Matching the Discrete BFKL Pomeron to DGLAP *

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Abstract

The discrete BFKL formalism which accounts for the running of the coupling and incorporates information about the phase of the oscillations at small transverse momentum, arising from the infrared properties of QCD, leads to a QCD pomeron consisting of a set of discrete Regge poles. Here we discuss under what circumstances this discrete pomeron leads to an amplitude which matched the prediction of a DGLAP analysis in the double leading logarithm limit.

1 Introduction

The BFKL formalism [1] (in the case of zero momentum transfer) considers an amplitude, $\mathcal{A}(y,t)$, for the forward scattering of a gluon with transverse momentum, k_T ($t = \ln(k_T^2/\Lambda_{QCD}^2)$), with rapidity gap y, which can be treated in terms of its Mellin transform

$$\mathcal{A}(y,t) \;=\; \int_{\mathcal{C}} d\omega e^{\omega y} \mathcal{A}_\omega(t) \tag{1}$$

where $\mathcal{C}$ represents a contour taken to the right of any singularities of the Mellin transform function $\mathcal{A}_\omega(t)$. The purely perturbative BFKL formalism generates a Mellin transform function with a cut along the real axis in the ω-plane. However, it has been shown [2] that if the strong coupling runs with the transverse momentum, t, a phase-matching boundary condition is automatically imposed at a value, t_c, of t where the t-dependence of $\mathcal{A}_\omega(t)$ changes from an oscillatory function to a decaying one. If, moreover, one assumes that the infrared properties of QCD impose a further constraint on the phase of these oscillations at some small value of transverse momentum, then the two constraints can only be

*Presented at the Low x workshop, May 30 - June 4 2013, Rehovot and Eilat, Israel simultaneously satisfied for a discrete set, ω_n, of ω, and the singularity structure of $\mathcal{A}_\omega(t)$ becomes a set of poles, akin to Regge poles. In such cases the amplitude $\mathcal{A}(y,t)$ may be written

$$\mathcal{A}(y,t) \; = \; \sum_n \mathcal{A}_{\omega_n}(t) e^{\omega_n y} \tag{2}$$

The positions of the poles ω_n as well as their residues, depend not only on the infrared phases, selected by the infrared properties of QCD, but also on the precise running of the coupling. In this way, the discrete BFKL formalism acts as a communicator between high and low energy scales and the predictions of these amplitudes with large rapidity gaps are affected by physics beyond the Standard Model (BSM) even at thresholds which are considerably higher than the energies at which such amplitudes are considered. In a recent paper [3] we have suggested that the quality of the fit of low-x structure functions at HERA is significantly improved by the presence of a supersymmmetry threshold at around 10 TeV.

This immediately poses the question as to how the results from the discrete BFKL formalism can match those of a DGLAP analysis [4] in the double leading-logarithm (DLL) limit where both y and t are large, for which the function $\mathcal{A}_\omega(t)$ obeys the DGLAP equation

$$e^{-t/2} \frac{\partial}{\partial t} \left\{ e^{t/2} \mathcal{A}_\omega(t) \right\} \; = \; \frac{C_A \alpha_s(t)}{\pi \omega} \mathcal{A}_\omega(t) \tag{3}$$

.

In the case of the purely perturbative BFKL formalism with a cut singularity in ω, this match is understood [5] from the fact that at large g and small ω, the Mellin transform function from the BFKL analysis is approximated by

$$\mathcal{A}_\omega(t) \; \sim \; \exp\left\{ - \int^t \frac{C_A \alpha_s(t')}{\pi \omega} dt' \right\}, \tag{4}$$

which is a solution to eq.(3) and the inverse Mellin transform (1) is dominated by a saddle-point at

$$\omega \; = \; \sqrt{\frac{C_A \alpha_s(t)}{\pi y}}. \tag{5}$$

2 The Green function

The BFKL equation (at leading order) with running coupling is given by

$$\frac{\partial}{\partial y} \mathcal{A}(y,t) \; = \; \int dt' \alpha_s(t) \mathcal{K}_0(t,t') \mathcal{A}(y,t') \tag{6}$$

and is solved in terms of a *universal* (i.e. process independent) Green function, $\mathcal{G}_\omega(t,t')$, which obeys the equation

$$\int dt'' \left[\omega \delta(t-t'' - \alpha_s(t) \mathcal{K}_0(t,t'') \right] \mathcal{G}_\omega(t'',t) \; = \delta(t-t'). \tag{7}$$

This Green function is only uniquely defined once certain boundary conditions are imposed. The first of these is the requirement that for physically sensible results

$$\mathcal{G}_\omega(t,t') \stackrel{t(t')\to\infty}{\to} 0. \tag{8}$$

There must also be a condition on the Green function for small t, (t') which is imposed by the infrared properties of QCD.

The kernel, $\mathcal{K}_0(t,t')$ may be expressed in terms of its Fourier transform

$$\mathcal{K}_0(t,t') = \int d\nu e^{i\nu(t-t')}\chi(\nu). \tag{9}$$

For simplicity, we start with a simplified model in which the characteristic function is a quadratic function of ν, i.e.

$$\chi(\nu) = a - b\nu^2, \tag{10}$$

and writing the leading order running coupling as

$$\alpha_s(t) = \frac{1}{\beta_0 t}, \tag{11}$$

the Green function obeys Airy's equation

$$\frac{1}{\beta_0 t}\left(a + b\frac{\partial^2}{\partial t^2}\right)\mathcal{G}_\omega(t,t') = \omega\mathcal{G}_\omega(t,t'), \tag{12}$$

so that it may be expressed in terms of Airy functions $A_i(z_\omega(t))$ and $B_i(z_\omega(t))$, where

$$z_\omega(t) = \left(\frac{\omega\beta_0}{b}\right)^{1/3}\left(t - \frac{a}{\omega\beta_0}\right) \tag{13}$$

This Green function oscillates if $t < a/(\omega\beta_0)$ whereas for $t > a/(\omega\beta_0)$, A_i decreases, whereas B_i increases. Thus a Green function, which is finite as $t\,(t') \to \infty$ may be written as

$$\mathcal{G}_\omega(t,t') = A_i(z_\omega(t))B_i(z_\omega(t'))\theta(t-t') + B_i(z_\omega(t))A_i(z_\omega(t'))\theta(t'-t) \tag{14}$$

However, this is *not* unique. A more general solution whch satisfies the ultraviolet boundary conditions is

$$\mathcal{G}_\omega(t,t') = A_i(z_\omega(t))\overline{B_i}(z_\omega(t'))\theta(t-t') + \overline{B_i}(z_\omega(t))A_i(z_\omega(t'))\theta(t'-t), \tag{15}$$

where

$$\overline{B_i}(z_\omega(t)) = B_i(z_\omega(t)) + c_\omega A_i(z_\omega(t)). \tag{16}$$

The coefficint c_ω encodes the infrared properties of QCD and thereby determines the behaviour of the Green function for small $t\,(t')$.

If we write $c_\omega = \cot(\phi(\omega))$, then for sufficiently small t ($z_\omega(t) \ll 0$) we have

$$\overline{B_i}(z_\omega(t) \to \frac{1}{\sin(\phi(\omega))}\sin\left(\frac{2}{3}\left(z_\omega(t)\right)^{2/3} + \frac{\pi}{4} + \phi(\omega)\right) \tag{17}$$

This has poles whenever $\phi(\omega) = n\pi$ and these are the Regge poles of the BFKL pomeron. determined by the phase of the oscillatory part of the Green function at small t.

This means that the Green function can be written in terms of a the discrete eigenfunctions, $f_\omega(t)$ of the BFKL operator, $\alpha_s(t)\mathcal{K}_0(t,t')$ with eigenvalues ω_n as

$$\mathcal{G}_\omega(t,t') \;=\sum_n \frac{f_{\omega_n}(t) f*_{\omega_n}(t')}{\omega-\omega_n} \;+\; \text{terms analytic in } \omega \tag{18}$$

The sum generates the discrete poles of the BFKL pomeron but the analytic remainder is *crucial* for the matching of the $t-$*dependence* of the large rapidity-gap amplitudes to DGLAP in the DLL limit.

In the case of the real BFKL characteristic function

$$\chi(\nu) \;=\; 2\left[\Psi(1) - \Re e\left\{\Psi(1/2+i\nu)\right\}\right], \tag{19}$$

the Green function may still be written in terms of Airy functions in the semi-classical approximation (in which the oscillation frequency is treated as a slowly varying function of t). The Green function is once again given by eq.(15), but in this case the argument $z_\omega(t)$ of the Airy functions is given by

$$\frac{2}{3}\left(z_\omega(t)\right)^{3/2} \;=\; \int_{t_c}^{t} dt' \nu_\omega(t'), \tag{20}$$

where $\nu_\omega(t)$ is the solution to

$$\alpha_s(t)\chi\left(\nu_\omega(t)\right) \;=\; \omega, \tag{21}$$

and t_c is the value of t at which $\nu_\omega(t_c) = 0$. Note that this value of t_c depends in the exact nature of the running of the coupling and is therefore sensitive to any thresholds for BSM physics - even for very large values of such thresholds.

3 Matching to DGLAP

For sufficiently large t, $\nu_\omega(t)$ may be approximated by

$$\nu_\omega(t) \;\stackrel{t\to\infty}{\to}\; i\left(\frac{1}{2} - \frac{C_A\alpha_s(t)}{\pi\omega}\right) \tag{22}$$

and the Airy function, A_i, approximates to

$$A_i(z_\omega(t)) \;\sim\; e^{-t/2}\exp\left\{\int^t dt' \frac{C_A\alpha_s(t')}{\pi\omega}\right\} \tag{23}$$

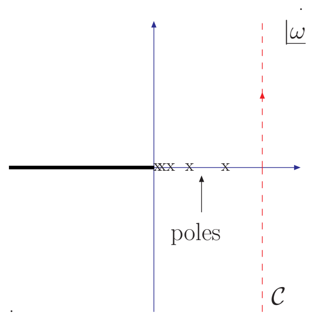

Figure 1: Contour for the inverse mellin transform of the scattering amplitude

4 Application to Deep Inelastic Scattering

For deep-inelastic scattering $t = \ln\left(Q^2/\Lambda^2\right)$, where Q^2 is the photon virtuality, and the rapidity y is replaced by $\ln(1/x)$. In the BFKL formalism, the unintegrated gluon density, $\dot{g}(x,t)$ which is derivative w.r.t. t of the gluon density, $g(x,t)$ is given by

$$\dot{g}(x,t) \;=\; \int_{\mathcal{C}} d\omega \int dt' x^{-\omega} \mathcal{G}_\omega(t,t') \Phi_P(t'), \tag{24}$$

where $\Phi_P(t)$ is the impact factor that describes the coupling of the QCD pomeron to the proton and is the only process-dependent factor.

The integral over ω goes over a contour $\mathcal{C}$ taken to the right of all the poles of $\mathcal{G}_\omega$

On the other hand the integrand possesses a saddle-point at $\omega = \omega_s$ given by

$$-\ln(x) \;=\; \frac{\partial}{\partial\omega} \ln\left[A_i\left(z_\omega(t)\right)\right]_{\omega=\omega_s} \tag{25}$$

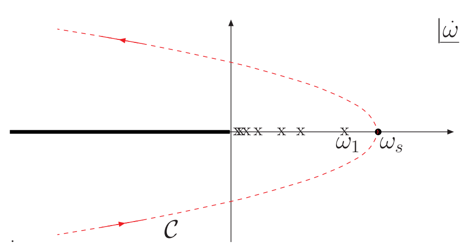

Figure 2: Deformed contour for the inverse Mellin transform of the scattering amplitude in the case where the saddle-point, ω_s lies to the right of all the poles of $\mathcal{G}_\omega$.

For sufficiently large t, this saddle-point lies to the right of all the poles of $\mathcal{G}_\omega$. i.e.

$$\omega_s \;\gg\; \frac{4\ln 2 C_A \alpha_s(t)}{\pi} \tag{26}$$

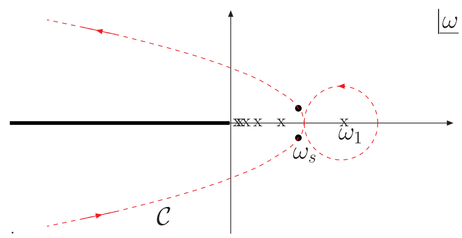

Figure 3: Deformed contour for the inverse Mellin transform of the scattering amplitude in the case where the saddle-point, ω_s lies to the left of the leading pole of $\mathcal{G}_\omega$.

and the contour may be deformed without crossing a singularity, such that the saddle-point approximation is valid and yields

$$\dot{g}(x,t) \;\sim\; x^{-\omega_s} \int dt' \mathcal{G}_{\omega_s}(t,t')\Phi_P(t'). \tag{27}$$

For large t for which the saddle-point is given by (5), we recover the DGLAP expression on the DLL limit.

The discrete poles affect the overall normalization of the unintegrated gluon density, but *not* its t-dependence.

However, if t is *not* sufficiently large then this saddle-point lies to the left of one or more of the poles of $\mathcal{G}_\omega$ and so the saddle-point approximation must be supplemented by the contribution from the contour surrounding these poles, giving rise to an expression for the unintegrated gluon density

$$\dot{g}(x,t) \;\sim\; x^{-\omega_s} \int dt' \mathcal{G}_{\omega_s}(t,t')\Phi_P(t') + \sum_{\omega_n > \omega_s} A_n x^{-\omega_n} A_i\left(z_{\omega_n}(t)\right). \tag{28}$$

The contribution from the poles, which does *not*, in any way, match the DGLAP expression dominates at sufficiently low x. Therefore the DLL limit of DGLAP is *not* a good approximation in this region of t.

5 Summary

The (discrete) BFKL universal Green function has poles whose positions and residues are controlled by

1. The infrared properties of QCD which fixes the phase of the oscillation at small t.
2. The precise running of the coupling including the effects of any BSM thresholds.

The Green function consists of a set of poles supplemented by an part which is analytic in ω in such a way that

1. For sufficiently large t, when the saddle-point in the inverse Mellin transform lies to the right of all the discrete poles, the t-dependence of the BFKL amplitude matched that of a DGLAP analysis in the DLL limit.

2. As t is reduced such that the saddle-point lies to the left of one or more of these poles, the saddle-point approximation for the BFKL amplitude must be supplemented by the contribution from the poles to the right of the saddle-point and a match to a DGLAP analysis is no longer obtained.

References

[1] I. I. Balitsky and L. N. Lipatov, Sov. J. Nucl. Phys. **28** (1978) 822; E. A. Kuraev, L. N. Lipatov and V. S. Fadin, Sov. Phys. JETP **44** (1976) 443; V. S. Fadin, E. A. Kuraev and L. N. Lipatov, Phys. Lett. B **60** (1975) 50.

[2] L. N. Lipatov, Sov. Phys. JETP **63** (1986) 904.

[3] H. Kowalski, L.N. Lipatov,and D. A. Ross, Phys. Part. Nucl. **44** (2013) 547
H. Kowalski, L.N. Lipatov, D. A. Ross, and G. Watt, Eur. Phys. J **C70** (2010) 983; Nucl. Phys **A854** (2011) 45

[4] G. Altarelli and G. Parisi, Nucl. Phys. **B126** (1977) 298
Yu. L. Dokshitzer, Sov. Phys. JETP **46** (1977) 46
V.N. Gribov and L.N. Lipatov, Sov. Nucl. Phys. **15** (1972) 438

[5] M. Ciafaloni and D. Colferai, Phys. Lett. **B452** (1999) 372
M. Ciafaloni, D. Colferai and D.P. Salam, Phys. Rev. **D60** (1999) 114036
M. Ciafaloni, D. Colferai, D.P. Salam and A. Stasto, Phys. Rev. **D66** (2002) 054014