Rare Configurations at High Energy

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Motivation: Why (another) toy model?

- Toy model : Suppress transverse dimensions
- Boost invariance + multiple scattering $\rightsquigarrow$ Uniquely fix model
  Contains 0-dim versions of
  - BFKL
  - Balitsky-JIMWLK
  - Gluon saturation
  - Pomeron Loops

Toy model construction (1/2)

- Consider two onia: hadrons composed of dipoles
  Configurations characterized by number of dipoles
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    τ: All orders elementary dipole-dipole scattering amplitude

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  - All possible interactions $\Leftrightarrow$ multiple scattering (Wilson Lines)

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- All kinds of number changing Pomeron transitions

“Observables” and evolution

- Target evolved by Y. Projectile m dipoles: $P_k^{\mathrm{L}}(0) = \delta_{mk}$

$$\frac{\mathrm{d}\langle s^m \rangle}{\mathrm{d}Y} = f_m \left[\langle s^{m+1} \rangle - \langle s^m \rangle\right]$$

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def: Continuum limit in $n \gg 1$

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- Would like to calculate
  - $\langle s \rangle, \langle s^2 \rangle, \ldots \langle t \rangle, \ldots \langle n \rangle, \langle n(n-1) \rangle, \ldots$
  - Probability distribution $P_n(Y)$: Much more insight

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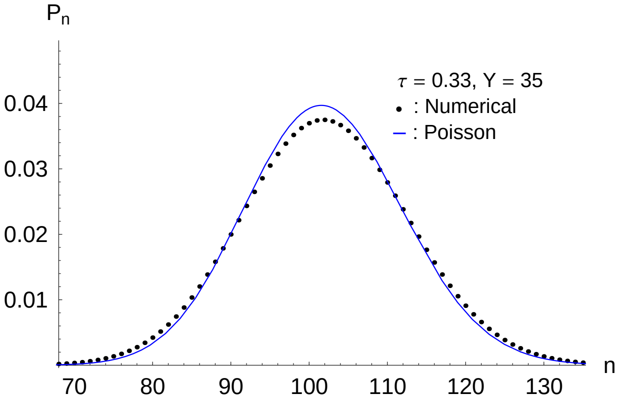
  - Contains the "whole" probability; bulk distribution

Bulk distribution (2/2)



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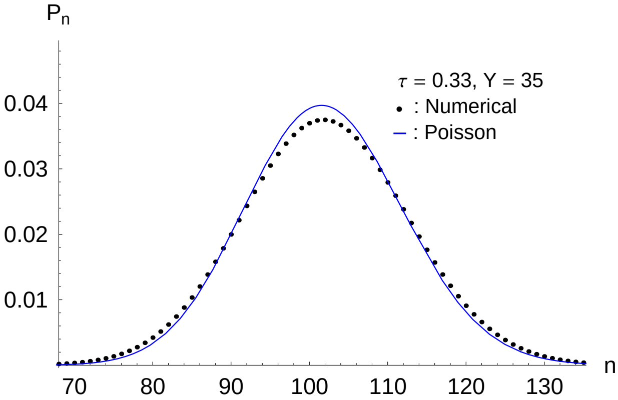
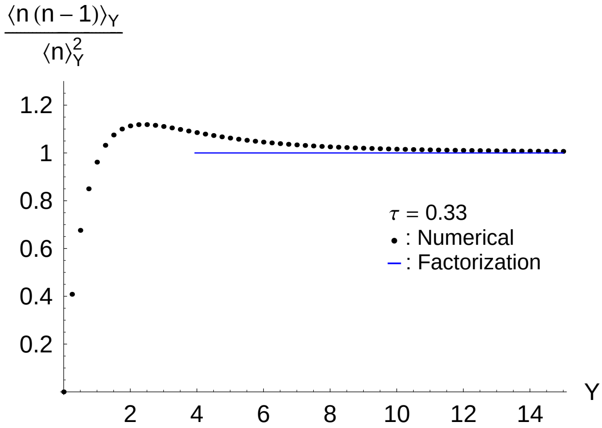


- Maximum at $n \simeq \langle n \rangle$, height $\sim 1/\sqrt{\langle n \rangle}$, width $\sim \sqrt{\langle n \rangle}$
- Emission rate saturates $\rightsquigarrow \langle n \rangle \sim Y$ (like in QCD)

Factorization - "mean field"



- Average number, pair-number determined by bulk distribution

Scattering off the bulk

- Contribution of bulk distribution to S-matrix

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- Wrong result

  S-matrix : projectile survival probability dominated by (target) configurations with few particles

Rare configurations (1/2)

- Probability to find one dipole in target $P_1(Y) = \mathrm{e}^{-Y}$
  Splitting of one dipole "fuels" configurations with "few" dipoles

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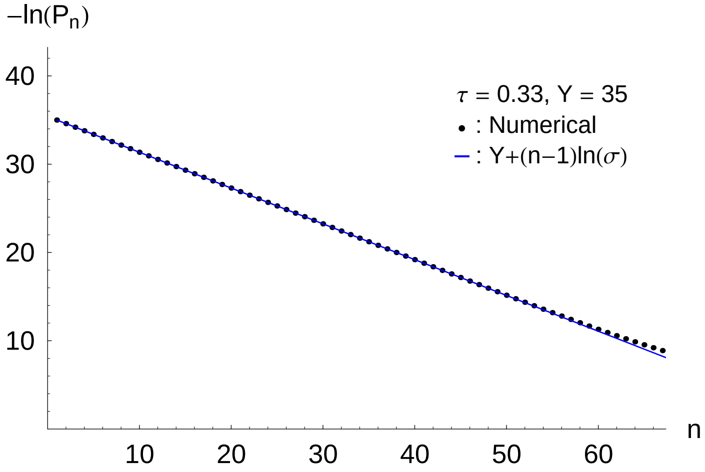
- Very rare configurations
- Low-n distribution matches with bulk at $n \sim n_{\mathrm{cr}}$

Rare configurations (2/2)



Scattering two dipoles off the target (1/2)

- Average value of S-matrix for onium – two dipoles scattering

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- Rare configurations $n \lesssim 1/\tau$ dominate the sum

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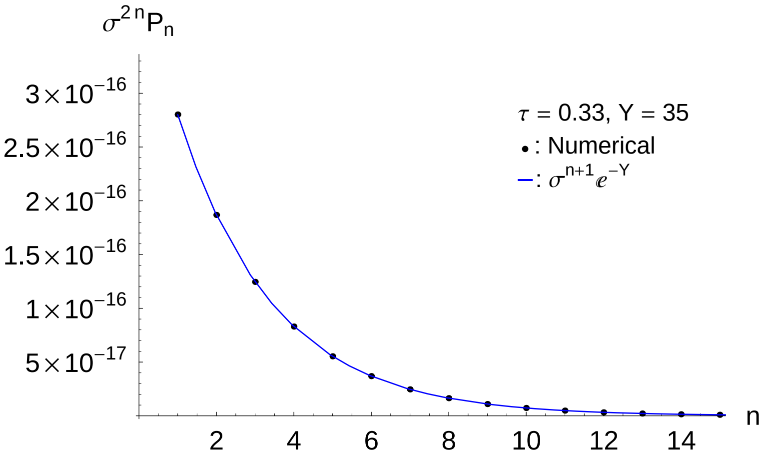

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  - Black disk limit
  - Same Y-dependence for any m, NO "mean-field" $\neq$ B-JIMWLK
  - An exact solution to Pomeron loop hierarchy
  - Almost same result from dipole picture (in target)

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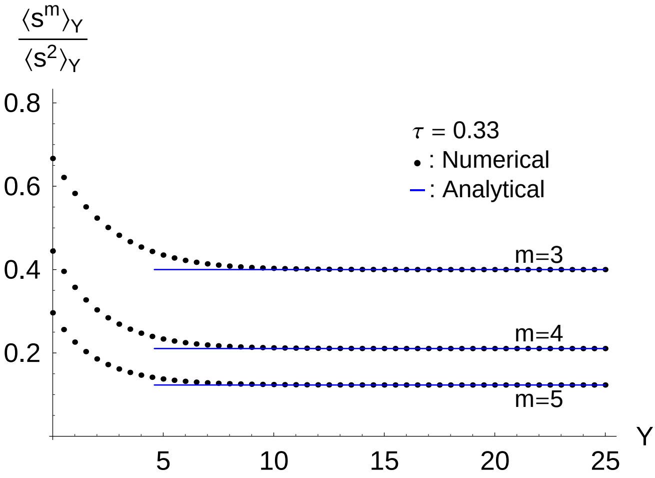

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- Asymptotic solution

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  - Calculate in other frame, e.g. COM $\rightsquigarrow$ same result
  - Mueller–Salam result (up to σ^2 prefactor)

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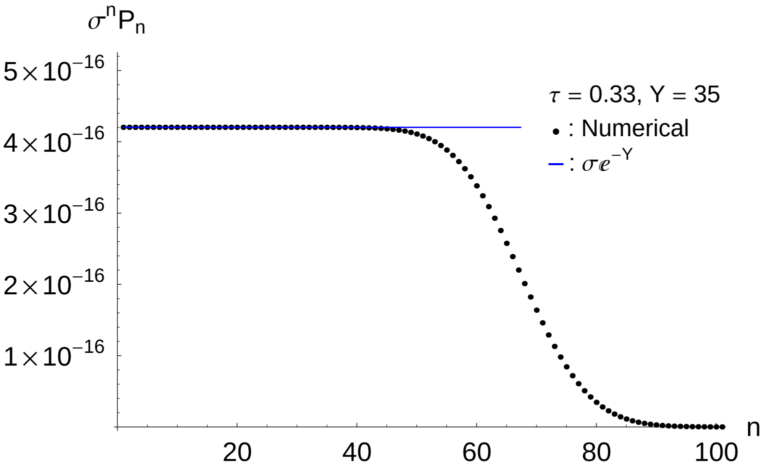
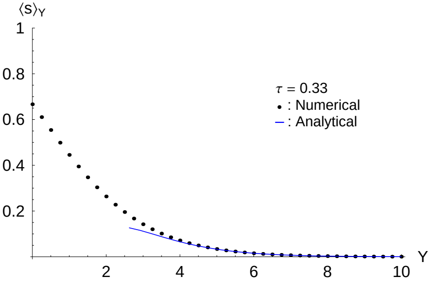
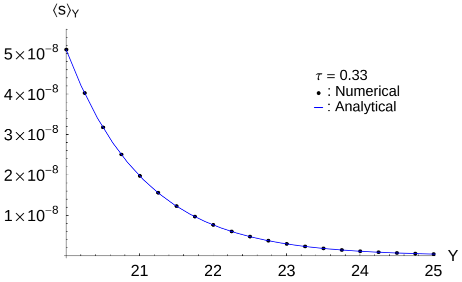


Scattering one dipole (3/3)





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  - Pomeron loop hierarchy $\supset$ B-JIMWLK hierarchy
  - Average dipole number linear in Y (marginal saturation);
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- Outlook
  - QCD