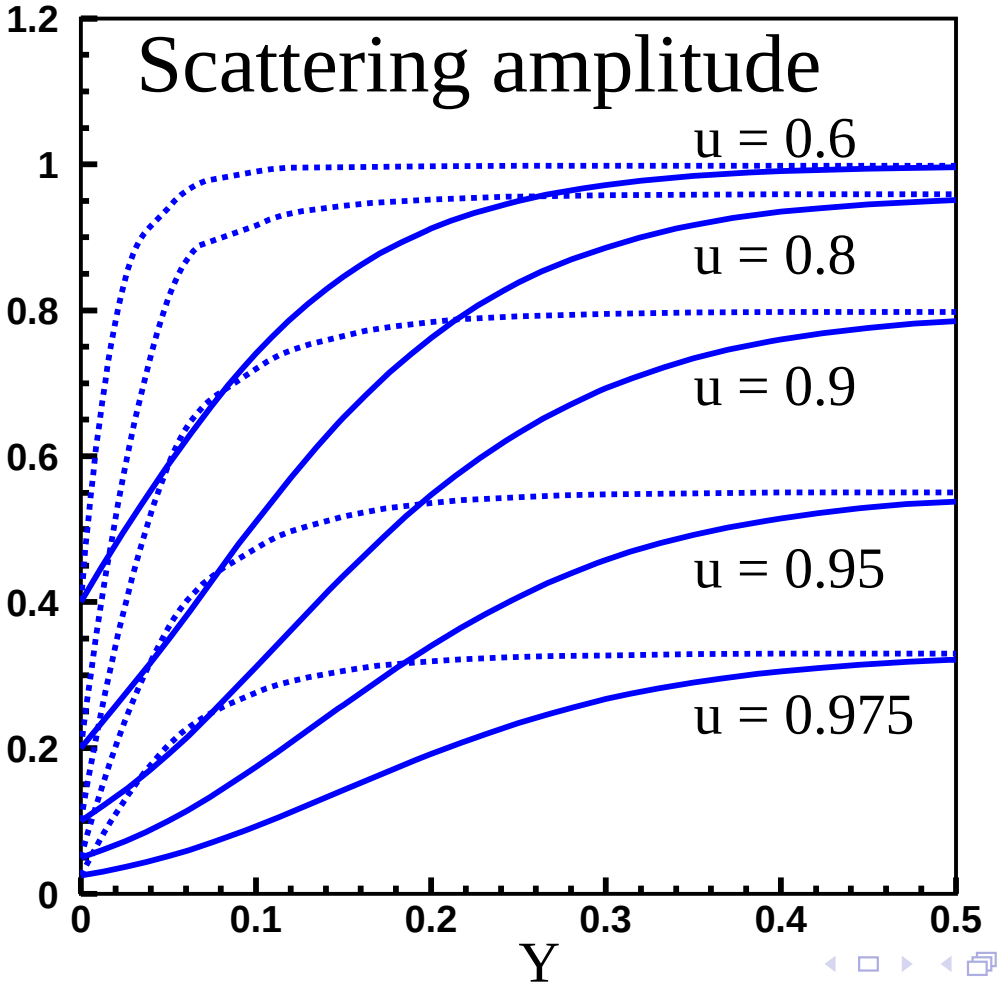
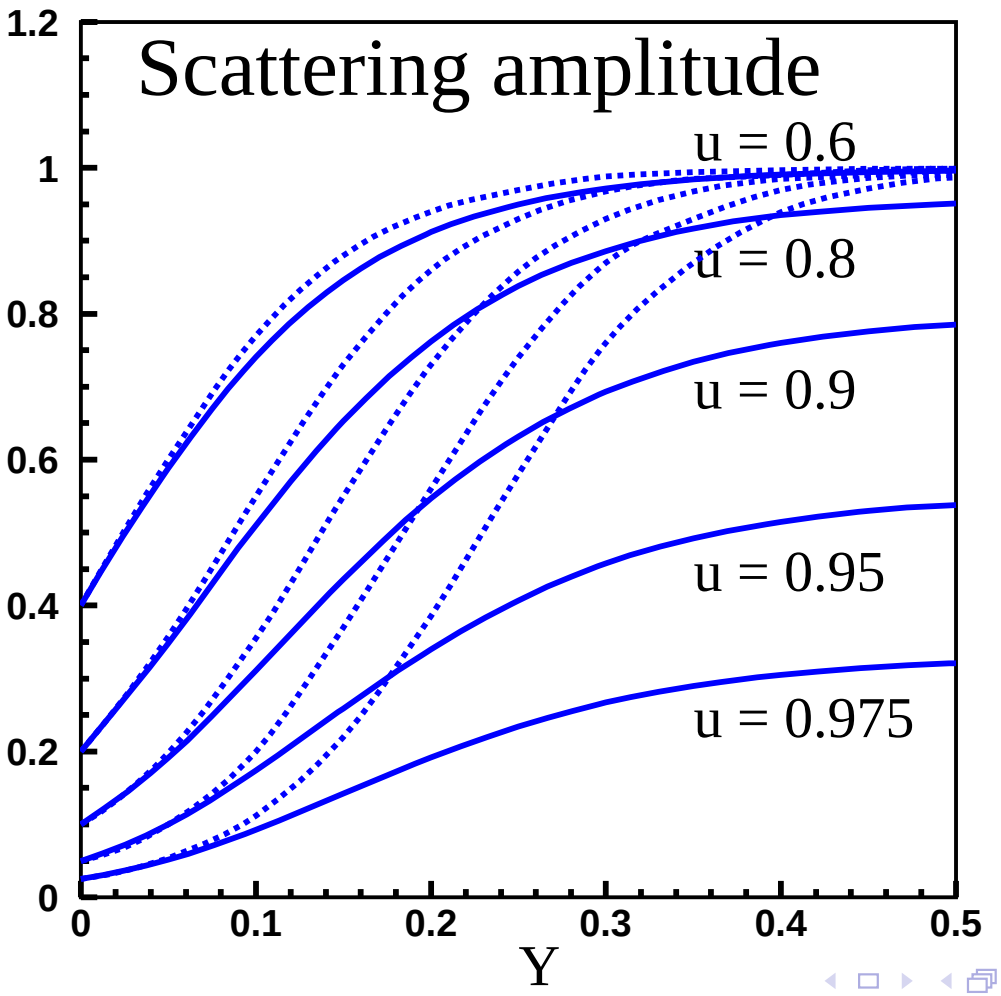
Exact solution and semi-classical solution

If $u \geq \frac{1}{2}$, $\quad N^{SC}(Y;u) = 1 - \exp(\kappa(u-1)) \sum_{n=0}^{n=[\kappa]+1} \tau(u)^n L_n(\kappa)\, e^{-n\kappa Y}$



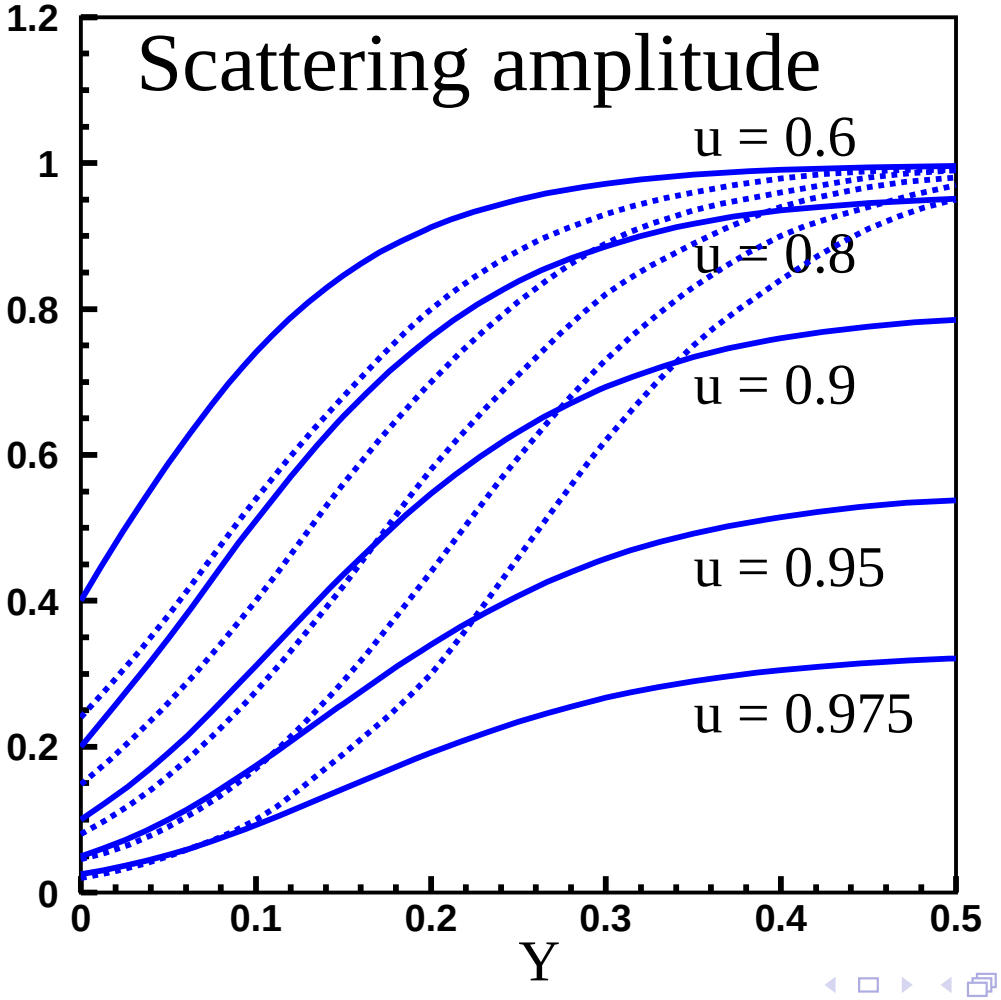
Exact solution and mean field approximation

$$N(Y;u) = 1 - \frac{u\,e^{-\kappa Y}}{1 + u\left(e^{-\kappa Y} - 1\right)}$$



Exact solution and MPSI approximation

$$N(Y,u) \;=\; 1 - \frac{1}{(1-u)\,e^{\kappa Y}}\,\exp\left(\frac{1}{(1-u)\,e^{\kappa Y}}\right)\,\Gamma\left(0,\,\frac{1}{(1-u)\,e^{\kappa Y}}\right)$$



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- We found exact solution in terms of appropriate eigenfunctions (Prolate Angular Spheroidal Functions) for the BFKL Pomeron Calculus in zero transverse dimensions.
- This solution behaves quite different in comparison with previously known solutions. Approximate methods are not trustable for $Y < 10$.
- Exact solution demonstrates Gray Disk behavior for the scattering amplitude.

Open Problems

- Which properties of BFKL Pomeron Calculus in zero transverse dimensions will be preserved in higher dimensions (real QCD problem) ?
- How we can extrapolate the new solution to more complicated cases (i.e. when we take into account also $\Gamma(1 \to 3)$ contributions) ?

Extra - The scalar equation and spheroidal harmonics (1)

Scalar wave equation

$$(\nabla^2 + k^2)\psi = 0$$

Prolate and oblate coordinates

The prolate and oblate coordinates are two of the eleven coordinate systems for which the (scalar) wave equation separates into three ordinary differential equations:

$$\psi = S_{mn}(c)(\eta)\, R_{mn}(c)(\xi)\, e^{\pm i\pi\varphi}$$

Denoting: d $\equiv$ Interfocal Distance and c $\equiv \frac{1}{2}k \cdot d$

Prolate coordinates

$$x = \frac{d}{2}\sqrt{(1-\eta^2)(\xi^2-1)}\, cos(\varphi)$$
$$y = \frac{d}{2}\sqrt{(1-\eta^2)(\xi^2-1)}\, sin(\varphi)$$
$$z = \frac{d}{2}\,\eta\,\xi$$

$-1 \leq \eta \leq 1, \quad 1 \leq \xi < \infty, \quad 0 \leq \varphi < 2\pi$

Oblate coordinates

$$x = \frac{d}{2}\sqrt{(1-\eta^2)(\xi^2+1)}\, cos(\varphi)$$
$$y = \frac{d}{2}\sqrt{(1-\eta^2)(\xi^2+1)}\, sin(\varphi)$$
$$z = \frac{d}{2}\,\eta\,\xi$$

$-1 \leq \eta \leq 1, \quad 0 \leq \xi < \infty, \quad 0 \leq \varphi < 2\pi$

Extra - Scalar wave equation and spheroidal harmonics (2)

Scalar wave equation in prolate coordinates:

$$\frac{\partial}{\partial\eta}\left((1-\eta^2)\,\frac{\partial\psi}{\partial\eta}\right) + \frac{\partial}{\partial\xi}\left((\xi^2-1)\,\frac{\partial\psi}{\partial\xi}\right) + \frac{\xi^2-\eta^2}{(\xi^2-1)\cdot(1-\eta^2)}\,\frac{\partial^2\psi}{\partial\varphi^2} + c^2(\xi^2-\eta^2)\psi = 0$$

- $\frac{\partial}{\partial\eta}\left((1-\eta^2)\,\frac{\partial S_{mn}(c)(\eta)}{\partial\eta}\right) + \left(\lambda_{mn}(c) - c^2\eta^2 - \frac{m^2}{(1-\eta^2)}\right) S_{mn}(c)(\eta) = 0$
- $\frac{\partial}{\partial\xi}\left((\xi^2-1)\,\frac{\partial R_{mn}(c)(\xi)}{\partial\xi}\right) - \left(\lambda_{mn}(c) - c^2\xi^2 + \frac{m^2}{(\xi^2-1)}\right) R_{mn}(c)(\xi) = 0$

Scalar wave equation in oblate coordinates:

$$\frac{\partial}{\partial\eta}\left((1-\eta^2)\,\frac{\partial\psi}{\partial\eta}\right) + \frac{\partial}{\partial\xi}\left((\xi^2+1)\,\frac{\partial\psi}{\partial\xi}\right) + \frac{\xi^2+\eta^2}{(\xi^2+1)\cdot(1-\eta^2)}\,\frac{\partial^2\psi}{\partial\varphi^2} + c^2(\xi^2+\eta^2)\psi = 0$$

- $\frac{\partial}{\partial\eta}\left((1-\eta^2)\,\frac{\partial S_{mn}(ic)(\eta)}{\partial\eta}\right) + \left(\lambda_{mn}(ic) + c^2\eta^2 - \frac{m^2}{(1-\eta^2)}\right) S_{mn}(ic)(\eta) = 0$
- $\frac{\partial}{\partial\xi}\left((\xi^2-1)\,\frac{\partial R_{mn}(ic)(\xi)}{\partial\xi}\right) - \left(\lambda_{mn}(ic) - c^2\xi^2 - \frac{m^2}{(\xi^2+1)}\right) R_{mn}(ic)(\xi) = 0$

Prolate $\longleftrightarrow$ Oblate $\quad\Longleftrightarrow\quad \{c \longleftrightarrow ic,\ \xi \longleftrightarrow i\xi\}$

Extra - Scalar wave equation and spheroidal harmonics (3)

$S^{1,2}_{mn}(c,\eta)$ are generalized Associated Legendre Functions

- $S^1_{mn}(c,\eta) \;=\; \sum\limits_{2k=-\infty}^{\infty} d^{mn}_k(c)\, P^m_{n+k}(\eta)$
- $S^2_{mn}(c,\eta) \;=\; \sum\limits_{2k=-\infty}^{\infty} d^{mn}_k(c)\, Q^m_{n+k}(\eta)$

$R^{1,2}_{mn}(c,\xi)$ are generalized Bessel Functions

- $R^1_{mn}(c,\xi) \;=\; (\xi^2-1)^{m/2}\,(c\xi)^{-m} \sum\limits_{2k=-\infty}^{\infty} a^{mn}_k(c)\, J^m_{n+k}(\xi) \xrightarrow{c\xi\to\infty} \frac{1}{c\xi} cos[c\xi - \frac{1}{2}(n+1)\pi]$
- $R^2_{mn}(c,\xi) \;=\; (\xi^2-1)^{m/2}\,(c\xi)^{-m} \sum\limits_{2k=-\infty}^{\infty} a^{mn}_k(c)\, N^m_{n+k}(\xi) \xrightarrow{c\xi\to\infty} \frac{1}{c\xi} sin[c\xi - \frac{1}{2}(n+1)\pi]$

$R^{3,4}_{mn}(c,\xi)$ are generalized Hankel Functions

- $R^3_{mn}(c,\xi) \;=\; R^1_{mn}(c,\xi) \;+\; i\, R^2_{mn}(c,\xi)$
- $R^3_{mn}(c,\xi) \;=\; R^1_{mn}(c,\xi) \;-\; i\, R^2_{mn}(c,\xi)$

Extra - Forward and Backward master equations (1)

Discrete Birth and Dead random process, continuous in time:

$$P_{n,m}(t') \;=\; \{\, state \;\; m \;\; at \; time \;\; t=0 \quad \longrightarrow \quad state \;\; n \;\; at \; time \;\; t=t' \,\}$$

- Forward master equation:
  $$\frac{d}{dt}P_{n,m}(t) \;=\; \lambda_{n-1}\cdot P_{n-1,m}(t) - (\lambda_n+\mu_n)\cdot P_{n,m}(t) + \mu_{n+1}\cdot P_{n+1,m}(t)$$
- Backward master equation:
  $$\frac{d}{dt}P_{n,m}(t) \;=\; \mu_m\cdot P_{n,m-1}(t) - (\lambda_m+\mu_m)\cdot P_{n,m}(t) + \lambda_m\cdot P_{n,m+1}(t)$$

Transformation to PDE

Introducing the small parameter h: $\quad h \longrightarrow 0$

- $h\cdot\;(\lambda_n(h)-\mu_n(h)) \;=\; a(n\cdot h)+O(h)$
- $h^2\cdot(\lambda_n(h)-\mu_n(h)) \;=\; b(n\cdot h)+O(h)$

Denoting $\{\, x \;=\; n\cdot h,\;\; y \;=\; n\cdot h \,\}$ we get $P_{n,m}(t') \;=\; P(x|y,t)$

Extra - Forward and Backward master equations (2)

Stochastic PDEs

$P(x|y,t)$ - probability that the random variable has the value x at time t given that it had the value at $t=0$:

$$P(x|y,t') \;=\; \{\, y \;\; at\ time \;\; t=0 \quad \longrightarrow \quad x \;\; at\ time \;\; t=t' \,\}$$

- Forward Kolmogorov diffusion (Fokker-Planck) equation:

$$\frac{\partial P(x|y,t)}{\partial t} \;=\; -\,\frac{\partial}{\partial x}\,[a(x)\cdot P(x|y,t)] \;+\; \frac{1}{2}\,\frac{\partial^2}{\partial x^2}\,[b(x)\cdot P(x|y,t)]$$

- Backward Kolmogorov diffusion equation:

$$\frac{\partial P(x|y,t)}{\partial t} \;=\; a(y)\cdot\frac{\partial P(x|y,t)}{\partial y} \;+\; \frac{1}{2}\,b(y)\cdot\frac{\partial^2 P(x|y,t)}{\partial y^2}$$