The BFKL Pomeron Calculus: Probabilistic Interpretation and High Energy Amplitude

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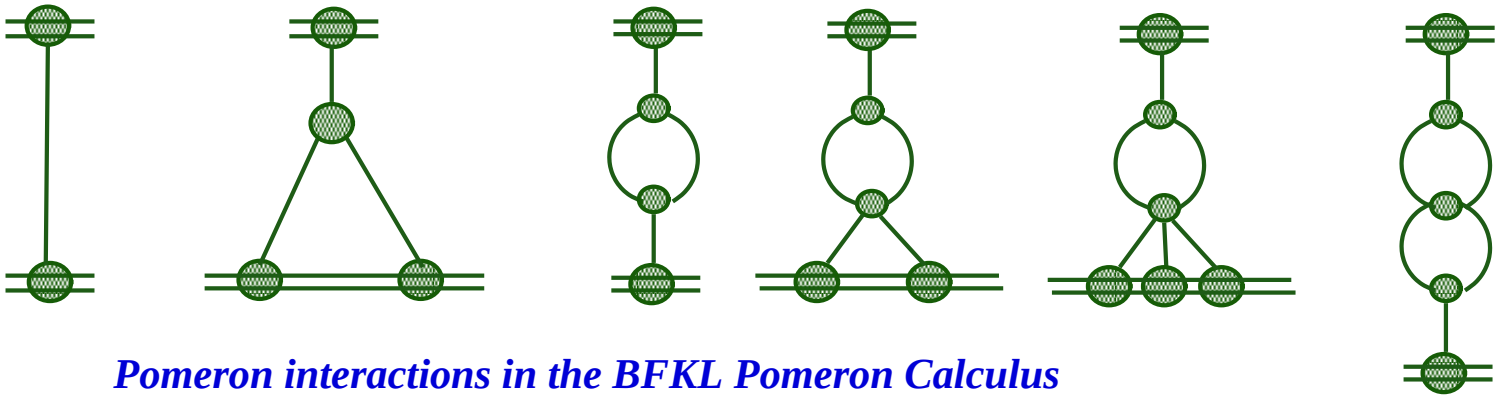
Outline:

- The BFKL Pomeron Calculus: main ideas and assumptions;
- Generating functional with Pomeron loops and Markov's processes;
- Generating functional $\equiv$ the BFKL Pomeron Calculus;
- Two dipoles $\Longrightarrow$ one dipole process in QCD;
- Scattering amplitude in the toy-model;
- High energy amplitude in QCD: solution to the functional equation;
- High energy amplitude in QCD: semi-classical approach (Mueller -Shosi band and solution in the saturation region).

The BFKL Pomeron Calculus



Pomeron interactions in the BFKL Pomeron Calculus

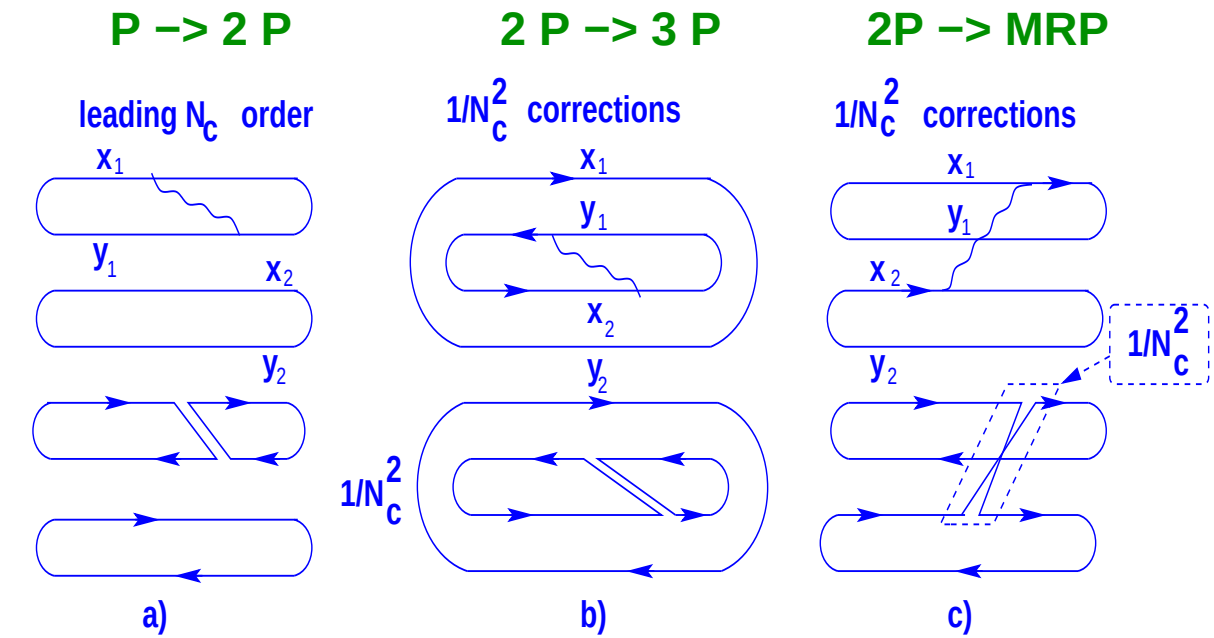
- $$L = \frac{\bar{\alpha}_S^2}{N_c^2} e^{\omega_{BFKL} Y} \; ; \quad l = \frac{\bar{\alpha}_S}{N_c^2} Y \; ; \quad \omega_{BFKL} \propto \bar{\alpha}_S \; ;$$

$$D1 \propto L \; ; \quad D2 \propto L^2 \quad D3 \propto L^2 \; ; \quad D4 \propto l \times L^2 \; ;$$

$$D5 \propto \frac{1}{N_c^2} \times L^3 \; ; \quad D6 \propto l \times L^2 \; ;$$

Assumptions:

- **The intercept of multi (n) -Reggeon Pomeron is smaller than the intercept of n-BFKL Pomeron exchange**



1 BFKL Pomeron topology **2 BFKL Pomeron topology** **Multi Reggeon Pomeron**

$$\Delta_2 \gg \Delta_{MRP}$$

Path integral formalism for the BFKL Pomeron Calculus

(Braun, 1999)

- $$Z[\Phi, \Phi^+] \;=\; \int D\Phi\, D\,\Phi^+ \exp\left(S_0 \,+\, S_I \,+\, S_E\right)$$

- $$S_0 \;=$$

$$\int dY\, dY'\, d^2x_1\, d^2x_2\, d^2x'_1\, d^2x'_2\, \Phi^+(x_1, x_2; Y)\, G^{-1}(x_1, x_2; Y | x'_1, x'_2; Y')\, \Phi(x'_1, x'_2; Y')$$

- $$S_I \;=$$

$$\frac{\pi \tilde{\alpha}_S^2}{N_c} \int dY \int \frac{d^2x_1\, d^2x_2\, d^2x_3}{x_{12}^2\, x_{23}^2\, x_{13}^2} \{ \left(\overrightarrow{L}_{1,3}\, \Phi(x_1, x_3; Y) \right) \cdot \Phi^+(x_1, x_2; Y)\, \Phi^+(x_2, x_3; Y) + h.c. \}$$

where $\overleftarrow{L}_{1',3'} \;=\; r'^4_{13}\, p^2_{1'}\, p^2_{3'}$ with $p^2 \;=\; -\nabla^2$

- $$S_E \;=$$

$$-\int dy\, d^2x_1\, d^2x_2\, \{ \Phi(x_1, x_2; y)\, \tau_{pr}(x_1, x_2; y) \;+\; \Phi^+(x_1, x_2; y)\, \tau_{tar}(x_1, x_2; y)$$

- **Probabilistic interpretation:**

" Reggeon field theory is equivalent to a chemical process where a radical can undergo diffusion, absorption, recombination, and autocatalytic production. Physically, these "radicals" are wee partons (colour dipoles)."

(P. Grassberger & K. Sundermeyer, 1978: "Reggeon Field Theory and Markov processes")

- **Colour dipoles (Mueller, 1994):**

  - **the wee partons of the BFKL Pomeron;**
  - **the correct degrees of freedom at high energy.**

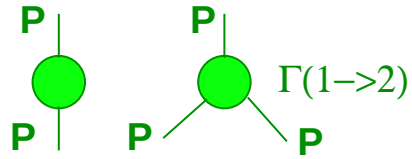
Generating Functional and Markov's processes

- $$Z(Y;[u]) = \sum_i \int P_n(...x_i, y_i...) \prod_{i=1}^{n} u(x_i, y_i)\, d^2x_i\; d^2\, y_i$$

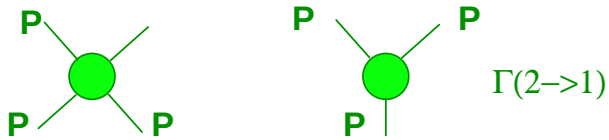
Initial conditions:

- $Z(Y, [u = 1]) = 1$;
- $Z(Y = Y_0, [u]) = u$;

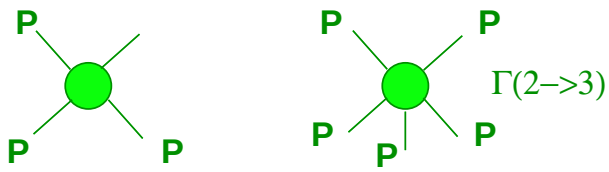
- $\frac{\partial P_n}{\partial Y} = -\sum_i V(1 \to 2) \otimes (P_n(...x_i, y_i...) - P_{n-1}(...x_i, y_i...))$



$-\sum_{i,k} V(2 \to 1) \otimes (P_n(...x_i, y_i, ..., x_k, y_k, ...) - P_{n+1}(...x_i, y_i, ..., x_k, y_k, ...))$



$-\sum_{i,k} V(2 \to 3) \otimes (P_n(...x_i, y_i, ..., x_k, y_k, ...) - P_{n-1}(...x_i, y_i, ..., x_k, y_k, ...))$



The typical death-birth process (Markov's chain)

$$\frac{\partial Z(Y;[u])}{\partial Y} = \chi[u]\, Z(Y;[u])$$

$$\chi[u] =$$

$$V(1 \to 2) \bigotimes (-u(x,y) + u(x,z)u(z,y)) \frac{\delta}{\delta u(x,y)} +$$

$$V(2 \to 1) \bigotimes (u(x,y) - u(x_1,y_1)u(x_2,y_2)) \frac{1}{2}\frac{\delta^2}{\delta u(x_1,y_1)\ \delta u(x_2,y_2)} +$$

$$V(2 \to 3) \bigotimes (1 - u(x_1,y_2))\,(u(x_2,y_1) - u(z,y_1)\,u(x_2,z))\frac{1}{2}\frac{\delta^2}{\delta u(x_1,y_1)\ \delta u(x_2,y_2)}$$

GF $\equiv$ BFKL PC

- $$Z(Y;u) \;=\; e^{H(u)\,(Y-Y_0)}\,Z(Y_0;u)$$

with operator H defined as

- $$H(u) \;=\; -\,\Gamma(P\to 2P)\,u(1-u)\,\frac{\partial}{\partial u} + \Gamma(2P\to P)\,u(1-u)\,\frac{\partial^2}{(\partial u)^2}$$

and

- $$Z(Y_0;u) \;=\; e^{\tau_{tr}(u-1)}$$

Introducing operators of creation (a^+) and annihilation (a)

- $\hat{a} \;=\; \dfrac{\partial}{\partial u}$; $\quad \hat{a}^+ \;=\; u$ that satisfy $[\hat{a},\hat{a}^+] = 1$ at fixed Y

- $$\mathcal{H} \;=\; -\,\Gamma(P \to 2P)\,\hat{a}^+\,(1-\hat{a}^+)\,\hat{a} \;+\; \Gamma(2 \to 1)\,\hat{a}^+\,(1-\hat{a}^+)\,\hat{a}^2$$

with the initial sate $|Y_0> \;=\; e^{\tau_{tr}(\hat{a}^+ - 1)}|0>$

- $$e^{\mathcal{H}\,(Y-Y_0)} = e^{\mathcal{H}\,\Delta Y}\,\ldots\,e^{\mathcal{H}\,\Delta Y} = \prod_{j=1}^{N}\,(1 + \mathcal{H}\,\Delta Y)$$

Introducing the coherent states **(Ciafaloni & Onodri, 1979)**

- $|\phi_j> \;=\; e^{\phi_j\,\hat{a}^+ - \phi_j}\,|0>$ **with completeness equation**

- $$1 \;=\; \left(\int \frac{d\phi_j d\phi_j^*}{\pi i}\,e^{-\phi_j\,\phi_j^* + \phi_j + \phi_j^*}\right)|\phi_j><\phi_j|$$

Redefining $\Phi_j = -\phi_j, \qquad \Phi^+_j = \phi^*_j - 1$

- $$< Y|A|Y_0 > = \frac{\int \mathcal{D}\Phi^+ \mathcal{D}\Phi A(Y) e^S}{\int \mathcal{D}\Phi^+ \mathcal{D}\Phi e^S}$$

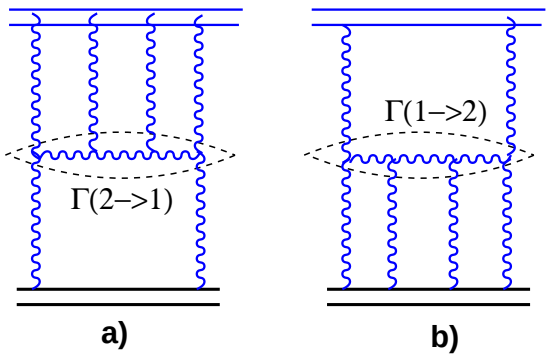
with

$$S = \int \left(\Phi^+ \frac{d}{dY} \Phi + \mathcal{H}(\Phi^+ + 1, -\Phi) \right) dY = \int \Big(\Phi^+ \frac{d}{dY} \Phi -$$

$$- \Gamma(P \to 2P) \Phi^+ \Phi + \Gamma(P \to 2P) \Phi^+ (\Phi)^2 + \Gamma(2P \to P) (\Phi^+)^2 \Phi$$

$$- \Gamma(2P \to P) (\Phi^+)^2 (\Phi)^2 \Big) dY$$

Two dipoles $\Longrightarrow$ one dipole process in QCD



(Iancu & Triantafyllopoulos - 2004)

$$\Gamma_{2P\to P}\left(x_1, y_1 + x_2, y_2 \to x, y\right) = \frac{\bar{\alpha}_S^2 \pi^2}{N_c^2} \Delta_x \Delta_y \int d^2x'_1\, d^2y'_1\, d^2x'_2\, d^2y'_2 \times \Gamma_{1\to 2}\left(x', y' \to x'_1, y'_1 + x'_2, y'_2\right) G_0(x_1, y_1; x'_1, y'_1)\, G_0(x_2, y_2; x'_2, y'_2)$$

However for the BFKL Pomeron

$$L_{13}\, G\left(x_1, x_3; Y | x'_1, x'_2; Y'\right) = \frac{1}{\lambda(n=0, \nu\to 0)}\, G\left(x_1, x_3; Y | x'_1, x'_2; Y'\right)$$

$$= G\left(x_1, x_3; Y | x'_1, x'_2; Y'\right)$$

Therefore,

$$\Gamma(2P \to P) = \int d^2z K(x, y; z)\, G_0(x_1, y_1 | x, z)\, G_0(x_1, y_1 | z, y)$$

with $K(x, y|z) = \frac{(x-y)^2}{(x-z)^2\,(z-y)^2}$

$V(2 \to 1(?))$

$$V_{1\to 2} = \frac{\bar{\alpha}_S}{2\,\pi}\Gamma_{P\to 2P} = \frac{\bar{\alpha}_S}{2\,\pi} K\,(x, y; z)\ ;$$

$$V_{2\to 1} = (\frac{4\pi^2\bar{\alpha}_S}{N_c})^2 \frac{\bar{\alpha}_S}{2\,\pi}\left(-\frac{\Gamma_{2P\to P}(x_1, y_1 + x_2, y_2 \to x, y)}{(x-y)^2} + \right.$$

$$+ \int \frac{d^2x\, d^2y}{(x-y)^2}\Gamma_{2P\to P}(x_1, y_1 + x_2, y_2 \to x, y)\cdot$$

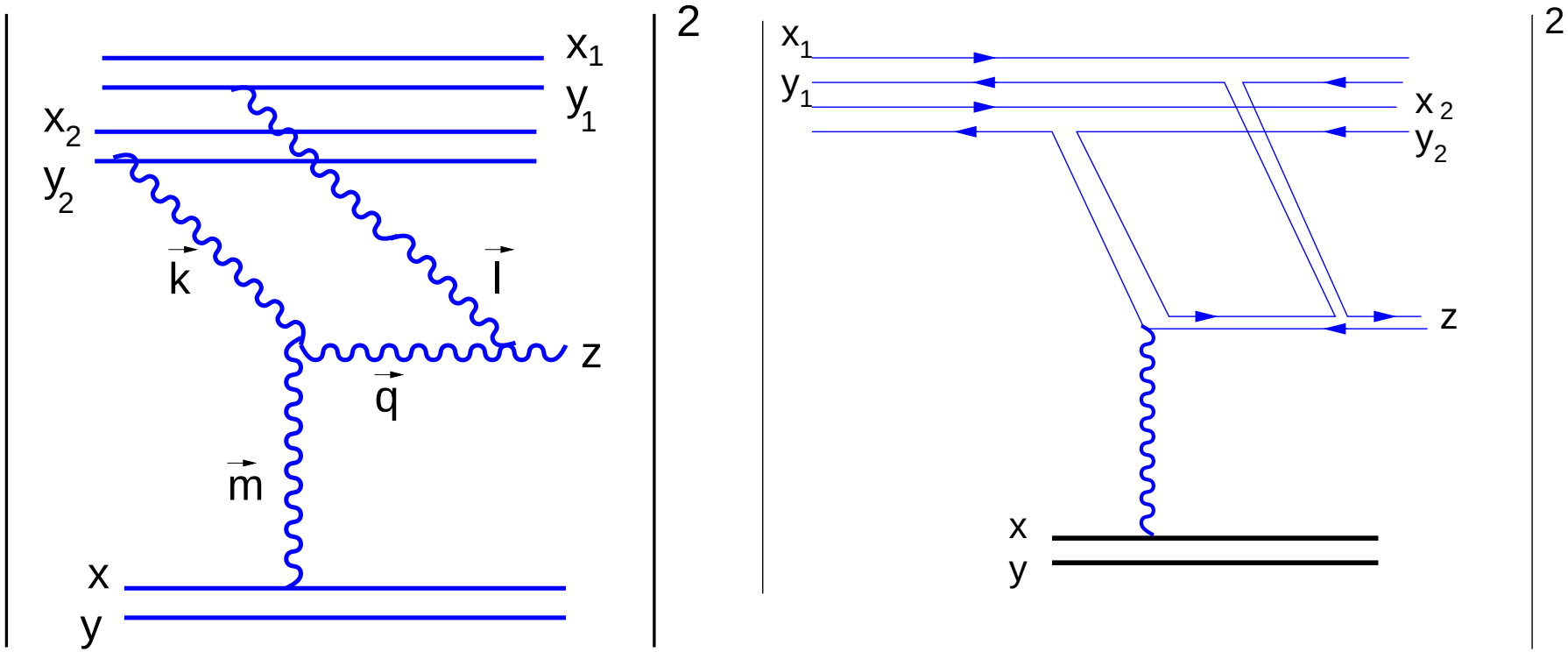
$$\left.\cdot \left(\delta^{(2)}(x_1 - x)\delta^{(2)}(y_1 - y) + \delta^{(2)}(x_2 - x)\delta^{(2)}(y_2 - y)\right)\right)$$

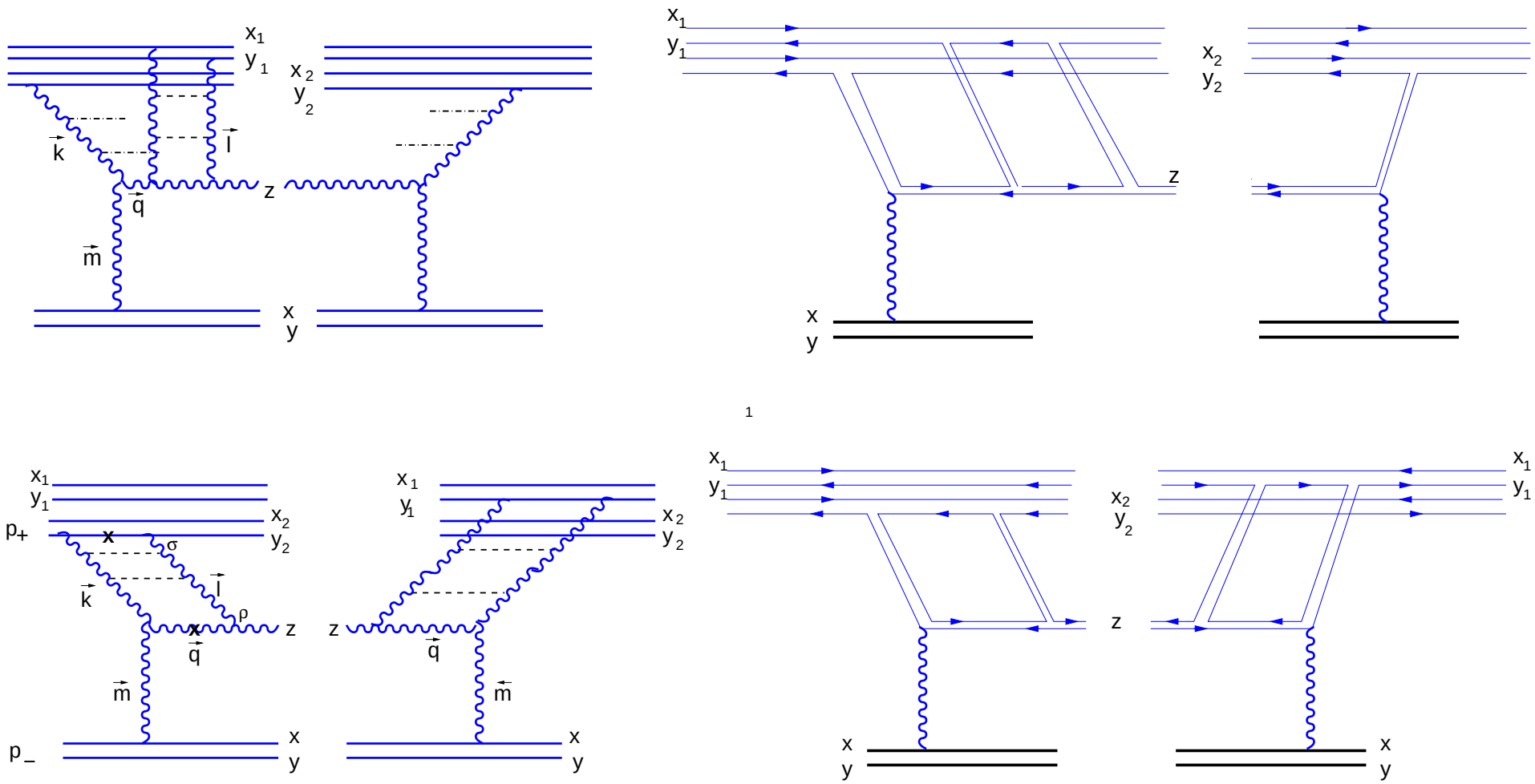
- $\int d^x\, d^2y\ V_{1\to 2} > 0$

However, $V_{1\to 2}$ is negative in some regions.

Two dipoles to one dipole process ?

$$|\Psi|^2 \otimes \gamma^{BA} = V(2 \to ?)\ \gamma^{BA}$$





$$\sum \text{ diagrams } < 0$$

- **Diagrams show the two stage process:**

Stage 1. 1 $\Longrightarrow$ 2 decay:

$$(x_1, y_1) \; + \; (x_2, y_2) \;\; \Longrightarrow \;\; (x_1, y_1) \; + \; (x_2, z) \; + \; (z, y_2)$$

Stage 2. 2 $\Longrightarrow$ 2 rescattering:

$$(x_1, y_1) \; + \; (x_2, z) \; + \; (z, y_2) \;\; \Longrightarrow \;\; (z, y_1) \; + \; (x_2, z) \; + \; (x_1, y_2)$$

- $$|\psi^{(0)}(x_1, y_1)|^2 |\psi^{(0)}(x_2, y_2)|^2 |\psi^{(0)}(x_3, y_3)|^2 \frac{\alpha_S N_c}{2\pi^2} \times \int_0^Y dY' \int d^2 z$$

$$\{2\gamma^{BA}(x_2, y_2|z) K(x_1, y_1|z) [\gamma^{BA}(x_1, z|x_3, y_3) + \gamma^{BA}(y_1, z|x_3, y_3)] + (1 \leftrightarrow 2)\}$$

Contribution to the GF

$$Z\left(Y,[u_i]\right) = -\frac{1}{2}\Gamma_{2P\to P}$$

$$u(x_1,z)\,u(z,y_1)\,u(x_2,z)\,u(z,y_2)\,\frac{\delta}{\delta u(x_1,y_1)}\,\frac{\delta}{\delta u(x_2,y_2)}$$

Therefore

Vertex $V(2\to 4)$ can be calculated in the JIMWLK approach.

Conclusions:

- **The generating functional approach is equivalent to the BFKL Pomeron calculus at high energies and gives a natural generalization of the BFKL Pomeron calculus at ultra high energies;**

- **The BFKL Pomeron calculus generates by the decay terms in the generating functional and, therefore, all of them can be calculated in the JIMWLK approach;**

- $GF + JIMWLK =$ **the selfconsistent and complete theory of high energy interaction in QCD**

- **The equations for the generating functional describes Markov's chain and can be a basis for a new Monte Carlo code which will solve the problem of the high energy behaviour of the scattering amplitude as as well as it will lead to the description of the inclusive processes;**

- **Being the simplest and the easiest for the direct physical interpretation, the GF approach has a major disadvantage: we have not learned yet, what vertices have to be taken into account and ,if all, how to incorporate them in an elegant scheme.**

THINK

Toy model:

$$\frac{\partial Z}{\partial Y} = -\Gamma(1 \to 2)\; u(1-u)\,\frac{\partial Z}{\partial u} +$$

$$\Gamma(2 \to 1)\; u^2(1-u)^2\,\frac{\partial^2 Z}{(\partial u)^2} + \Gamma(2 \to 3) u\,(1-u)^2 \frac{\partial^2 Z}{(\partial u)^2}$$

$$\Gamma(1 \to 2) \propto \bar{\alpha}_S\,; \quad \Gamma(2 \to 1) \propto \bar{\alpha}_S^3/N_c^2; \quad \Gamma(2 \to 3) \propto \bar{\alpha}_S/N_c^2;$$

$$\boxed{\frac{\Gamma(1\to 2)}{\Gamma(2\to 1)} \equiv \kappa_1 \propto \frac{N_c^2}{\bar{\alpha}_S^2} \gg 1}\;; \qquad \boxed{\frac{\Gamma(2\to 1)}{\Gamma(2\to 1)} \equiv \kappa_2 \propto \alpha_S^2\, N_c^2 \gg 1}$$

Asymptotic solution:

Semi-classical approach:

$$Z(Y=\infty;u) = e^{\Phi(\infty;u)}; \qquad \frac{\partial^2 Z_{asymp}(u)}{(\partial u)^2} \Longrightarrow (\Phi'_u)^2\, e^{\Phi(\infty;u)}$$

Solution:

- $\Phi_u(u) = \frac{\kappa_1}{u(1+u)+\kappa_2(1-u)}$;
- $Z(\infty,u) = \left(\frac{(\kappa_2-u)(\frac{2}{\kappa_2})}{(\kappa_2-1)(1+\frac{2}{\kappa_2}-u)}\right)^{\frac{\kappa_1}{\kappa_2}}$

$\Gamma(2P \to P)$ **gives a contribution only in a very limitted part of the kinematic region, namely, for** $1-u < 2/\kappa_2$

Semi-classical solution:

- $$Z\left(\mathcal{Y}, u\right) \; = = \; \int_{a-i\infty}^{a+i\infty} \frac{d\,\omega}{2\pi i}\; e^{\omega\,\mathcal{Y}}\; \left(e^{\Phi^{+}(\omega,u)}\,\phi^{+}(\omega) \; + \; e^{\Phi^{-}(\omega,u)}\,\phi^{-}(\omega)\right)$$

where

- $$\Phi^{\pm}_{u} \; = \; \frac{\kappa_1}{2\;L(u)}\left\{ 1 \; \pm \; \sqrt{1 \; + \; \frac{4\;\omega\;L(u)}{\kappa_1\;u\,(1-u)}}\;\right\}$$

$$L(u) = u(1+u) \; + \; \kappa_2\,(1-u)$$

and

$$\phi^{+}(\omega) = \frac{1}{\omega} - \frac{\Gamma\left(\alpha \; - \sqrt{\alpha\;\omega}\right)\Gamma\left(\alpha \; + \sqrt{\alpha\;\omega}\right)}{\Gamma\left(2\,\alpha\right)}$$

$$+ \; \frac{\Gamma\left(\frac{\alpha}{2} - \sqrt{\alpha\,\omega}\right)\Gamma\left(\frac{\alpha}{2} + \sqrt{\alpha\;\omega}\right)}{\Gamma\left(\alpha\right)}\;{}_2F_1\left(-2, \frac{\alpha}{2} - \sqrt{\alpha\;\omega}, \alpha, 2\right)$$