# BFKL and dipole model

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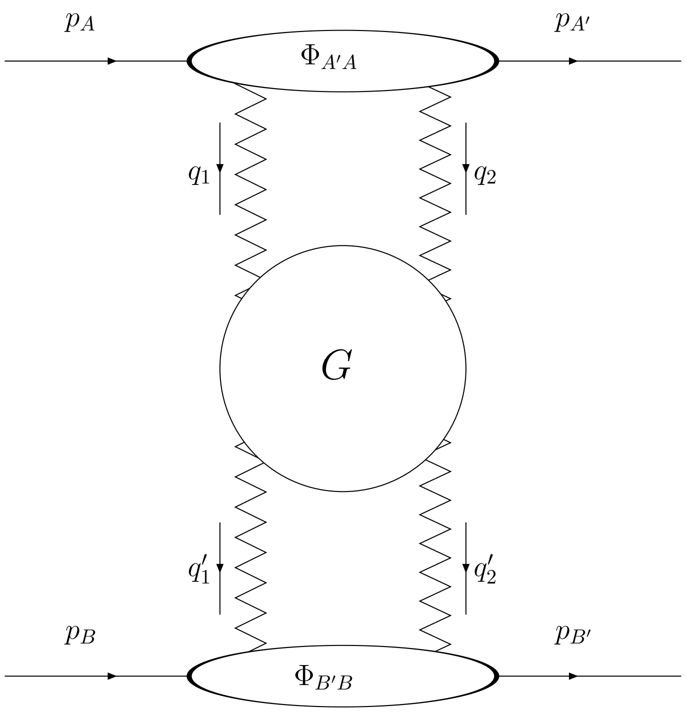
# Contents

## BFKL approach

In the BFKL approach scattering amplitudes $\mathcal{A}_{AB}^{A'B'}$ are presented in the form :

$$\Phi_{A'A} \otimes G \otimes \Phi_{B'B}.$$




Impact factors $\Phi_{A'A}$ and $\Phi_{B'B}$ describe transitions $A \to A'$ and $B \to B'$ ,
$G$ – Green's function for two interacting Reggeized gluons,

$$\widehat{\mathcal{G}} = e^{Y\widehat{\mathcal{K}}},$$

$Y = \ln(s/s_0)$ , $\widehat{\mathcal{K}}$ – BFKL kernel,

$$\widehat{\mathcal{K}} = \widehat{\Omega} + \widehat{\mathcal{K}}_r$$

## BFKL approach

$$\widehat{\mathcal{K}} = \widehat{\Omega} + \widehat{\mathcal{K}}_r$$

$\widehat{\Omega} = \omega(\hat{\vec{q}}_1) + \omega(\hat{\vec{q}}_2)$ — the “virtual” part, $\langle \vec{q}_i | \hat{\omega}_i | \vec{q}_i{}' \rangle = \delta(\vec{q}_i - \vec{q}_i{}')\omega(\vec{q}_i)$ ,

$\omega(\vec{q})$– the gluon Regge trajectory;

In the leading order at $D = 4 + 2\epsilon$:

$$\omega^{(1)}(\vec{q}) = -\frac{g^2 N_c \Gamma(1-\epsilon)}{(4\pi)^{2+\epsilon}} \frac{2}{\epsilon} (\vec{q})^\epsilon$$

In the next-to-leading order in the limit $\epsilon \to 0$

$$\omega(\vec{q}) = \omega^{(1)}(\vec{q}) \left( 1 + \frac{\omega^{(1)}(\vec{q})}{4} \left[ \frac{11}{3} + \left( 2\zeta(2) - \frac{67}{9} \right) \epsilon + \left( \frac{404}{27} - 2\zeta(3) \right) \epsilon^2 \right] \right)$$

V.S. F., R. Fiore and M.I. Kotsky, 1996;

J. Bluemlein, V. Ravindran, W.L. van Neerven, 1998;

V.Del Duca, E.W.N. Glover, 2001.

## BFKL approach

$$\widehat{\mathcal{K}} = \widehat{\Omega} + \widehat{\mathcal{K}}_r$$

$\widehat{\mathcal{K}}_r$ — the “real” part,

$$\langle \vec{q}_1, \vec{q}_2 | \hat{\mathcal{K}}_r | \vec{q}_1{}', \vec{q}_2{}' \rangle = \delta(\vec{q} - \vec{q}\,') \frac{1}{\vec{q}_1{}^2 \vec{q}_2{}^2} \mathcal{K}_r(\vec{q}_1, \vec{q}_1{}'; \vec{q}), \;\; \vec{q} = \vec{q}_1 + \vec{q}_1{}' = \vec{q}_2 + \vec{q}_2{}'.$$

In the leading order

$$\mathcal{K}_r^B(\vec{q}_1, \vec{q}_2; \vec{q}) = \frac{g^2 N_c c_{\mathcal{R}}}{(2\pi)^{D-1}} \left( \frac{\vec{q}_1^{\,2} \vec{q}_2{}'^{\,2} + \vec{q}_2^{\,2} \vec{q}_1{}'^{\,2}}{(\vec{q}_1 - \vec{q}_1{}')^2} - \vec{q}^{\,2} \right)$$

Only one structure with coefficients depending from $t$–channel colour states.

Possible representations of the colour group in the $t$–channel

$$\underline{1}, \underline{8_a}, \underline{8_s}, \underline{10}, \underline{\overline{10}}, \underline{27}.$$

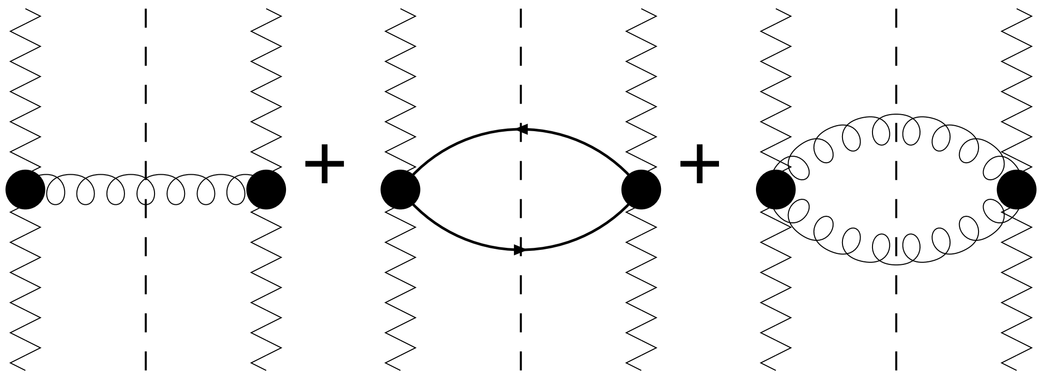
$$c_1 = 1\,, \;\; c_{8_a} = c_{8_s} = \frac{1}{2}\,, \;\; c_{10} = c_{\overline{10}} = 0\,, \;\; c_{27} = -\frac{1}{4N_c}$$

## BFKL approach

$$\widehat{\mathcal{K}} = \widehat{\Omega} + \widehat{\mathcal{K}}_r$$

In the next-to-leading order

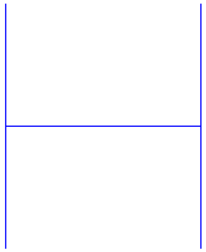
$$\widehat{\mathcal{K}}_r = \widehat{\mathcal{K}}_G + \widehat{\mathcal{K}}_{Q\overline{Q}} + \widehat{\mathcal{K}}_{GG}$$

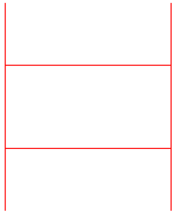


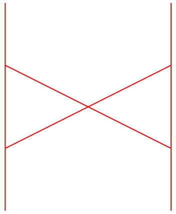
Each of the $\widehat{\mathcal{K}}_{Q\overline{Q}}$ and $\widehat{\mathcal{K}}_{GG}$ has two independent terms with different colour coefficients.

## BFKL approach

The colour group diagrams







For the colour singlet in the $t$-channel the colour coefficients

$$c_1 = \frac{1}{N_c(N_c^1 - 1)} Tr\left(T^a T^a\right)$$

$$a_1 = \frac{1}{N_c^2(N_c^1 - 1)} Tr\left(T^a T^a T^b T^b\right), \quad b_1 = \frac{1}{N_c^2(N_c^1 - 1)} Tr\left(T^a T^b T^a T^b\right)$$

## BFKL approach

The group generators

$T^a = t^a$ for quarks,

$T^a_{bc} = -if^{abc}$ for gluons.

Instead of contributions with the colour coefficient $a_1$ and $b_1$ it is convenient to consider the contributions with the coefficients $a_1 - b_1$ and $b_1$. We call them "non-Abelian" and "Abelian", or "symmetric", contributions. We have

$a_1 - b_1 = \frac{1}{2}$ both for the fundamental and adjoint representation;

$b_1 = -\frac{1}{4N_c^3}$ and $b_1 = \frac{1}{2}$ for the fundamental and adjoint representations.

$\widehat{\mathcal{K}}_r$ is found in the NLO both for the forward scattering

V.S. F., L.N. Lipatov, 1998, M. Ciafaloni and G. Camici, 1998

i.e. for $t = 0$ and color singlet (Pomeron) in the $t$-channel, and for any fixed $t$ and any possible color state in the $t$-channel

V.S. F.,R. Fiore and A. Papa, 1999, V.S. F. and D.A. Gorbachev, 2000, V.S. F. and R. Fiore, 2005.

## BFKL approach

The Pomeron channel is the most important for phenomenological applications, although from the theoretical point of view the color octet case seems to be even more important because of the gluon Reggeization.

In the Pomeron channel infrared divergencies of “virtual” and “real” parts cancel.

The remarkable property of the LO colour singlet kernel – invariance with respect to the conformal transformations of coordinates in the transverse two-dimensional space $\vec{r} = (x, y)$

$$z \to \frac{az + b}{cz + d} \, ,$$

where $z = x + iy, \;\; a, b, c, d$ are complex numbers

L.N. Lipatov, 1986.

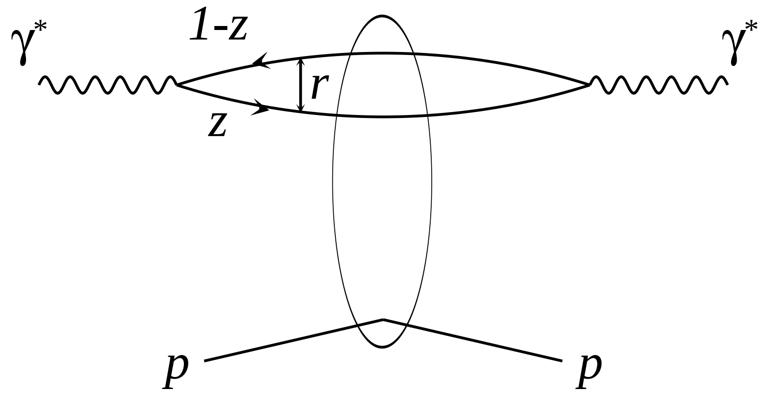
## Colour dipole picture

In the color dipole approach

N.N. Nikolaev and B.G. Zakharov, 1994,

A. H. Mueller, 1994

$\gamma^*$ scattering is considered as $\gamma^*$ splitting into a $q\bar{q}$ colour dipole with subsequent $q\bar{q}$ scattering. The important point is conservation of transverse coordinates of the dipole components.




$$\sigma_{\gamma^*}(x, Q^2) = \int d^2r \int_0^1 dz |\Psi_{\gamma^*}(r, z, Q^2)|^2 \sigma_{dp}(r, x),$$

$x = Q^2/s$, $\Psi_{\gamma^*}(r, z, Q^2)$ is the photon wave function, $z$ is the longitudinal momentum fraction carried by the quark, $\vec{r} = \vec{r}_1 - \vec{r}_2$, $\vec{r}_1$ and $\vec{r}_2$ are the quark and antiquark transverse coordinates, $\sigma_{dp}(r, x)$ is the dipole cross section,

## Colour dipole picture

$$\sigma_{dp}(r,x) = 2\int d^2b\,\mathcal{N}(\vec{r}_1,\vec{r}_2;Y)\,;$$

$\vec{b} = (\vec{r}_1+\vec{r}_2)/2$ is the impact parameter, $Y = \log(1/x)$, $\mathcal{N}(\vec{r}_1,\vec{r}_2;Y)$ is the the imaginary part of the dipole scattering amplitude obeying the equation

$$\frac{\partial\mathcal{N}}{\partial Y} = \widehat{\mathcal{K}}_{dip}\mathcal{N},$$

$$\langle\vec{r}_1\vec{r}_2|\hat{\mathcal{K}}_{dip}|\vec{r}_1{}'\vec{r}_2{}'\rangle = \frac{\alpha_s N_c}{2\pi^2}\int d^2\rho\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{1\rho}^{\,2}\vec{r}_{2\rho}^{\,2}}\left(\delta(\vec{r}_{11'})\delta(\vec{r}_{2\rho})+\delta(\vec{r}_{22'})\delta(\vec{r}_{1\rho})-\delta(\vec{r}_{11'})\delta(\vec{r}_{22'})\right),$$

$\vec{r}_{ij} = \vec{r}_i - \vec{r}_j,\ \ \vec{r}_{i\rho} = \vec{r}_i - \vec{\rho},\ \ \vec{r}_{ij'} = \vec{r}_i - \vec{r}_j{}',$

with the non-linear extension (BK equation) for $S = 1-\mathcal{N}$:

$$\frac{\partial S(\vec{r}_1,\vec{r}_2;Y)}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{\rho}\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{1\rho}^{\,2}\vec{r}_{2\rho}^{\,2}}\left[S(\vec{r}_1,\vec{\rho},Y)S(\vec{\rho},\vec{r}_2,Y)-S(\vec{r}_1,\vec{r}_2,Y)\right]$$

Ia. Balitsky, 1996, Yu. Kovchegov, 1999.

## Motivation

A clear understanding of the relation between the BFKL and colour dipole approaches is very important. It could help in further development of the theoretical description of small-$x$ processes.

Since the colour dipole approach is developed in the coordinate representation, the relation can be investigated by transformation of the BFKL approach in this representation.

There are additional reasons for considering the BFKL kernel in the coordinate representation in the transverse space:

It reveals conformal properties of the kernel.

Evidently, the conformal invariance is violated by renormalization. One may wonder, however, whether the renormalization is the only source of the violation. If so, one can expect the conformal invariance of the NLO BFKL kernel in supersymmetric extensions of QCD.

The complexity of the NLO BFKL kernel in the momentum representation. The colour singlet kernel for $t \neq 0$ is found in the NLO in the form of the intricate two-dimensional integral. The hope was on its simplification.

## Motivation

Relation between the BFKL and the color dipole approaches in the LO was discussed many times.

With the advent of the color dipole approach it was affirmed that it is equivalent to the BFKL.

Recently, the relation of the non-linear generalizations was analyzed

J. Bartels, L. N. Lipatov, G. P. Vacca, 2004,

J. Bartels, L. N. Lipatov, M. Salvadore, G. P. Vacca, 2005.

The extension of the analysis to the NLO was started in 2006 both from the dipole

Ya. Balitsky, 2006,

Yu. Kovchegov, 2006

and the BFKL

V.S. F, R. Fiore, A. Papa, 2006,

V.S. F, R. Fiore, A.V. Grabovsky, A. Papa, 2007

side.

## The dipole (Möbius) representation of the BFKL kernel in the LO

In the LO the comparison of two approaches can be performed absolutely rigorously, at $D = 4 + 2\epsilon$. Since the dipole kernel is determined by the probability of the soft gluon emission by $\bar{q}q$ pair

$$2\alpha_s N_c \, | \int \frac{d^{D-2}k}{(2\pi)^{D-2}} \frac{\vec{k}}{\vec{k}^2} (e^{i\vec{k}\vec{r}_{1\rho}} - e^{i\vec{k}\vec{r}_{2\rho}}) \, |^{\,2}$$

$$= \frac{\alpha_s N_c}{2\pi^2} \left( \frac{\Gamma(1+\epsilon)}{\pi^\epsilon} \right)^2 \left( \frac{\vec{r}_{1\rho}}{\vec{r}_{1\rho}^{\,2(1+\epsilon)}} - \frac{\vec{r}_{2\rho}}{\vec{r}_{2\rho}^{\,2(1+\epsilon)}} \right)^2 ,$$

we have

$$\langle \vec{r}_1 \vec{r}_2 | \widehat{\mathcal{K}}_{dip} | \vec{r}_1{}' \vec{r}_2{}' \rangle = \frac{g^2 N_c \Gamma^2(1+\epsilon)}{8\pi^{3+2\epsilon}} \int d^{2+2\epsilon}\rho \left( \frac{\vec{r}_{1\rho}}{\vec{r}_{1\rho}^{\,2(1+\epsilon)}} - \frac{\vec{r}_{2\rho}}{\vec{r}_{2\rho}^{\,2(1+\epsilon)}} \right)^2$$

$$\left( \delta(\vec{r}_{11'})\delta(\vec{r}_{2\rho}) + \delta(\vec{r}_{22'})\delta(\vec{r}_{1\rho}) - \delta(\vec{r}_{11'})\delta(\vec{r}_{22'}) \right) .$$

## The dipole (Möbius) representation of the BFKL kernel in the LO

The direct Fourier transform of the BFKL kernel gives

$$\langle \vec{r}_1 \vec{r}_2 | \widehat{\mathcal{K}} | \vec{r}_1{}' \vec{r}_2{}' \rangle = \langle \vec{r}_1 \vec{r}_2 | \widehat{\mathcal{K}}_{dip} | \vec{r}_1{}' \vec{r}_2{}' \rangle - \frac{g^2 N_c \Gamma^2(1+\epsilon)}{8\pi^{3+2\epsilon}} \left[ \frac{\delta(\vec{r}_{11'})}{\vec{r}_{12'}^{\,2(1+2\epsilon)}} + \frac{\delta(\vec{r}_{22'})}{\vec{r}_{21'}^{\,2(1+2\epsilon)}} - 2 \frac{\delta(\vec{r}_{1'2'}) \vec{r}_{11'} \vec{r}_{22'}}{\vec{r}_{11'}^{\,2(1+\epsilon)} \vec{r}_{22'}^{\,2(1+\epsilon)}} \right].$$

The BFKL kernel is not equivalent to the dipole one. Actually the first one is more general than the second. This is clear, because the BFKL kernel can be applied not only in the case of scattering of colourless objects.

However, when applied to the latter case, we can use the “dipole” and “gauge invariance” properties of targets and projectiles
L. N. Lipatov, 1989,
and omit the terms in the kernel proportional to $\delta(\vec{r}_{1'2'})$, as well as change the terms independent either of $\vec{r}_1$ or of $\vec{r}_2$ in such a way that the resulting kernel becomes conserving the “dipole” property, i.e. the property which provides the vanishing of cross-sections for scattering of zero-size dipoles. The coordinate representation of the kernel obtained in such a way is what we call the dipole form of the BFKL kernel.

## Uncertainty in the NLO BFKL kernel

For colourless objects the impact factors in the scattering amplitudes

$$\delta(\vec{q}_A - \vec{q}_B)\mathcal{A}_{AB}^{A'B'} = \frac{i}{8(2\pi)^{D-2}}\langle A'\bar{A}|e^{Y\widehat{\mathcal{K}}}\frac{1}{\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}}|\bar{B}'B\rangle$$

are "gauge invariant": $\langle A'\bar{A}|\vec{q},0\rangle = \langle A'\bar{A}|0,\vec{q}\rangle = 0$ . Therefore $\langle A'\bar{A}|\Psi\rangle = 0$ if $\langle \vec{r}_1,\vec{r}_2|\Psi\rangle$ does not depend either on $\vec{r}_1$ or on $\vec{r}_2$. $\langle A'\bar{A}|\widehat{\mathcal{K}}$ is "gauge invariant" as well, because $\langle \vec{q}_1,\vec{q}_2|\widehat{\mathcal{K}}_r|\vec{q}_1^{\,\prime},\vec{q}_2^{\,\prime}\rangle$ vanishes at $\vec{q}_1^{\,\prime}=0$ or $\vec{q}_2^{\,\prime}=0$. It means that we can change $|In\rangle \equiv \left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{-1}|\bar{B}'B\rangle$ for $|In_d\rangle$, where $|In_d\rangle$ has the "dipole" property $\langle\vec{r},\vec{r}|In_d\rangle = 0$. After this one can omit the terms in the kernel proportional to $\delta(\vec{r}_{1'2'})$, as well as change the terms independent either of $\vec{r}_1$ or of $\vec{r}_2$ in such a way that the resulting kernel becomes conserving the "dipole" property.

## Uncertainty in the NLO BFKL kernel

The scattering amplitudes

$$\delta(\vec{q}_A - \vec{q}_B)\mathcal{A}_{AB}^{A'B'} = \frac{i}{8(2\pi)^{D-2}} \langle A'\bar{A}| e^{Y\widehat{\mathcal{K}}} \frac{1}{\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}} |\bar{B}'B\rangle$$

are invariant under the transformation

$$\widehat{\mathcal{K}} \to \widehat{\mathcal{O}}^{-1}\widehat{\mathcal{K}}\widehat{\mathcal{O}}, \ \langle A'\bar{A}| \to \langle A'\bar{A}|\widehat{\mathcal{O}}, \ \frac{1}{\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}}|\bar{B}'B\rangle \to \widehat{\mathcal{O}}^{-1}\frac{1}{\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}}|\bar{B}'B\rangle \ .$$

After fixation of the LO kernel transformations with $\hat{\mathcal{O}} = 1 - \hat{O}$, where $\hat{O} \sim g^2$, are still possible. At the NLO we get

$$\hat{\mathcal{K}} \to \hat{\mathcal{K}} - [\hat{\mathcal{K}}^{(B)}, \hat{O}] \ ,$$

where $\hat{\mathcal{K}}^{(B)}$ is the leading order kernel.
We will use

$$\hat{O} = -\frac{\alpha_s(\mu)}{8\pi}(\frac{11}{3}N_c - \frac{2}{3}n_f)\ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)$$

## The form of the kernel in the dipole representation

In the NLO the dipole form can be written as

$$\langle \vec{r}_1 \vec{r}_2 | \hat{\mathcal{K}}_d^{NLO} | \vec{r}_1{}' \vec{r}_2{}' \rangle = \frac{\alpha_s^2(\mu) N_c^2}{4\pi^3} \Bigg[ \delta(\vec{r}_{11'}) \delta(\vec{r}_{22'}) \int d\vec{\rho}\, g^0(\vec{r}_1, \vec{r}_2; \vec{\rho})$$

$$+ \delta(\vec{r}_{11'}) g(\vec{r}_1, \vec{r}_2; \vec{r}_2{}') + \delta(\vec{r}_{22'}) g(\vec{r}_2, \vec{r}_1; \vec{r}_1{}') + \frac{1}{\pi} g(\vec{r}_1, \vec{r}_2; \vec{r}_1{}', \vec{r}_2{}') \Bigg]$$

with the functions $g$ turning into zero when their first two arguments coincide.

The first three terms contain ultraviolet singularities which cancel in their sum, as well as in the LO, with account of the “dipole” property of the “target” impact factors. The coefficient of $\delta(\vec{r}_{11'})\delta(\vec{r}_{22'})$ is written in the integral form in order to make the cancellation evident.

## The “non-Abelian” part of the quark contribution

The “non-Abelian” (leading in $N_c$) part of the quark contribution is known at arbitrary $D$ (V.S. F., R. Fiore, A. Papa, 1999).
Its dipole form is found (V.S. F., R. Fiore, A. Papa, 2006) at arbitrary $D$ as well.
However, the transformation is rather complicated. In the physical space-time dimension $D = 4$ the dipole form can be obtained in a much easier way, starting from the renormalized BFKL kernel at $D = 4$ in a specific form.

$$\omega^Q(\vec{q}_i) = \int d^2k F_\omega^Q(\vec{k}, \vec{q}_i) \ ,$$

$$F_\omega^Q(\vec{k}, \vec{q}_i) = -\frac{\alpha_s^2(\mu)}{16\pi^2} \frac{2N_c n_f}{3} \frac{\vec{q}_i^{\,2}}{\vec{k}^{\,2}(\vec{q}_i - \vec{k})^2} \left( \ln \frac{\vec{k}^{\,2}(\vec{q}_i - \vec{k})^2}{\mu^2 \vec{q}_i^{\,2}} - \frac{5}{3} \right) ,$$

## The “non-Abelian” part of the quark contribution

$$\langle \vec q_1 \vec q_2 | \widehat{\mathcal{K}}_r^Q | \vec q_1^{\,'} \vec q_2^{\,'} \rangle = \delta(\vec q - \vec q^{\,'}) F_r^Q(\vec q_1, \vec q_1^{\,'}; \vec q) \; ,$$

$$F_r^Q(\vec q_1, \vec q_1^{\,'}; \vec q) = \frac{\alpha_s^2(\mu)}{16\pi^2} \frac{4N_c n_f}{3} \left( \frac{\vec q_2^{\,2}}{\vec q_2^{\,2} \vec k^{\,2}} \left[ \frac{\vec q_2^{\,\prime 2}}{\vec q_2^{\,2} \vec k^{\,2}} \left( \ln \frac{\vec k^{\,2} \vec q_2^{\,2}}{\mu^2 \vec q_2^{\,2}} - \frac{5}{3} \right) \right.\right.$$

$$\left.\left. + \frac{\vec q_1^{\,\prime 2}}{\vec q_1^{\,2} \vec k^{\,2}} \left( \ln \frac{\vec k^{\,2} \vec q_1^{\,2}}{\mu^2 \vec q_1^{\,\prime 2}} - \frac{5}{3} \right) - \frac{\vec q^{\,2}}{\vec q_1^{\,2} \vec q_2^{\,2}} \left( \ln \frac{\vec q_1^{\,2} \vec q_2^{\,2}}{\vec q^{\,2} \mu^2} - \frac{5}{3} \right) \right] \right) \, .$$

The singularities must be regularized by limitations on integration regions $\vec k^{\,2} \geq \lambda^2$ and $(\vec q_i - \vec k)^2 \geq \lambda^2$ with $\lambda$ tending to zero or in an equivalent way.

The contribution to the dipole form

$$g_Q(\vec r_1, \vec r_2; \vec\rho) = -g_Q^0(\vec r_1, \vec r_2; \vec\rho) = \frac{n_f}{3N_c} \left( \frac{\vec r_{12}^{\,2}}{\vec r_{1\rho}^{\,2} \vec r_{2\rho}^{\,2}} \ln \frac{\vec r_\mu^{\,2}}{\vec r_{12}^{\,2}} + \frac{\vec r_{1\rho}^{\,2} - \vec r_{2\rho}^{\,2}}{\vec r_{1\rho}^{\,2} \vec r_{2\rho}^{\,2}} \ln \frac{\vec r_{1\rho}^{\,2}}{\vec r_{2\rho}^{\,2}} \right) \; ,$$

$\ln \vec r_\mu^{\,2} = -\frac{5}{3} + 2\psi(1) - \ln \frac{\mu^2}{4}; \quad g_{QNA}(\vec r_1, \vec r_2; \vec r_1^{\,\prime}, \vec r_2^{\,\prime}) = 0.$

The result agrees with J. Balitsky, 2006.

## The “Abelian” part of the quark contribution

In the momentum representation the “Abelian” contribution was calculated many years ago in the framework of QED.

H. Cheng, T.T. Wu, Phys. Rev. D10 (1970) 2775

V.N. Gribov, L.N. Lipatov, G.V. Frolov, Yad. Fiz 12 (1970) 994

It is given by the “box” and “cross-box” diagrams and is suppressed by the factor $1/N_c^2$.

It does not contain neither ultraviolet nor infrared divergencies, so that from the beginning it can be taken at $D = 4$.

In the momentum representation this contribution is the most complicated one.

It this representation it is obtained only in the form of the two-dimensional integral over Feynman parameters.

It is better to start with this contribution before integration over momenta of produced quarks.

## The “Abelian” part of the quark contribution

$$\langle \vec{q}_1, \vec{q}_2 | \widehat{\mathcal{K}}^a | \vec{q}_1^{\,\prime}, \vec{q}_2^{\,\prime} \rangle = \delta(\vec{q} - \vec{q}^{\,\prime}) \frac{\alpha_s^2 n_f}{(2\pi)^2 N_c} \frac{-2}{\vec{q}_1^{\,2} \vec{q}_2^{\,2}} \int_0^1 dx \int \frac{d^2 k_1}{(2\pi)^2} F(\vec{q}_1, \vec{q}_2; \vec{k}_1, \vec{k}_2)$$

$$F(\vec{q}_1, \vec{q}_2; \vec{k}_1, \vec{k}_2) = x(1-x) \left( \frac{2(\vec{q}_1 \vec{k}_1) - \vec{q}_1^{\,2}}{\sigma_{11}} + \frac{2(\vec{q}_1 \vec{k}_2) - \vec{q}_1^{\,2}}{\sigma_{21}} \right) \left( \frac{2(\vec{q}_2 \vec{k}_1) + \vec{q}_2^{\,2}}{\sigma_{12}} + \frac{2(\vec{q}_2 \vec{k}_2) + \vec{q}_2^{\,2}}{\sigma_{22}} \right)$$

$$+ \frac{x \vec{q}^{\,2} \big(2(\vec{q}_1 \vec{k}_1) - \vec{q}_1^{\,2}\big)}{2\sigma_{11}} \left( \frac{1}{\sigma_{22}} - \frac{1}{\sigma_{12}} \right) + \frac{x \vec{q}^{\,2} \big(2(\vec{q}_2 \vec{k}_1) + \vec{q}_2^{\,2}\big)}{2\sigma_{12}} \left( \frac{1}{\sigma_{11}} - \frac{1}{\sigma_{21}} \right) +$$

$$+ \frac{1}{\sigma_{11}\sigma_{12}} \left( -2(\vec{q}_1 \vec{k}_1)(\vec{q}_2 \vec{q}_2^{\,\prime}) - 2(\vec{q}_2 \vec{k}_1)(\vec{q}_1 \vec{q}_1^{\,\prime}) + \big(\vec{q}_2^{\,2} - \vec{q}_1^{\,2}\big)(\vec{k}_1 \vec{k}) + \vec{q}_1^{\,2} \vec{q}_2^{\,\prime 2} - \frac{\vec{k}^{\,2} \vec{q}^{\,2}}{2} \right)$$

where $\vec{k}_1 + \vec{k}_2 = \vec{k} = \vec{q}_1 - \vec{q}_1^{\,\prime} = \vec{q}_2^{\,\prime} - \vec{q}_2$

$$\sigma_{11} = \big(\vec{k}_1 - x\vec{q}_1\big)^2 + x(1-x)\vec{q}_1^{\,2}\,, \; \sigma_{21} = \big(\vec{k}_2 - (1-x)\vec{q}_1\big)^2 + x(1-x)\vec{q}_1^{\,2}$$

$$\sigma_{12} = \big(\vec{k}_1 + x\vec{q}_2\big)^2 + x(1-x)\vec{q}_2^{\,2}\,, \; \sigma_{22} = \big(\vec{k}_2 + (1-x)\vec{q}_2\big)^2 + x(1-x)\vec{q}_2^{\,2}$$

## The “Abelian” part of the quark contribution

It contributes only to $g_(\vec{r}_1, \vec{r}_2; \vec{r}_1{}', \vec{r}_2{}')$:

$$g_Q(\vec{r}_1, \vec{r}_2; \vec{r}_1{}', \vec{r}_2{}') = \frac{n_f}{N_c^3} \frac{1}{\vec{r}^{\,4}_{1'2'}} \left[ \left( \frac{\vec{r}^{\,2}_{12'} \vec{r}^{\,2}_{1'2} + \vec{r}^{\,2}_{11'} \vec{r}^{\,2}_{22'} - \vec{r}^{\,2}_{12} \vec{r}^{\,2}_{1'2'}}{2\left(\vec{r}^{\,2}_{12'} \vec{r}^{\,2}_{1'2} - \vec{r}^{\,2}_{11'} \vec{r}^{\,2}_{22'}\right)} \ln \frac{\vec{r}^{\,2}_{12'} \vec{r}^{\,2}_{1'2}}{\vec{r}^{\,2}_{11'} \vec{r}^{\,2}_{22'}} - 1 \right) \right] .$$

It coincides with the corresponding part of the quark contribution to the dipole kernel I. Balitsky, 2006.

It turns out that the dipole form of the Abelian” part of the quark contribution is quite simple as compared with the very complicated form in the momentum representation.

Moreover, it is conformal invariant.

It could be especially interesting for the QED Pomeron.

However, one has to remember that in QED the use of the dipole form is limited to scattering of neutral objects, as well as that the conformal invariance is broken by masses.

## The gluon contribution

Evidently the main and the most important part of the BFKL kernel is given by thegluon contribution.

In the colour singlet channel it can written as

$$\hat{\mathcal{K}} = \hat{\mathcal{K}}_p + \hat{\mathcal{K}}_s \ ,$$

where the “planar” part

$$\hat{\mathcal{K}}_p = \hat{\omega}_1 + \hat{\omega}_2 + 2\hat{\mathcal{K}}_r^{(8)} + \frac{11\alpha_s(\mu)N_c}{24\pi}\left[\hat{\mathcal{K}}^{(B)}, \ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)\right],$$

$\hat{\mathcal{K}}_r^{(8)}$ is the real part of the colour octet kernel;

the “symmetric” part $\langle\vec{q}_1, \vec{q}_2|\hat{\mathcal{K}}_s|\vec{q}_1{}', \vec{q}_2{}'\rangle$ is finite in the limit $\epsilon = 0$. Moreover, it does not give terms divergent in $\epsilon = 0$ by action of the kernel, since it has no non-integrable singularities in the limit $\epsilon = 0$.

## The gluon contribution

Omitting terms with $\delta(\vec{r}_{1'2'})$, we reduce the NLO piece of the "planar" part to the form:

$$\langle\vec{q}_1,\vec{q}_2|\hat{\mathcal{K}}_p^{NLO}|\vec{q}_1^{\,\prime},\vec{q}_2^{\,\prime}\rangle \to \frac{\alpha_s^2(\mu)N_c^2}{4\pi^3}\Bigg[-\delta(\vec{q}_{11'})\delta(\vec{q}_{22'})\left(\int d\vec{k}\left(V(\vec{k})+V(\vec{k},\vec{k}-\vec{q}_1)\right)-3\pi\zeta(3)\right)$$

$$+\delta(\vec{q}-\vec{q}^{\,\prime})\Bigg\{V(\vec{k})+2V(\vec{k},\vec{q}_1)+\frac{(\vec{q}_1\vec{q}_2)}{4\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\left[\ln\left(\frac{\vec{q}_1^{\,\prime\,2}}{\vec{q}^{\,2}}\right)\ln\left(\frac{\vec{q}_2^{\,\prime\,2}}{\vec{q}^{\,2}}\right)+\ln^2\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_1^{\,\prime\,2}}\right)\right]$$

$$-\frac{1}{2\vec{k}^2}\ln^2\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_1^{\,\prime\,2}}\right)+\left[\frac{(\vec{q}_1\vec{k})^{\,2}}{\vec{q}_1^{\,2}\vec{k}^{\,2}}-1-\frac{(\vec{q}_1+\vec{k})\vec{q}_2}{\vec{q}_2^{\,2}}+\left(\frac{(\vec{k}\vec{q}_2)}{\vec{k}^{\,2}}+\frac{(\vec{q}_1\vec{q}_2)}{\vec{q}_1^{\,2}}\right)\frac{(\vec{q}_1\vec{k})}{\vec{q}_2^{\,2}}\right]I(\vec{q}_1^{\,2},\vec{q}_1^{\,\prime\,2},\vec{k}^{\,2})$$

$$+\frac{(\vec{k}\vec{q}_1)}{2\vec{k}^{\,2}\vec{q}_1^{\,2}}\left[\ln^2\left(\frac{\vec{q}_2^{\,2}}{\vec{q}_2^{\,\prime\,2}}\right)+\frac{1}{2}\ln\left(\frac{\vec{q}_2^{\,2}}{\vec{q}_2^{\,\prime\,2}}\right)\ln\left(\frac{\vec{q}_2^{\,2}\vec{q}_2^{\,\prime\,2}}{\vec{k}^{\,4}}\right)+\ln^2\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_1^{\,\prime\,2}}\right)-\frac{1}{2}\ln\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_1^{\,\prime\,2}}\right)\ln\left(\frac{\vec{q}_1^{\,2}\vec{q}_1^{\,\prime\,2}}{\vec{k}^{\,4}}\right)\right]$$

$$+\frac{1}{4\vec{q}_1^{\,2}}\left[\ln\left(\frac{\vec{q}_1^{\,\prime\,2}}{\vec{k}^{\,2}}\right)\ln\left(\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,\prime\,2}}{\vec{q}_2^{\,2}\vec{q}_1^{\,\prime\,2}}\right)+\ln\left(\frac{\vec{q}_2^{\,2}}{\vec{q}^{\,2}}\right)\ln\left(\frac{\vec{q}_1^{\,\prime\,2}\vec{q}_2^{\,\prime\,2}}{\vec{q}^{\,4}}\right)\right]\Bigg\}+(\vec{q}_1\leftrightarrow\vec{q}_2,\ \ \vec{q}_1^{\,\prime}\leftrightarrow\vec{q}_2^{\,\prime})\Bigg],$$

## The gluon contribution

$$V(\vec{k}) = \frac{1}{2\vec{k}^{\,2}}\left(\frac{67}{9} - 2\zeta(2) - \frac{11}{3}\ln\left(\frac{\vec{k}^{\,2}}{\mu^2}\right)\right) \ ,$$

$$V(\vec{k},\vec{q}) = \frac{\vec{k}\vec{q}}{2\vec{k}^{\,2}\vec{q}^{\,2}}\left(\frac{11}{3}\ln\left(\frac{\vec{k}^{\,2}\vec{q}^{\,2}}{\mu^2(\vec{k}-\vec{q})^2}\right) - \frac{67}{9} + 2\zeta(2)\right) - \frac{11}{12\vec{k}^{\,2}}\ln\left(\frac{\vec{q}^{\,2}}{(\vec{k}-\vec{q})^2}\right)$$

$$-\frac{11}{12\vec{q}^{\,2}}\ln\left(\frac{\vec{k}^{\,2}}{(\vec{k}-\vec{q})^2}\right) \ ,$$

$$I(\vec{q}_1^{\,2},\vec{q}_1^{\,\prime\,2},\vec{k}^{\,2}) = \int_0^1 \frac{dx}{\vec{q}_1^{\,2}(1-x)+\vec{q}_1^{\,\prime\,2}x - \vec{k}^{\,2}x(1-x)}\ln\left(\frac{\vec{q}_1^{\,2}(1-x)+\vec{q}_1^{\,\prime\,2}x}{\vec{k}^{\,2}x(1-x)}\right) \ .$$

The singularities must be regularized by limitations on integration regions $\vec{k}^{\,2} \geq \lambda^2$ and $(\vec{q}_i - \vec{k})^2 \geq \lambda^2$ with $\lambda$ tending to zero or in an equivalent way.

## The gluon contribution

The “symmetric ” part does not contain neither ultraviolet nor infrared divergencies, so that from the beginning it can be taken at $D = 4$.

In the momentum representation this contribution is the most complicated one.

It this representation it is obtained only in the form of the two-dimensional integral over Feynman parameters.

It is better to start with this contribution before integration over momenta of produced quarks.

For the total gluon contributions we obtain

$$g^0(\vec{r}_1, \vec{r}_2; \rho) = \frac{3}{2} \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{1\rho}^{\,2} \vec{r}_{2\rho}^{\,2}} \ln\left(\frac{\vec{r}_{1\rho}^{\,2}}{\vec{r}_{12}^{\,2}}\right) \ln\left(\frac{\vec{r}_{2\rho}^{\,2}}{\vec{r}_{12}^{\,2}}\right) - \frac{11}{12} \left[ \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{1\rho}^{\,2} \vec{r}_{2\rho}^{\,2}} \ln\left(\frac{\vec{r}_{1\rho}^{\,2} \vec{r}_{2\rho}^{\,2}}{r_\mu^4}\right) \right.$$

$$\left. + \left(\frac{1}{\vec{r}_{2\rho}^{\,2}} - \frac{1}{\vec{r}_{1\rho}^{\,2}}\right) \ln\left(\frac{\vec{r}_{2\rho}^{\,2}}{\vec{r}_{1\rho}^{\,2}}\right) \right] \ ,$$

## The gluon contribution

$$g(\vec{r}_1, \vec{r}_2; \vec{r}_2{}') = \frac{11}{6} \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{22'}^{\,2} \vec{r}_{12'}^{\,2}} \ln\left(\frac{\vec{r}_{12}^{\,2}}{r_\mu^2}\right) + \frac{11}{6}\left(\frac{1}{\vec{r}_{22'}^{\,2}} - \frac{1}{\vec{r}_{12'}^{\,2}}\right) \ln\left(\frac{\vec{r}_{22'}^{\,2}}{\vec{r}_{12'}^{\,2}}\right)$$

$$+ \frac{1}{2\vec{r}_{22'}^{\,2}} \ln\left(\frac{\vec{r}_{12'}^{\,2}}{\vec{r}_{22'}^{\,2}}\right) \ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{12'}^{\,2}}\right) - \frac{\vec{r}_{12}^{\,2}}{2\,\vec{r}_{22'}^{\,2} \vec{r}_{12'}^{\,2}} \ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{22'}^{\,2}}\right) \ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{12'}^{\,2}}\right),$$

$$\ln r_\mu^2 = 2\psi(1) - \ln\frac{\mu^2}{4} - \frac{3}{11}\left(\frac{67}{9} - 2\zeta(2)\right).$$

Both $g^0(\vec{r}_1, \vec{r}_2; \vec{\rho})$ and $g(\vec{r}_1, \vec{r}_2; \vec{\rho})$ vanish at $\vec{r}_1 = \vec{r}_2$. Then, these functions turn into zero for $\vec{\rho}^{\,2} \to \infty$ faster than $(\vec{\rho}^{\,2})^{-1}$ to provide the infrared safety. The ultraviolet singularities of these functions at $\vec{\rho} = \vec{r}_2$ and $\vec{\rho} = \vec{r}_1$ cancel on account of the “dipole” property of the “target” impact factors.

The last term is the most complicated one:

## The gluon contribution

$$g(\vec{r}_1,\vec{r}_2;\vec{r}_1^{\,\prime},\vec{r}_2^{\,\prime}) = \left[\frac{(\vec{r}_{22'}\ \vec{r}_{12})}{\vec{r}_{11'}^{\,2}\vec{r}_{22'}^{\,2}\vec{r}_{1'2'}^{\,2}} - \frac{2\,(\vec{r}_{22'}\ \vec{r}_{11'})}{\vec{r}_{11'}^{\,2}\vec{r}_{22'}^{\,2}\vec{r}_{1'2'}^{\,2}} + \frac{2\,(\vec{r}_{22'}\ \vec{r}_{12'})\,(\vec{r}_{11'}\ \vec{r}_{12'})}{\vec{r}_{11'}^{\,2}\vec{r}_{22'}^{\,2}\vec{r}_{1'2'}^{\,2}\vec{r}_{12'}^{\,2}}\right]\ln\left(\frac{\vec{r}_{12'}^{\,2}}{\vec{r}_{1'2'}^{\,2}}\right)$$

$$+\frac{1}{2\vec{r}_{1'2'}^{\,2}}\left[\frac{(\vec{r}_{11'}\,\vec{r}_{22'})}{\vec{r}_{11'}^{\,2}\,\vec{r}_{22'}^{\,2}} + \frac{(\vec{r}_{21'}\,\vec{r}_{12'})}{\vec{r}_{21'}^{\,2}\,\vec{r}_{12'}^{\,2}} - \frac{2(\vec{r}_{22'}\ \vec{r}_{21'})}{\vec{r}_{22'}^{\,2}\vec{r}_{21'}^{\,2}}\right]\ln\left(\frac{\vec{r}_{11'}^{\,2}\,\vec{r}_{22'}^{\,2}}{\vec{r}_{1'2'}^{\,2}\vec{r}_{12}^{\,2}}\right) + \frac{(\vec{r}_{11'}\,\vec{r}_{22'})}{2\vec{r}_{11'}^{\,2}\,\vec{r}_{22'}^{\,2}\vec{r}_{1'2'}^{\,2}}\ln\left(\frac{\vec{r}_{21'}^{\,2}\,\vec{r}_{12'}^{\,2}}{\vec{r}_{11'}^{\,2}\,\vec{r}_{22'}^{\,2}}\right)$$

$$+\frac{1}{d\,\vec{r}_{1'2'}^{\,2}}\left[\frac{(\vec{r}_{1'2'}\,\vec{r}_{12'})\vec{r}_{12}^{\,2}}{\vec{r}_{11'}^{\,2}} + \frac{2(\vec{r}_{22'}\,\vec{r}_{21'})(\vec{r}_{12}\,\vec{r}_{21'})}{\vec{r}_{21'}^{\,2}} + \frac{(\vec{r}_{22'}\,\vec{r}_{12'})(\vec{r}_{11'}\,\vec{r}_{21'})}{\vec{r}_{11'}^{\,2}\,\vec{r}_{22'}^{\,2}}\vec{r}_{1'2'}^{\,2} - \vec{r}_{1'2'}^{\,2}\right]\ln\left(\frac{\vec{r}_{12'}^{\,2}\,\vec{r}_{21'}^{\,2}}{\vec{r}_{11'}^{\,2}\,\vec{r}_{22'}^{\,2}}\right)$$

$$+\frac{1}{2\vec{r}_{1'2'}^{\,4}}\left(\frac{\vec{r}_{11'}^{\,2}\,\vec{r}_{22'}^{\,2}}{d}\ln\left(\frac{\vec{r}_{12'}^{\,2}\,\vec{r}_{21'}^{\,2}}{\vec{r}_{11'}^{\,2}\vec{r}_{22'}^{\,2}}\right) - 1\right) + \frac{1}{\vec{r}_{11'}^{\,2}}\left(\frac{(\vec{r}_{12}\,\vec{r}_{21'})}{\vec{r}_{1'2'}^{\,2}\vec{r}_{21'}^{\,2}} - \frac{(\vec{r}_{11'}\,\vec{r}_{12})}{\vec{r}_{1'2'}^{\,2}\vec{r}_{22'}^{\,2}} - \frac{(\vec{r}_{11'}\,\vec{r}_{21'})}{\vec{r}_{22'}^{\,2}\vec{r}_{21'}^{\,2}}\right)\ln\left(\frac{\vec{r}_{12'}^{\,2}}{\vec{r}_{11'}^{\,2}}\right)$$

$$-\frac{(\vec{r}_{12}\ \vec{r}_{22'})}{\vec{r}_{1'2'}^{\,2}\vec{r}_{22'}^{\,2}\vec{r}_{12'}^{\,2}}\ln\left(\frac{\vec{r}_{11'}^{\,2}}{\vec{r}_{1'2'}^{\,2}}\right) + (1\leftrightarrow 2),$$

$$d = \vec{r}_{12'}^{\,2}\vec{r}_{21'}^{\,2} - \vec{r}_{11'}^{\,2}\vec{r}_{22'}^{\,2}.$$

This term also vanishes at $\vec{r}_1 = \vec{r}_2$, so that it possesses the "dipole" property. It has ultraviolet singularity only at $\vec{r}_{1'2'} = 0$ and tends to zero at large $\vec{r}_1^{\,\prime\,2}$ and $\vec{r}_2^{\,\prime\,2}$ suffi ciently quickly in order to provide the infrared safety.

# BFKL in SUSY

SUSY extensions of QCD contain gluons and Maiorana fermions in the adjoint representation of the colour group. The gluon contribution does not change. The fermion one can be obtained by change of the group coefficients:

$$n_f \longrightarrow n_M N_c$$

for the "non-Abelian" part, and

$$n_f \longrightarrow -n_M N_c^3$$

for the "Abelian" part; $n_M$ is the number of flavours of Maiorana quarks. For $N$–extended SUSY $n_M = N$.

At $N > 1$ besides quarks there are $n_s$ scalar particles; $n_s = 2$ at $N = 2$ and $n_s = 6$ at $N = 4$.

The contribution of the scalar particles to the BFKL kernel can be easily calculated.

## BFKL in SUSY

Analogously to the quark case it is convenient to divide it into two parts, with the same colour group coefficients. As well as in the quark case, after the transformation

$$\hat{\mathcal{K}} \rightarrow \hat{\mathcal{K}} - [\hat{\mathcal{K}}^{(B)}, \hat{O}] \; ,$$

$$\hat{O} = -\frac{\alpha_s(\mu)N_c}{8\pi}(\frac{11}{3} - \frac{2}{3}n_M - \frac{1}{6}n_M)\ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)$$

The "non-Abelian" part contributes only to $g_Q(\vec{r}_1, \vec{r}_2; \vec{\rho})$ and $g_Q^0(\vec{r}_1, \vec{r}_2; \vec{\rho})$. At that

$$g_Q(\vec{r}_1, \vec{r}_2; \vec{\rho}) = -g_Q^0(\vec{r}_1, \vec{r}_2; \vec{\rho}) = \frac{n_s}{12}\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{1\rho}^{\,2}\vec{r}_{2\rho}^{\,2}}\ln\frac{\vec{r}_{\mu}^{\,2}}{\vec{r}_{12}^{\,2}} + \frac{\vec{r}_{1\rho}^{\,2} - \vec{r}_{2\rho}^{\,2}}{\vec{r}_{1\rho}^{\,2}\vec{r}_{2\rho}^{\,2}}\ln\frac{\vec{r}_{1\rho}^{\,2}}{\vec{r}_{2\rho}^{\,2}}\right) \; ,$$

where $\ln r_{\mu}^{\,2} = -\frac{8}{3} + 2\psi(1) - \ln\frac{\mu^2}{4}$.

## BFKL in SUSY

The "Abelian" part contributes only to $g_(\vec{r}_1, \vec{r}_2; \vec{r}_1{}', \vec{r}_2{}')$:

$$g_Q(\vec{r}_1, \vec{r}_2; \vec{r}_1{}', \vec{r}_2{}') = \frac{n_s}{2} \frac{1}{\vec{r}^4_{1'2'}} \left[ \left( \frac{\vec{r}^2_{12'} \vec{r}^2_{1'2}}{\left( \vec{r}^2_{12'} \vec{r}^2_{1'2} - \vec{r}^2_{11'} \vec{r}^2_{22'} \right)} \ln \frac{\vec{r}^2_{12'} \vec{r}^2_{1'2}}{\vec{r}^2_{11'} \vec{r}^2_{22'}} - 1 \right) \right]$$

It is evidently conformal invariant.

## Summary

- The coordinate representation of the colour singlet BFKL kernel permits to understand relation between the BFKL and the colour dipole approaches and conformal properties of the BFKL kernel.
- The colour singlet BFKL kernel is more general than the dipole one.
- In the case of scattering of colourless objects the BFKL kernel can be written in the dipole form (Möbius representation).
- The dipole form is greatly simplified in comparison with the BFKL kernel in the momentum representation.
- The quark contribution to the dipole form agrees with corresponding contribution to the BK kernel.
- The “Abelian” part of the quark contribution is conformal invariant. The same is the scalar particle contribution in the SUSY QCD extensions.