Gluon saturation from DIS to AA collisions

II – DIS and proton-nucleus collisions

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General outline

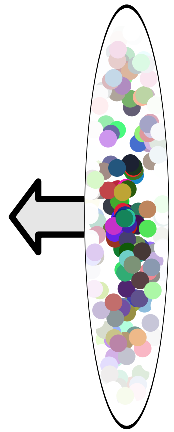
- Lecture I : Gluon saturation, Color Glass Condensate
- Lecture II : DIS and proton-nucleus collisions
- Lecture III : Saturation in nucleus-nucleus collisions

Lecture II : DIS and pA collisions

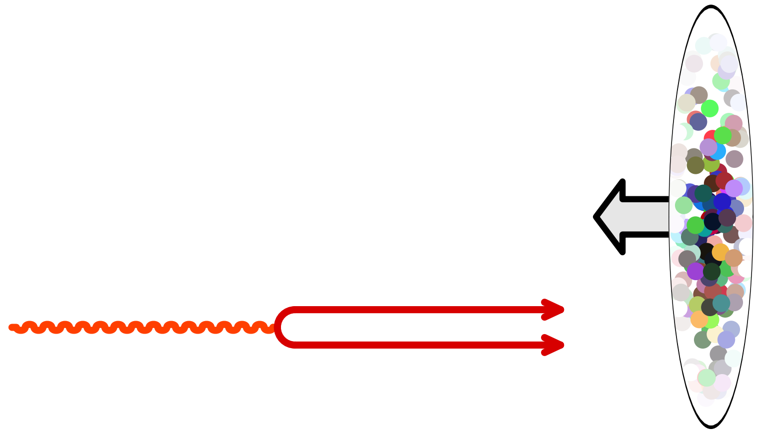
- Eikonal scattering
- Energy dependence
- Geometrical scaling
- Fits of DIS data
- Proton-Nucleus collisions

Introduction

Probing saturation in ideal conditions

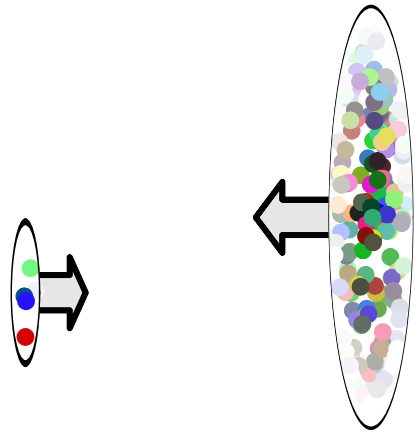


Probing saturation in ideal conditions



- Ideally, one would like to collide the saturated nucleon/nucleus with a well known simple probe that does not involve QCD at all ▷ Deep Inelastic Scattering

Probing saturation in ideal conditions



- Ideally, one would like to collide the saturated nucleon/nucleus with a well known simple probe that does not involve QCD at all ▷ Deep Inelastic Scattering
- The next best thing is to probe a saturated hadron with another hadron which is not saturated
  - preferably, the probe should be a nucleon – not a nucleus – whose relevant parton content is not at small x

Light-cone coordinates

- Light-cone coordinates are defined by choosing a privileged axis (generally the z axis) along which particles have a large momentum. Then, for any 4-vector a^μ, one defines :

$$a^+ \equiv \frac{a^0 + a^3}{\sqrt{2}} \quad , \quad a^- \equiv \frac{a^0 - a^3}{\sqrt{2}}$$

$$a^{1,2} \quad \text{unchanged.} \qquad \text{Notation :} \quad \vec{\boldsymbol{a}}_\perp \equiv (a^1, a^2)$$

- Under a Lorentz boost in the z direction :

$$a^+ \to \Lambda\, a^+ \quad , \quad a^- \to \Lambda^{-1}\, a^- \quad , \quad a^{1,2} \to a^{1,2}$$

- Some useful formulas :

$$x \cdot y = x^+ y^- + x^- y^+ - \vec{\boldsymbol{x}}_\perp \cdot \vec{\boldsymbol{y}}_\perp$$

$$d^4x = dx^+ dx^- d^2\vec{\boldsymbol{x}}_\perp$$

$$\Box = 2\partial^+\partial^- - \vec{\boldsymbol{\nabla}}_\perp^2 \qquad \text{Notation :} \quad \partial^+ \equiv \frac{\partial}{\partial x^-} \; , \; \partial^- \equiv \frac{\partial}{\partial x^+}$$

Parton-nucleus cross-section

- The study of all the reactions of this type can be reduced to that of individual partons with the saturated target :

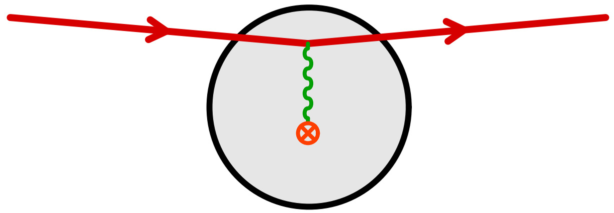
$$d\sigma = \underbrace{d\Phi_1 \cdots d\Phi_n}_{\text{invariant phase-space for the final state}} \frac{1}{2p^+} 2\pi\delta(p^+ - \sum_i q_i^+) \;\; |\mathcal{M}|^2$$

  - Invariant phase-space : $d\Phi \equiv \dfrac{d^3\vec{q}}{(2\pi)^3 2\omega_{\boldsymbol{q}}}$
  - $\mathcal{M} \equiv$ transition amplitude $\big\langle \vec{\boldsymbol{q}}_1 \cdots \vec{\boldsymbol{q}}_{n\,\mathrm{out}} \big| \vec{\boldsymbol{p}}_{\mathrm{in}} \big\rangle$ in the presence of the color field of the target
  - The delta function comes from the fact that a highly boosted target field (in the $-z$ direction) has no k^+ components

Eikonal scattering

Goal

- Consider the scattering amplitude off an external potential :



$$S_{\beta\alpha} \equiv \big\langle \beta_{\rm out} \big| \alpha_{\rm in} \big\rangle = \big\langle \beta_{\rm in} \big| U(+\infty, -\infty) \big| \alpha_{\rm in} \big\rangle$$

where $U(+\infty, -\infty)$ is the evolution operator from $t = -\infty$ to $t = +\infty$

$$U(+\infty, -\infty) = T \, \exp \left[i \int d^4x \, \mathcal{L}_{\rm int}(\phi_{\rm in}(x)) \right]$$

Note : $\mathcal{L}_{\rm int}$ contains the self-interactions of the fields and their interactions with the external potential

- We want to calculate its high energy limit :

$$S^{(\infty)}_{\beta\alpha} \equiv \lim_{\omega \to +\infty} \big\langle \beta_{\rm in} \big| e^{i\omega K^3} U(+\infty, -\infty) e^{-i\omega K^3} \big| \alpha_{\rm in} \big\rangle$$

where K^3 is the generator of boosts in the $+z$ direction

Eikonal scattering in a nutshell

- In a scattering at high energy, the collision time goes to zero as $s^{-1/2}$
- With scalar interactions, this implies a decrease of the scattering amplitude as $s^{-1/2}$
- With vectorial interactions, this decrease is compensated by the growth of the component J^+ of the vector current

$\triangleright$ the eikonal approximation gives the finite limit of the scattering amplitude in the case of vectorial interactions when $s \to +\infty$

Eikonal limit

- Consider an external vector potential, that couples via $e\,\mathcal{A}_\mu(x) J^\mu(x)$ (J^μ is the current associated to some conserved charge)
- We will assume that the external potential is non-zero only in a finite range in x^+, $x^+ \in [-L, +L]$
- The action of K^3 on states and (scalar) fields is :

$$e^{-i\omega K^3} \big|\vec{\boldsymbol{p}} \cdots {}_{\rm in}\big\rangle = \big|(e^\omega p^+, \vec{\boldsymbol{p}}_\perp) \cdots {}_{\rm in}\big\rangle$$

$$e^{i\omega K^3} \phi_{\rm in}(x) e^{-i\omega K^3} = \phi_{\rm in}(e^{-\omega} x^+, e^\omega x^-, \vec{\boldsymbol{x}}_\perp)$$

- K^3 does not change the ordering in x^+. Hence,

$$e^{i\omega K^3} U(+\infty, -\infty) e^{-i\omega K^3} = T \exp i \int d^4x \; \mathcal{L}_{\rm int}(e^{i\omega K^3} \phi_{\rm in}(x) e^{-i\omega K^3})$$

where $\mathcal{L}_{\rm int} = \mathcal{L}_{\rm self}(\phi) + e\,\mathcal{A}_\mu J^\mu$

Eikonal limit

- Split the evolution operator $U(+\infty,-\infty)$ into three factors :

$$U(+\infty,-\infty)=U(+\infty,+L)U(+L,-L)U(-L,-\infty)$$

Upon application of K^3, this becomes :

$$e^{i\omega K^3}U(+\infty,-\infty)e^{-i\omega K^3}=e^{i\omega K^3}U(+\infty,+L)e^{-i\omega K^3}$$
$$\times e^{i\omega K^3}U(+L,-L)e^{-i\omega K^3}e^{i\omega K^3}U(-L,-\infty)e^{-i\omega K^3}$$

- The external potential $\mathcal{A}_\mu(x)$ is unaffected by K^3

- The components of $J^\mu(x)$ are changed as follows :

$$e^{i\omega K^3}J^i(x)e^{-i\omega K^3}=J^i(e^{-\omega}x^+,e^{\omega}x^-,\vec{\boldsymbol{x}}_\perp)$$
$$e^{i\omega K^3}J^-(x)e^{-i\omega K^3}=e^{-\omega}\;J^-(e^{-\omega}x^+,e^{\omega}x^-,\vec{\boldsymbol{x}}_\perp)$$
$$e^{i\omega K^3}J^+(x)e^{-i\omega K^3}=e^{\omega}\;J^+(e^{-\omega}x^+,e^{\omega}x^-,\vec{\boldsymbol{x}}_\perp)$$

Eikonal limit

- The factors $U(+\infty,+L)$ and $U(-L,-\infty)$ do not contain the external potential. In order to deal with these factors, it is sufficient to change variables : $e^{-\omega}x^+ \to x^+$, $e^{\omega}x^- \to x^-$. This leads to :

$$\lim_{\omega\to+\infty} e^{i\omega K^3} U(+\infty,+L) e^{-i\omega K^3} = U_{\rm self}(+\infty,0)$$

$$\lim_{\omega\to+\infty} e^{i\omega K^3} U(-L,-\infty) e^{-i\omega K^3} = U_{\rm self}(0,-\infty)$$

where $U_{\rm self}$ is the same as U, but with the self-interactions only

- For the factor $U(L,-L)$, the change $e^{\omega}x^- \to x^-$ leads to :

$$e^{i\omega K^3} U(+L,-L) e^{-i\omega K^3} = \\ = T\exp i\int_{-L}^{+L} d^4x\; e^{-\omega}\Big[e\,\mathcal{A}^-(x^+,e^{-\omega}x^-,\vec{\boldsymbol{x}}_\perp) \\ \times e^{\omega} J^+(e^{-\omega}x^+,x^-,\vec{\boldsymbol{x}}_\perp) + \mathcal{O}(1)\Big]$$

Eikonal limit

- Therefore, in the limit $\omega \to +\infty$, we have :

$$\lim_{\omega\to+\infty} e^{i\omega K^3} U(+L,-L) e^{-i\omega K^3} = \exp\left[ie \int d^2\vec{\boldsymbol{x}}_\perp \chi(\vec{\boldsymbol{x}}_\perp)\rho(\vec{\boldsymbol{x}}_\perp)\right]$$

with

$$\begin{cases} \chi(\vec{\boldsymbol{x}}_\perp) \equiv \int dx^+ \; \mathcal{A}^-(x^+,0,\vec{\boldsymbol{x}}_\perp) \\ \rho(\vec{\boldsymbol{x}}_\perp) \equiv \int dx^- \; J^+(0,x^-,\vec{\boldsymbol{x}}_\perp) \end{cases}$$

- The high-energy limit of the scattering amplitude is :

$$S^{(\infty)}_{\beta\alpha} = \big\langle \beta_{\rm in} \big| U_{\rm self}(+\infty,0) \; \exp\left[ie \int_{\vec{\boldsymbol{x}}_\perp} \chi(\vec{\boldsymbol{x}}_\perp)\rho(\vec{\boldsymbol{x}}_\perp)\right] U_{\rm self}(0,-\infty) \big| \alpha_{\rm in} \big\rangle$$

  - Only the $-$ component of the vector potential matters
  - The self-interactions and the interactions with the external potential are factorized $\;\triangleright$ parton model
  - This is an exact result when $s \to +\infty$

Eikonal limit

- The previous formula still contains all the self-interactions of the fields. In order to perform the perturbative expansion, it is convenient to write first :

$$S^{(\infty)}_{\beta\alpha} = \sum_{\gamma,\delta} \big\langle \beta_{\rm in} \big| U_{\rm self}(+\infty, 0) \big| \gamma_{\rm in} \big\rangle \times \big\langle \gamma_{\rm in} \big| \exp\Big[ie\int_{\vec{\boldsymbol{x}}_\perp} \chi(\vec{\boldsymbol{x}}_\perp)\rho(\vec{\boldsymbol{x}}_\perp)\Big] \big| \delta_{\rm in} \big\rangle \big\langle \delta_{\rm in} \big| U_{\rm self}(0, -\infty) \big| \alpha_{\rm in} \big\rangle$$

- The factor

$$\sum_\delta \big| \delta_{\rm in} \big\rangle \big\langle \delta_{\rm in} \big| U_{\rm self}(0, -\infty) \big| \alpha_{\rm in} \big\rangle$$

is the Fock expansion of the initial state: the state prepared at $x^+ = -\infty$ may have fluctuated into another state before it interacts with the external potential

Eikonal limit

- We need to calculate matrix elements such as $\left\langle\gamma_{\rm in}\middle|\boldsymbol{F}\middle|\delta_{\rm in}\right\rangle$, with :

$$\boldsymbol{F} \equiv \exp\, ie \int \chi_a(\vec{\boldsymbol{x}}_\perp)\rho^a(\vec{\boldsymbol{x}}_\perp)$$

  - having QCD in mind, we have reinstated the color indices
  - the contribution of quarks and antiquarks to $\rho^a(\vec{\boldsymbol{x}}_\perp)$ is :

$$\rho^a(\vec{\boldsymbol{x}}_\perp) = t^a_{ij} \int \frac{dp^+}{4\pi p^+}\frac{d^2\vec{\boldsymbol{p}}_\perp}{(2\pi)^2}\frac{d^2\vec{\boldsymbol{q}}_\perp}{(2\pi)^2}\Big\{b^\dagger_{\rm in}(p^+,\vec{\boldsymbol{p}}_\perp;i)b_{\rm in}(p^+,\vec{\boldsymbol{q}}_\perp;j)e^{i(\vec{\boldsymbol{p}}_\perp-\vec{\boldsymbol{q}}_\perp)\cdot\vec{\boldsymbol{x}}_\perp}$$
$$-d^\dagger_{\rm in}(p^+,\vec{\boldsymbol{p}}_\perp;i)d_{\rm in}(p^+,\vec{\boldsymbol{q}}_\perp;j)e^{-i(\vec{\boldsymbol{p}}_\perp-\vec{\boldsymbol{q}}_\perp)\cdot\vec{\boldsymbol{x}}_\perp}\Big\}$$

  - Note : one should keep the ordering of the exponential in x^+
  - the contribution of gluons is similar, with a color matrix in the adjoint representation

- The action of $\boldsymbol{F}$ on a state $\left|\delta_{\rm in}\right\rangle$ gives a state with the same particle content, the same $+$ components for the momenta, but modified transverse momenta and colors

Light-cone wavefunction

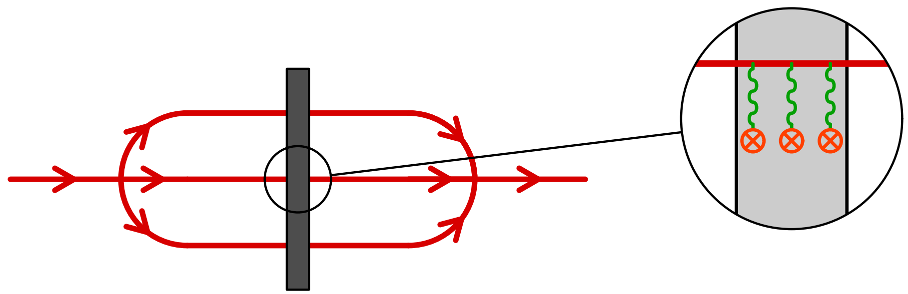
- For each intermediate state $\left\langle\delta_{\rm in}\right| \equiv \left\langle\{k_i^+, \vec{\boldsymbol{k}}_{i\perp}\}\right|$, define the corresponding light-cone wave function by :

$$\Psi_{\delta\alpha}(\{k_i^+, \vec{\boldsymbol{x}}_{i\perp}\}) \equiv \prod_i \int \frac{d^2\vec{\boldsymbol{k}}_{i\perp}}{(2\pi)^2} e^{-i\vec{\boldsymbol{k}}_{i\perp}\cdot\vec{\boldsymbol{x}}_{i\perp}} \left\langle\delta_{\rm in}\middle|U_{\rm self}(0,-\infty)\middle|\alpha_{\rm in}\right\rangle$$

- Each charged particle going through the external field acquires a phase proportional to its charge (antiparticles get an opposite phase) :

$$\Psi_{\delta\alpha}(\{k_i^+, \vec{\boldsymbol{x}}_{i\perp}\}) \longrightarrow \Psi_{\delta\alpha}(\{k_i^+, \vec{\boldsymbol{x}}_{i\perp}\}) \prod_i U_i(\vec{\boldsymbol{x}}_\perp)$$

$$U_i(\vec{\boldsymbol{x}}_\perp) \equiv T\exp\left[ig_i \int dx^+ \, \mathcal{A}_a^-(x^+, 0, \vec{\boldsymbol{x}}_\perp) t^a\right]$$



Light-cone wavefunction

- We have seen that the number and the nature of the particles is unchanged under the action of the operator $\boldsymbol{F}$. Moreover, in terms of the transverse coordinates, we simply have

$$\left\langle \gamma_{\rm in} \middle| \boldsymbol{F} \middle| \delta_{\rm in} \right\rangle = \delta_{NN'} \prod_i \left[4\pi k_i^+ \delta(k_i^+ - k_i^{+\prime}) \delta(\vec{\boldsymbol{x}}_{i\perp} - \vec{\boldsymbol{x}}'_{i\perp}) U_{R_i}(\vec{\boldsymbol{x}}_{i\perp}) \right]$$

  where $U_R(\vec{\boldsymbol{x}}_\perp)$ is a Wilson line operator, in the representation R appropriate for the particle going through the target

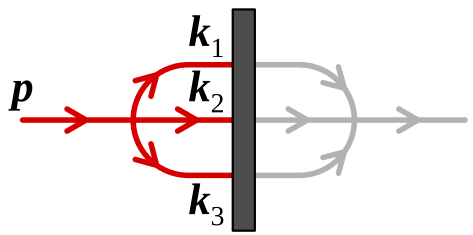
- Therefore, the high energy scattering amplitude can be written as :

$$S_{\beta\alpha}^{(\infty)} = \sum_\delta \int \left[\prod_{i\in\delta} d\Phi_i \right] \Psi^\dagger_{\delta\beta}(\{k_i^+, \vec{\boldsymbol{x}}_{i\perp}\}) \left[\prod_{i\in\delta} U_{R_i}(\vec{\boldsymbol{x}}_{i\perp}) \right] \Psi_{\delta\alpha}(\{k_i^+, \vec{\boldsymbol{x}}_{i\perp}\})$$

- As we shall see shortly, some loop corrections are enhanced by logs of the energy. They must be resummed and drive the energy evolution of the amplitude

Light-cone wave function

- The calculation of $\langle \delta_{\rm in} | U_{\rm self}(0,-\infty) | \alpha_{\rm in} \rangle$ is similar to that of scattering amplitudes in the vacuum. The only difference is that the integration over x^+ at each vertex runs only over half of the real axis $[-\infty, 0]$
  - In Fourier space, this means that the $-$ component of the momentum is not conserved at the vertices
  - Instead of a δ function, one gets an energy denominator
- Example with a single interaction :



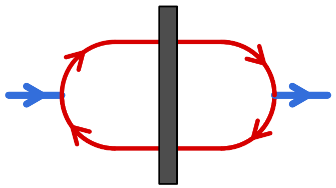
$$\langle \vec{\boldsymbol{k}}_1 \vec{\boldsymbol{k}}_2 \vec{\boldsymbol{k}}_{3\,{\rm in}} | U_{\rm self}(0,-\infty) | \vec{\boldsymbol{p}}_{\rm in} \rangle = -ig \int_{-\infty}^{0} d^4x \; e^{i(k_1+k_2+k_3-p)\cdot x}$$

$$= -g \, \frac{(2\pi)^3 \delta(\vec{\boldsymbol{k}}_{1\perp} + \vec{\boldsymbol{k}}_{2\perp} + \vec{\boldsymbol{k}}_{3\perp} - \vec{\boldsymbol{p}}_\perp) \delta(k_1^+ + k_2^+ + k_3^+ - p^+)}{k_1^- + k_2^- + k_3^- - p^- - i\epsilon}$$

Energy dependence

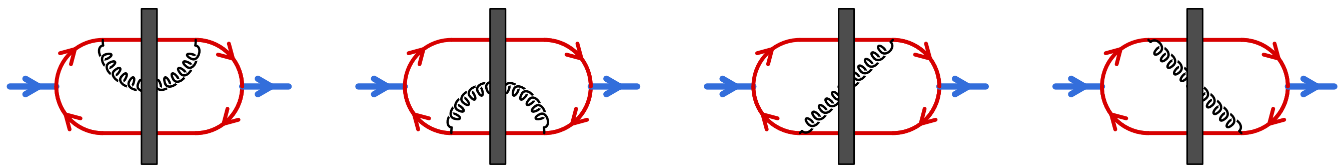
Scattering of a dipole

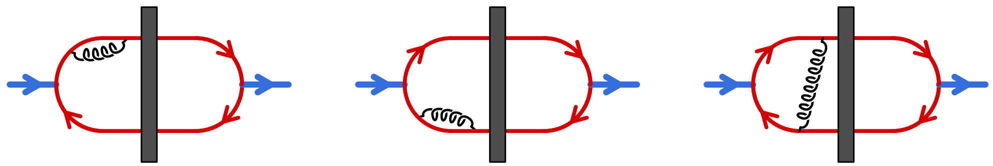
- Assume that the initial and final states α and β are a color singlet $Q\overline{Q}$ dipole. The bare scattering amplitude can be written as :



$$\propto \left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\right|^2 \mathrm{tr}\left[U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

- At one loop, the following diagrams must be evaluated :

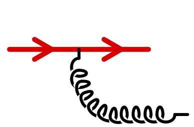




$+$ h.c.

Scattering of a dipole

- In the gauge $A^+ = 0$, the emission of a gluon of momentum k by a quark can be written as :



$$= 2gt^a \, \frac{\vec{\boldsymbol{\epsilon}}_\lambda \cdot \vec{\boldsymbol{k}}_\perp}{k_\perp^2}$$

- In coordinate space, this reads :

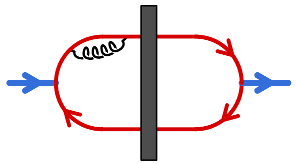
$$\int \frac{d^2\vec{\boldsymbol{k}}_\perp}{(2\pi)^2} \, e^{i\vec{\boldsymbol{k}}_\perp\cdot(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)} \, 2gt^a \, \frac{\vec{\boldsymbol{\epsilon}}_\lambda \cdot \vec{\boldsymbol{k}}_\perp}{k_\perp^2} = \frac{2ig}{2\pi} t^a \frac{\vec{\boldsymbol{\epsilon}}_\lambda \cdot (\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$

- When connecting two gluons, one must use :

$$\sum_\lambda \vec{\boldsymbol{\epsilon}}_\lambda^i \vec{\boldsymbol{\epsilon}}_\lambda^j = -g^{ij}$$

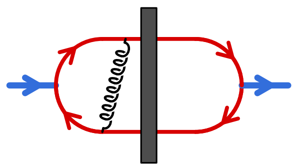
Virtual corrections

- Consider first the loop corrections inside the wavefunction of the incoming or outgoing dipole
- Examples :



$$= \left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\right|^2 \mathrm{tr}\left[t^a t^a U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

$$\times -2\alpha_s \int \frac{dk^+}{k^+} \int \frac{d^2\vec{\boldsymbol{z}}_\perp}{(2\pi)^2} \frac{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)\cdot(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2}$$



$$= \left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\right|^2 \mathrm{tr}\left[t^a U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)t^a\right]$$

$$\times 4\alpha_s \int \frac{dk^+}{k^+} \int \frac{d^2\vec{\boldsymbol{z}}_\perp}{(2\pi)^2} \frac{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)\cdot(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)^2}$$

- Reminder : $t^a t^a = (N_c^2-1)/2N_c \equiv C_F$

Virtual corrections

- The sum of all virtual corrections is :

$$-\frac{C_F\alpha_s}{\pi^2}\int\frac{dk^+}{k^+}\int d^2\vec{\boldsymbol{z}}_\perp\frac{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)^2}$$

$$\times\left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\right|^2\mathrm{tr}\left[U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

- The integral over k^+ is divergent. It should have an upper bound at p^+ :

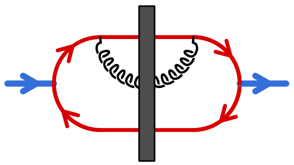
$$\int^{p^+}\frac{dk^+}{k^+}=\ln(p^+)=Y$$

  $\triangleright$ When Y is large, $\alpha_s Y$ may not be small. By differentiating with respect to Y, we will get an evolution equation in Y whose solution resums all the powers $(\alpha_s Y)^n$

- Note : the integral over $\vec{\boldsymbol{z}}_\perp$ is divergent when $\vec{\boldsymbol{z}}_\perp=\vec{\boldsymbol{x}}_\perp$ or $\vec{\boldsymbol{y}}_\perp$

Real corrections

- There are also real corrections, for which the state that interacts with the target has an extra gluon
- Example :



$$= \left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)\right|^2 \mathrm{tr}\left[t^a U(\vec{\boldsymbol{x}}_\perp) t^b U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

$$\times 4\alpha_s \int \frac{dk^+}{k^+} \int \frac{d^2\vec{\boldsymbol{z}}_\perp}{(2\pi)^2} \widetilde{U}_{ab}(\vec{\boldsymbol{z}}_\perp) \frac{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)\cdot(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$

  - $\widetilde{U}_{ab}(\vec{\boldsymbol{z}}_\perp)$ is a Wilson line in the adjoint representation
- In order to simplify the color structure, first recall that :

$$t^a \widetilde{U}_{ab}(\vec{\boldsymbol{z}}_\perp) = U(\vec{\boldsymbol{z}}_\perp) t^b U^\dagger(\vec{\boldsymbol{z}}_\perp)$$

- Then use the $SU(N_c)$ Fierz identity :

$$t^b_{ij} t^b_{kl} = \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2N_c}\delta_{ij}\delta kl$$

Real corrections

- The Wilson lines can be rearranged into :

$$\mathrm{tr}\left[t^a U(\vec{\boldsymbol{x}}_\perp) t^b U^\dagger(\vec{\boldsymbol{y}}_\perp)\right] \widetilde{U}_{ab}(\vec{\boldsymbol{z}}_\perp) = \frac{1}{2}\mathrm{tr}\left[U^\dagger(\vec{\boldsymbol{z}}_\perp) U(\vec{\boldsymbol{x}}_\perp)\right] \mathrm{tr}\left[U(\vec{\boldsymbol{z}}_\perp) U^\dagger(\vec{\boldsymbol{y}}_\perp)\right] - \frac{1}{2N_c}\mathrm{tr}\left[U(\vec{\boldsymbol{x}}_\perp) U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

  - The term in $1/2N_c$ cancels against a similar term in the virtual contribution
  - All the real terms have the same color structure

- When we sum all the real terms, we generate the same kernel as in the virtual terms :

$$\frac{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2 (\vec{\boldsymbol{y}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$

- In order to simplify the notations, let us denote :

$$\boldsymbol{S}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \equiv \frac{1}{N_c}\mathrm{tr}\left[U(\vec{\boldsymbol{x}}_\perp) U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

Evolution equation

- The 1-loop scattering amplitude reads :

$$-\frac{\alpha_s N_c^2 Y}{2\pi^2}\left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\right|^2\int d^2\vec{\boldsymbol{z}}_\perp\,\frac{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)^2}$$
$$\times\Big\{\boldsymbol{S}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)-\boldsymbol{S}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{z}}_\perp)\boldsymbol{S}(\vec{\boldsymbol{z}}_\perp,\vec{\boldsymbol{y}}_\perp)\Big\}$$

- Reminder: the bare scattering amplitude was :

$$\left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\right|^2 N_c\,\boldsymbol{S}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)$$

- Hence, we have :

$$\frac{\partial\boldsymbol{S}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)}{\partial Y}=-\frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{\boldsymbol{z}}_\perp\,\frac{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)^2}$$
$$\times\Big\{\boldsymbol{S}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)-\boldsymbol{S}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{z}}_\perp)\boldsymbol{S}(\vec{\boldsymbol{z}}_\perp,\vec{\boldsymbol{y}}_\perp)\Big\}$$

  - since $\boldsymbol{S}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{x}}_\perp)=1$, the integral over $\vec{z}_\perp$ is now regular

BFKL equation

Kuraev, Lipatov, Fadin (1977), Balitsky, Lipatov (1978)

- The BFKL equation can be obtained by linearizing the previous equation

- Write $\boldsymbol{S}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\equiv 1-\boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)$ and assume that we are in the dilute regime, so that the scattering amplitude $\boldsymbol{T}$ is small. Drop the terms that are non-linear in $\boldsymbol{T}$:

$$\frac{\partial\,\boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)}{\partial Y}=\frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{\boldsymbol{z}}_\perp\;\frac{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)^2}$$
$$\times\Big\{\boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{z}}_\perp)+\boldsymbol{T}(\vec{\boldsymbol{z}}_\perp,\vec{\boldsymbol{y}}_\perp)-\boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\Big\}$$

- The solution of this equation grows exponentially when $Y\to+\infty$ $\quad\triangleright$ serious unitarity problem...

Non-linear evolution equation

- In fact, the first evolution equation we derived has a bounded solution. The unbounded solutions of BFKL are due to dropping the non-linear term. The full equation reads :

$$\frac{\partial \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{\boldsymbol{z}}_\perp \frac{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)^2}$$
$$\times\Big\{\boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{z}}_\perp)+\boldsymbol{T}(\vec{\boldsymbol{z}}_\perp,\vec{\boldsymbol{y}}_\perp)-\boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\underline{-\boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{z}}_\perp)\boldsymbol{T}(\vec{\boldsymbol{z}}_\perp,\vec{\boldsymbol{y}}_\perp)}\Big\}$$

  (Balitsky-Kovchegov equation)

- The r.h.s. vanishes when $\boldsymbol{T}$ reaches 1, and the growth stops. The non-linear term lets both dipoles interact after the splitting of the original dipole

- Both $\boldsymbol{T}=0$ and $\boldsymbol{T}=1$ are fixed points of this equation

$$\boldsymbol{T}=\epsilon\ : \qquad \text{r.h.s.} > 0 \qquad \Rightarrow \quad \boldsymbol{T}=0 \text{ is unstable}$$
$$\boldsymbol{T}=1-\epsilon\ : \qquad \text{r.h.s.} > 0 \qquad \Rightarrow \quad \boldsymbol{T}=1 \text{ is stable}$$

Caveats and improvements

- So far, we have studied the scattering amplitude between a color dipole and a “god given” patch of color field. This is too naive to describe any realistic situation
- We need to improve the treatment of the target
- An experimentally measured cross-section is an average over many collisions, and there is no reason why these fields should be the same in different collisions :

$$\boldsymbol{T} \quad \to \quad \langle \boldsymbol{T} \rangle$$

$\langle \cdots \rangle$ denotes the average over the target configurations, i.e.

$$\langle \cdots \rangle = \int [D\rho]\; W_Y[\rho] \cdots$$

Balitsky hierarchy

- Because of this average over the target configurations, the evolution equation we have derived should be written as :

$$\frac{\partial\,\langle \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\rangle}{\partial Y}=\frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{\boldsymbol{z}}_\perp\;\frac{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)^2}$$
$$\times\Big\{\langle \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{z}}_\perp)\rangle+\langle \boldsymbol{T}(\vec{\boldsymbol{z}}_\perp,\vec{\boldsymbol{y}}_\perp)\rangle-\langle \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\rangle\underline{-\langle \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{z}}_\perp)\boldsymbol{T}(\vec{\boldsymbol{z}}_\perp,\vec{\boldsymbol{y}}_\perp)\rangle}\Big\}$$

- As one can see, the equation is no longer a closed equation, since the equation for $\langle\boldsymbol{T}\rangle$ depends on a new object, $\langle\boldsymbol{T}\,\boldsymbol{T}\rangle$
- One can derive an evolution equation for $\langle\boldsymbol{T}\,\boldsymbol{T}\rangle$. Its right hand side contains objects with six Wilson lines
  - There is in fact an infinite hierarchy of nested evolution equations, whose generic structure is

$$\frac{\partial\,\left\langle(\boldsymbol{U}\boldsymbol{U}^\dagger)^n\right\rangle}{\partial Y}=\int\cdots\left\langle(\boldsymbol{U}\boldsymbol{U}^\dagger)^n\right\rangle\oplus\left\langle(\boldsymbol{U}\boldsymbol{U}^\dagger)^{n+1}\right\rangle$$

Balitsky-Kovchegov equation

- If one performs the large N_c approximation on all the equations of the Balitsky hierarchy, they can be rewritten in terms of the dipole operator $\boldsymbol{T} \equiv 1 - \frac{1}{N_c}\mathrm{tr}(UU^\dagger)$ only. But they still contain averages like $\langle \boldsymbol{T}^n \rangle$
- In order to truncate the hierarchy of equations, one may assume that
$$\langle \boldsymbol{T}\,\boldsymbol{T} \rangle \approx \langle \boldsymbol{T} \rangle\ \langle \boldsymbol{T} \rangle$$
- This approximation gives for $\langle \boldsymbol{T} \rangle$ the same evolution equation as the one we had for a fixed configuration of the target

Geometrical scaling

Analogy with reaction-diffusion

Munier, Peschanski (2003,2004)

- Assume translation and rotation invariance, and define :

$$N(Y,k_\perp) \equiv 2\pi \int d^2\vec{\boldsymbol{x}}_\perp \; e^{i\vec{\boldsymbol{k}}_\perp\cdot\vec{\boldsymbol{x}}_\perp} \; \frac{\left\langle \boldsymbol{T}(0,\vec{\boldsymbol{x}}_\perp)\right\rangle_Y}{x_\perp^2}$$

- From the Balitsky-Kovchegov equation for $\langle \boldsymbol{T}\rangle$, we obtain the following equation for N :

$$\frac{\partial N(Y,k_\perp)}{\partial Y} = \frac{\alpha_s N_c}{\pi}\Big[\chi(-\partial_L)N(Y,k_\perp) - N^2(Y,k_\perp)\Big]$$

with

$$L \equiv \ln(k_\perp^2/k_0^2)$$

$$\chi(\gamma) \equiv 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$$

Analogy with reaction-diffusion

- Expand the function $\chi(\gamma)$ to second order around its minimum $\gamma = 1/2$

- Introduce new variables :

$$t \sim Y$$
$$z \sim L + \frac{\alpha_s N_c}{2\pi} \chi''(1/2)\, Y$$

- The equation for N becomes :

$$\partial_t N = \partial_z^2 N + N - N^2$$

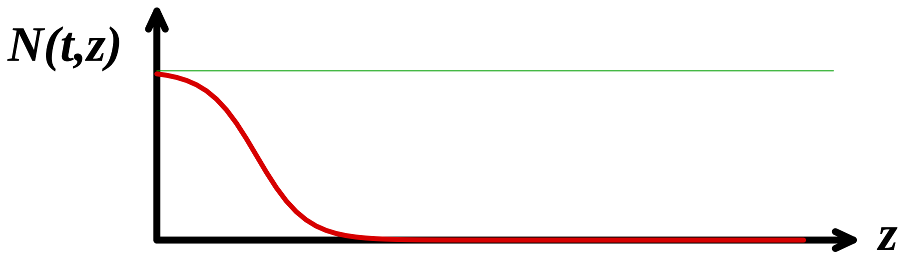
(known as the Fisher-Kolmogorov-Petrov-Piscounov equation)

Analogy with reaction-diffusion

- Interpretation : this equation is typical for all the diffusive systems in which a reaction $A \longleftrightarrow A + A$ takes place
  - $\partial_z^2 N$: diffusion term (the quantity under consideration can hop from a site to the neighboring sites)
  - $+N$: gain term corresponding to $A \to A + A$
  - $-N^2$: loss term corresponding to $A + A \to A$
- Note : this equation has two fixed points :
  - $N = 0$: unstable
  - $N = 1$: stable
- The stable fixed point at $N = 1$ exists only if one keeps the loss term. In other words, one would not have it from the BFKL equation

Traveling waves

- Assume an initial condition $N(t_0, z)$ that goes smoothly from 1 at $z = -\infty$ to 0 at $z = +\infty$, and behaves like $\exp(-\beta z)$ when $z \gg 1$



- The solution of the F-KPP equation is known to behave like a traveling wave at asymptotic times (Bramson, 1983) :

$$N(t, z) \underset{t\to+\infty}{\sim} N(z - m_\beta(t))$$

with $\quad m_\beta(t) = 2t - 3\ln(t)/2 + \mathcal{O}(1) \quad$ if $\quad \beta > 1$

$\triangleright$ universal front velocity for a large class of initial conditions

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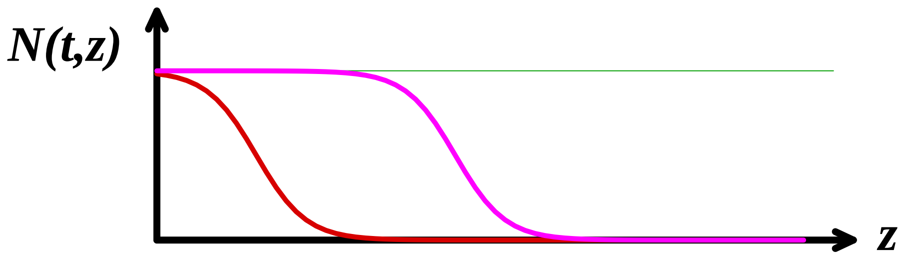

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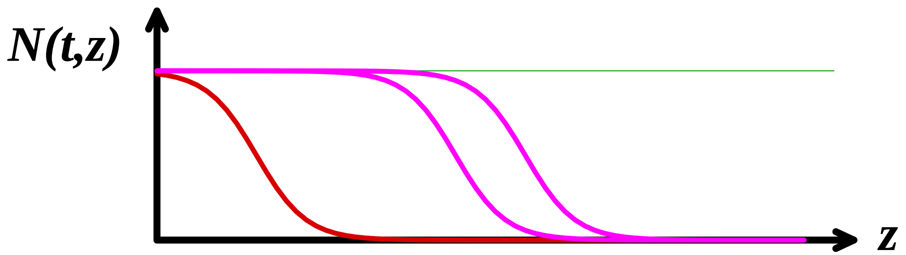

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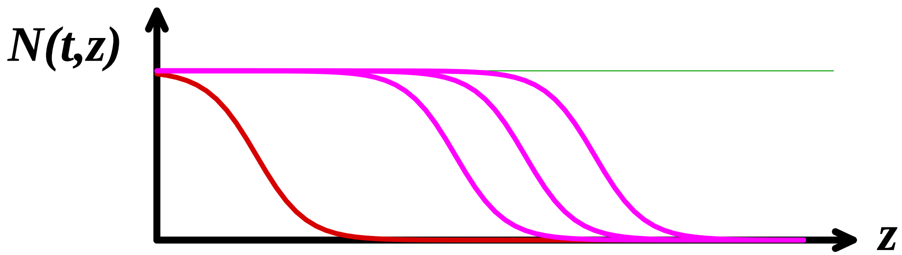

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▷ universal front velocity for a large class of initial conditions

Geometrical scaling in DIS

Iancu, Itakura, McLerran (2002)
Mueller, Triantafyllopoulos (2002)
Munier, Peschanski (2003)

- In QCD, the initial condition is of the required form, with $\beta > 1$
  ▷ front velocity independent of the initial condition

- Going back to the original variables, one gets :

$$N(Y,k_\perp) = N\left(k_\perp / Q_s(Y)\right)$$

with

$$Q_s^2(Y) = k_0^2 \; Y^{-\delta} \; e^{\lambda Y}$$

- Going from $N(Y,k_\perp)$ to $\left<\boldsymbol{T}(0,\vec{\boldsymbol{x}}_\perp)\right>_Y$, we obtain :

$$\left<\boldsymbol{T}(0,\vec{\boldsymbol{x}}_\perp)\right>_Y = T(Q_s(Y)x_\perp)$$

Geometrical scaling in DIS

- The total $\gamma^* p$ cross-section, measured in Deep Inelastic Scattering, can be written in terms of N:

$$\sigma^{\rm tot}_{\gamma^* p}(Y, Q^2) = 2\pi R^2 \int d^2\vec{\boldsymbol{x}}_\perp \int_0^1 dz \left|\psi(z, x_\perp, Q^2)\right|^2 \langle \boldsymbol{T}(0, \vec{\boldsymbol{x}}_\perp)\rangle_Y$$

  - The photon wavefunction ψ is calculable in QED. It depends on the dipole size $x_\perp$ only via

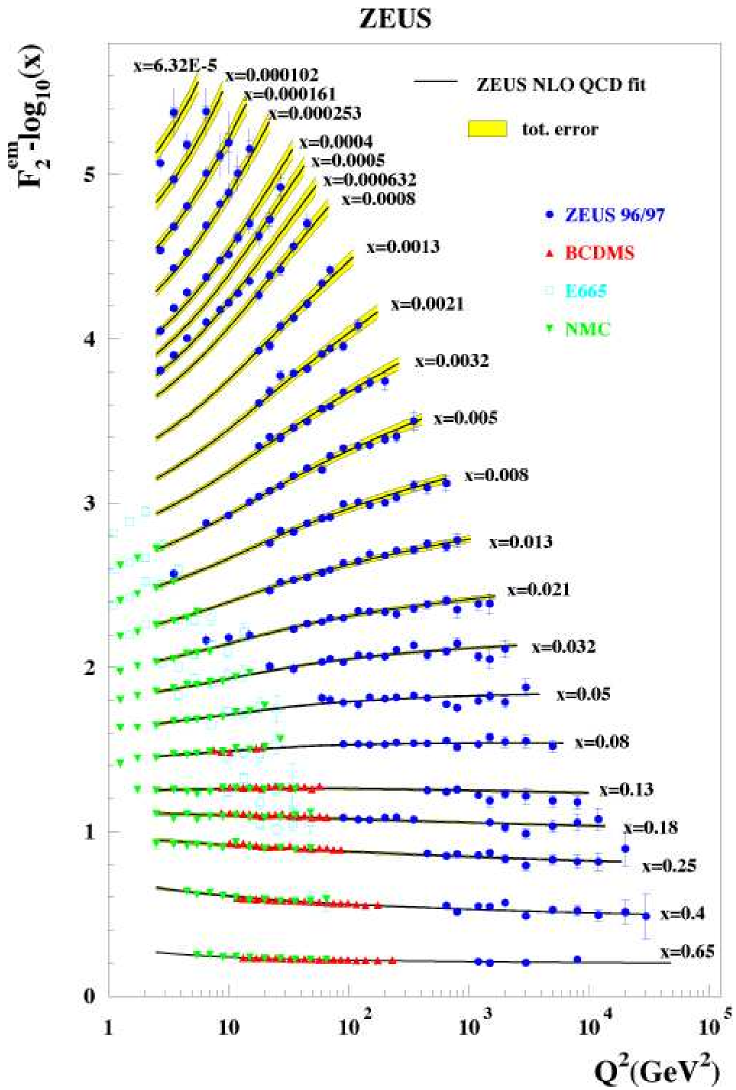
$$\left|\psi(z, x_\perp, Q^2)\right|^2 = f(\overline{Q}_f x_\perp)$$

    with $\overline{Q}_f^2 \equiv m_f^2 + Q^2 z^2 (1 - z^2)$

- If one neglects the quark masses, the scaling properties of $\langle \boldsymbol{T}\rangle_Y$ imply that $\sigma_{\gamma^* p}$ depends only on the ratio $Q^2/Q_s^2(Y)$, rather than on Q^2 and Y separately

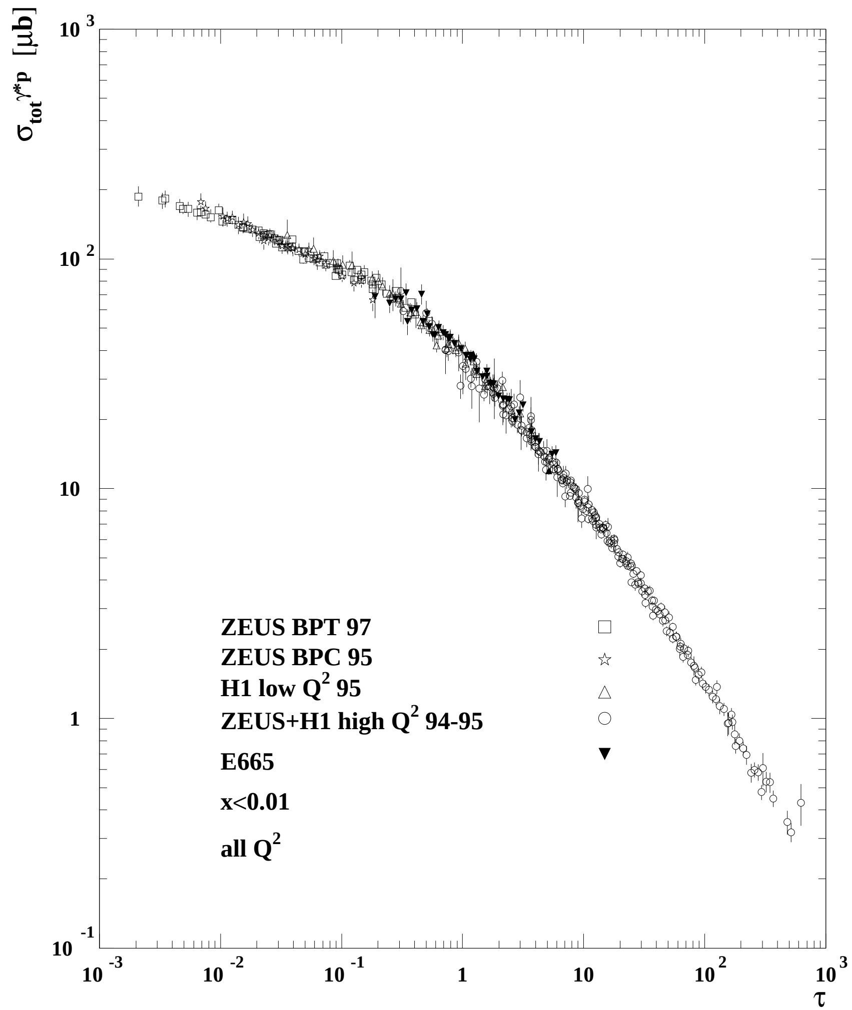
Geometrical scaling in DIS

- HERA data as a function of Q^2 and x :

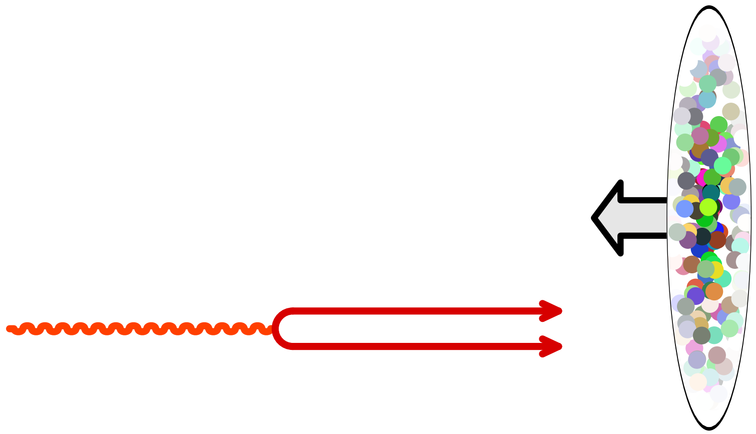


Geometrical scaling in DIS

Stasto, Golec-Biernat, Kwiecinski (2000)



Fits of DIS data



Dipole cross-section

- Computing F_2 requires to know $\left<\boldsymbol{T}(0, \vec{\boldsymbol{x}}_\perp)\right>_Y$ as a function of dipole size and energy
- This object is often presented in the form of the "dipole cross-section" :

$$\sigma_{\rm dip}(\vec{\boldsymbol{r}}_\perp, Y) \equiv 2\int d^2\vec{\boldsymbol{b}}\;\left<\boldsymbol{T}(\vec{\boldsymbol{b}} - \frac{\vec{\boldsymbol{r}}_\perp}{2}, \vec{\boldsymbol{b}} + \frac{\vec{\boldsymbol{r}}_\perp}{2})\right>_Y$$

Note : this formula assumes that the scattering amplitude is real

- In principle, the BK equation prescribes the energy dependence of the dipole cross-section once it is known at a certain energy
- Alternatively, one can model this cross-section (including its energy dependence)

Golec-Biernat–Wusthoff model

- GBW modeled the dipole cross-section as a Gaussian, with an energy dependence entirely contained in Q_s

$$\begin{cases} \sigma_{\rm dip}(\vec{\boldsymbol{r}}_\perp, Y) = \sigma_0 \left[1 - e^{-Q_s(Y)^2 r_\perp^2/4}\right] \\ Q_s^2(Y) = Q_0^2 \, e^{\lambda(Y-Y_0)} \end{cases}$$

- The exponential form in $\sigma_{\rm dip}$ is inspired of Glauber scattering
- The fit parameters are σ_0, Q_0, λ and possibly an effective quark mass in the photon wave-function
- Quite good for all small-x HERA data, with some problems at large Q^2

Bartels–Golec-Biernat–Kowalski model

- This model aims at improving the agreement at large Q^2, by having a more realistic cross-section at small dipole sizes :

$$\sigma_{\rm dip}(\vec{\boldsymbol{r}}_\perp, Y) = \sigma_0 \left[1 - e^{-\pi^2 r_\perp^2 \alpha_s(\mu^2) xG(x,\mu^2)/3\sigma_0}\right]$$

- The scale μ^2 is chosen of the form $\mu_0^2 + C/r_\perp^2$
- The gluon distribution $xG(x, \mu^2)$ obeys the DGLAP equation. Thus, the dipole cross-section has the correct behavior at small transverse distance
- This form improves the fit quality at large Q^2
- A saturation scale is also hidden in this dipole cross-section, if one recalls the formula

$$Q_s^2 \quad \sim \quad \frac{\alpha_s xG(x, Q_s^2)}{\pi R^2}$$

Iancu-Itakura-Munier model

- This model of the dipole cross-section is derived from LO BFKL :

$$\begin{cases} Q_s r_\perp \le 2 \quad : \quad \sigma_{\rm dip}(\vec{\boldsymbol{r}}_\perp, Y) = \dfrac{\sigma_0}{2}\left(\dfrac{Q_s(Y) r_\perp}{2}\right)^{2(\gamma_s + \ln(2/Q_s r_\perp)/\kappa\lambda Y)} \\ Q_s r_\perp \ge 2 \quad : \quad \sigma_{\rm dip}(\vec{\boldsymbol{r}}_\perp, Y) = \sigma_0 \left[1 - e^{a \ln^2(b Q_s r_\perp)}\right] \\ \\ Q_s^2(Y) = Q_0^2 \; e^{\lambda(Y - Y_0)} \end{cases}$$

  - Some parameters are fixed from LO BFKL : $\gamma_s = 0.63, \kappa = 9.9$
  - σ_0, Q_0 and λ must be fitted
  - a and b are adjusted for a smooth transition at $Q_s r_\perp = 2$

Exclusive processes

Kowalski, Motyka, Watt (2006)

- So far, we have only considered the total DIS cross-section, obtained from the forward dipole amplitude via the optical theorem
- In order to study more exclusive processes, one needs non-forward amplitudes. From our general eikonal formula, they read :

$$\big\langle \Omega_{\rm out} \big| \gamma^*{}_{\rm in} \big\rangle = \int d^2\vec{\boldsymbol{r}}_\perp \int_0^1 dz\; \Psi^*_\Omega \psi \underbrace{\int d^2\vec{\boldsymbol{b}}\; e^{i\vec{\boldsymbol{q}}_\perp\cdot\vec{\boldsymbol{b}}} \Big\langle \boldsymbol{T}\big(\vec{\boldsymbol{b}} - \frac{\vec{\boldsymbol{r}}_\perp}{2}, \vec{\boldsymbol{b}} + \frac{\vec{\boldsymbol{r}}_\perp}{2}\big) \Big\rangle_Y}_{\substack{\text{non-forward dipole cross-section} \\ \text{with momentum transfer } \vec{\boldsymbol{q}}_\perp}}$$

Note : this formula assumes that the relevant dipole sizes $r_\perp$ are small compared to the target radius (i.e. the typical $\boldsymbol{b}$)

Exclusive processes

- By squaring this amplitude, one gets the diffractive cross-section for the production of the state Ω with momentum transfer $\boldsymbol{q}_\perp$

$$\frac{d\sigma^{\rm diff}_{\gamma^* p\to\Omega p}}{d^2\vec{\boldsymbol{q}}_\perp} = \left|\left<\Omega_{\rm out}\big|{\gamma^*}_{\rm in}\right>\right|^2$$

- The relationship to the the inclusive DIS cross-section is

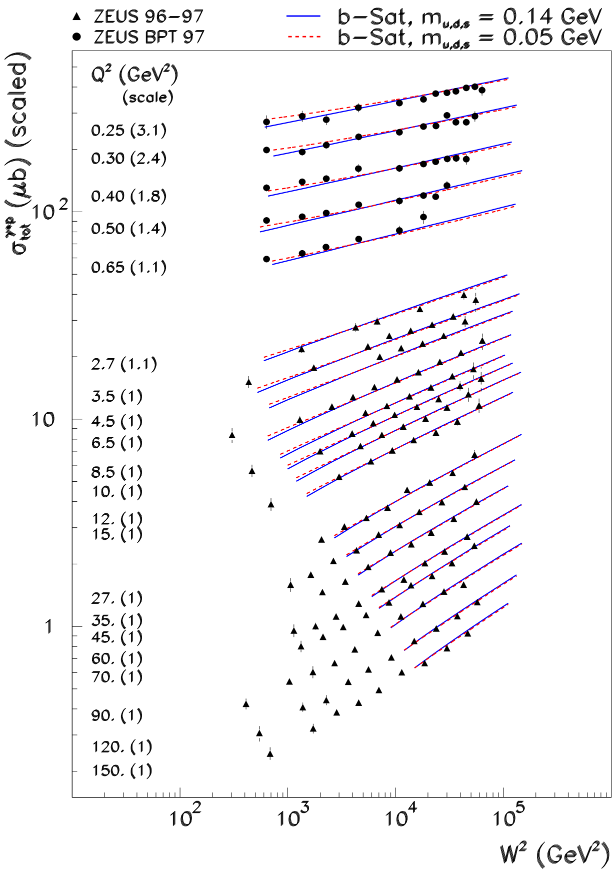
$$\sigma^{\rm tot}_{\gamma^* p}(Y, Q^2) = 2\,{\rm Re}\left<{\gamma^*}_{\rm out}\big|{\gamma^*}_{\rm in}\right>_{\vec{\boldsymbol{q}}_\perp=0}$$

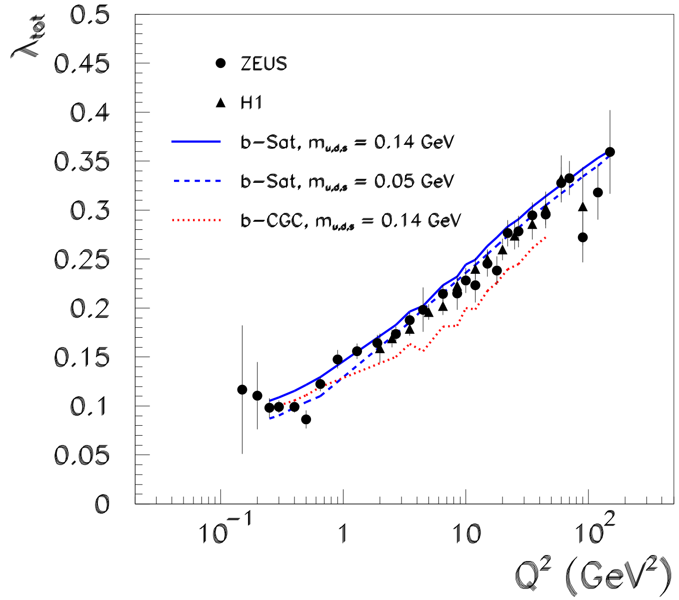
  Note : inclusive DIS only constrains the dipole amplitude averaged over impact parameter. However, if one measures the $\boldsymbol{q}_\perp$ dependence in exclusive reactions, one obtains informations about the $\boldsymbol{b}$ dependence of the dipole amplitude

- General strategy : extend the previous models in order to give them a $\boldsymbol{b}$-dependence, in a way that preserves F_2

Exclusive reactions

- For the total DIS cross-section, the fit is as good as before :

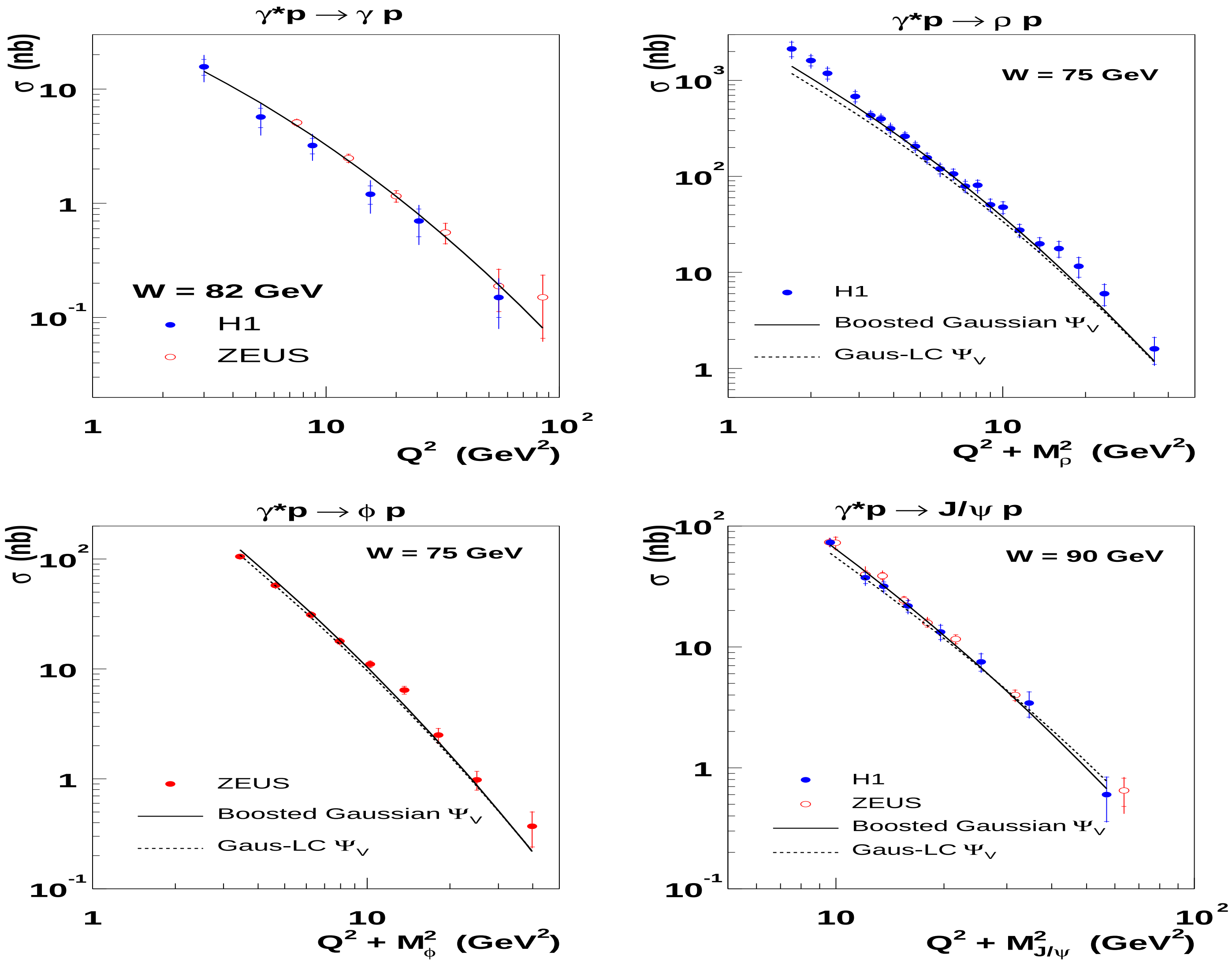




Exclusive reactions

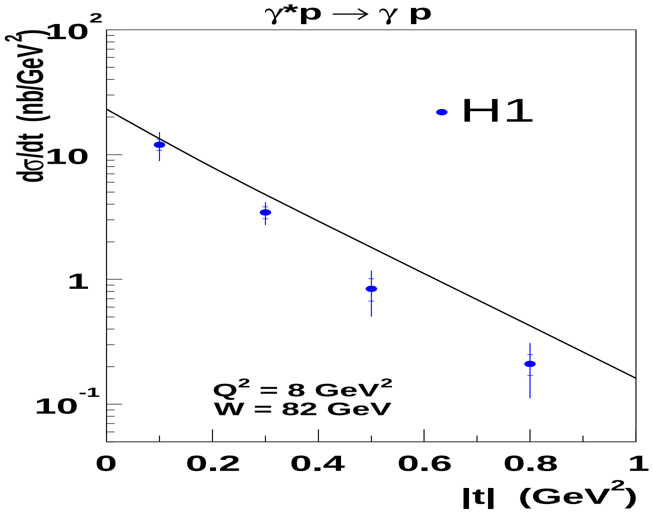
Introduction
Eikonal scattering
Energy dependence
Geometrical scaling
Fits of DIS data
● Dipole cross-section
● Dipole models
● Exclusive processes
Proton-Nucleus collisions

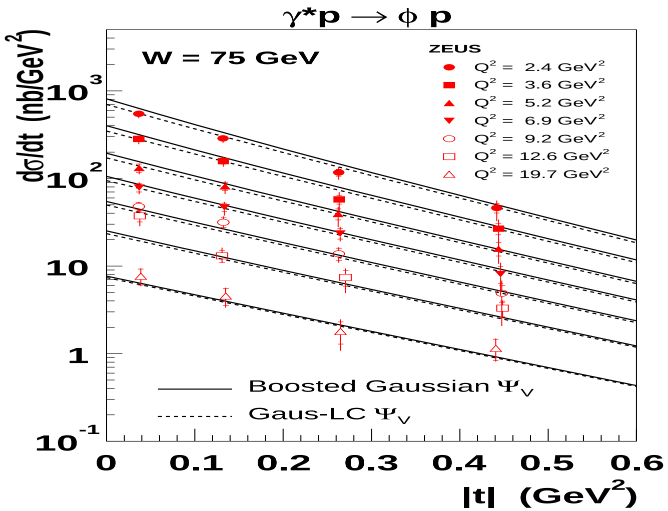
- Exclusive photon and vector meson production :

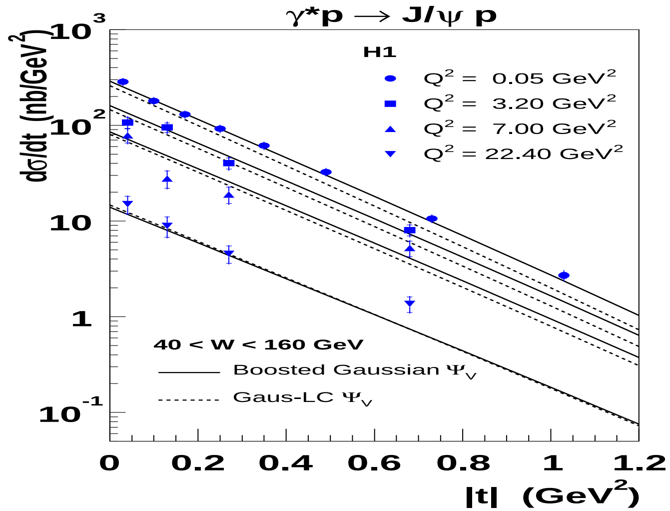


Exclusive reactions

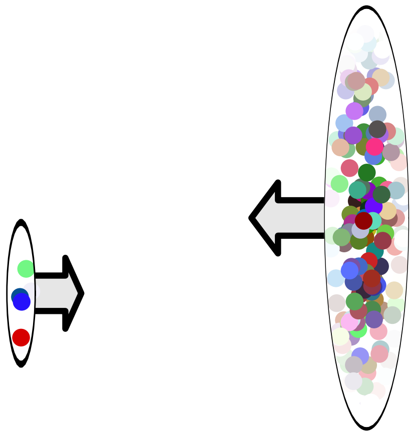
- Exclusive photon and vector meson production :





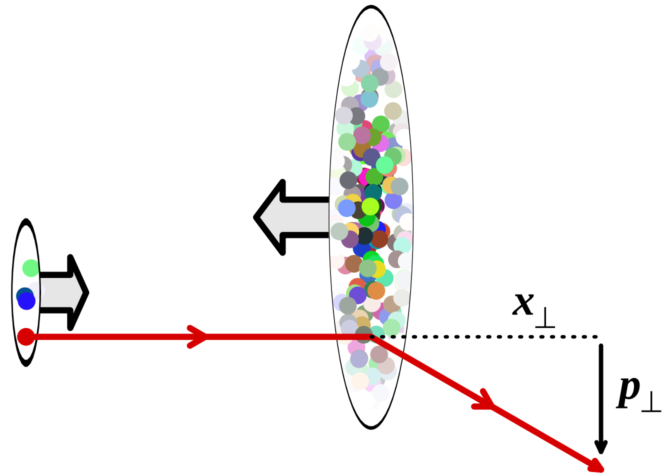


Proton-Nucleus collisions



Link to the dipole cross-section

- When the proton is dilute, a pA collision can be seen as a collision between one parton from the proton and the color fields of the nucleus :



- In a given configuration of the target, the scattering amplitude reads :

$$\mathcal{M} \propto \int d^2\vec{\boldsymbol{x}}_\perp \; e^{i\vec{\boldsymbol{p}}_\perp\cdot\vec{\boldsymbol{x}}_\perp} \; U(\boldsymbol{x}_\perp)$$

Note : U is in the representation of the incoming parton

Link to the dipole cross-section

- The squared amplitude, averaged over the target configurations, reads :

$$\begin{aligned} |\mathcal{M}|^2 &\propto \int d^2\vec{\boldsymbol{r}}_\perp \, e^{i\vec{\boldsymbol{p}}_\perp \cdot \vec{\boldsymbol{r}}_\perp} \int d^2\vec{\boldsymbol{b}} \, \left\langle U(\vec{\boldsymbol{b}} + \frac{\vec{\boldsymbol{r}}_\perp}{2}) U^\dagger(\vec{\boldsymbol{b}} - \frac{\vec{\boldsymbol{r}}_\perp}{2}) \right\rangle_Y \\ &\propto \int d^2\vec{\boldsymbol{r}}_\perp \, e^{i\vec{\boldsymbol{p}}_\perp \cdot \vec{\boldsymbol{r}}_\perp} \, \sigma_{\rm dip}(\vec{\boldsymbol{r}}_\perp, Y) \end{aligned}$$

▷ the $p_\perp$-spectrum of the scattered parton is given by the Fourier transform of the dipole cross-section

- For a dipole model to be usable in this context, the corresponding cross-section must have a positive definite Fourier transform. All the previous models have shortcomings in this respect:
  - GBW: exponential tail, ruled out at high $\boldsymbol{p}_\perp$ by pQCD
  - BGBK: negative Fourier transform at large $\boldsymbol{p}_\perp$
  - IIM: oscillatory Fourier transform

Description of the proton

- One can describe the partons of the proton without intrinsic $\boldsymbol{k}_\perp$ (collinear factorization) or with some intrinsic $\boldsymbol{k}_\perp$
- Collinear factorization :

$$\frac{d\sigma_{pA\to iX}}{d^2\vec{\boldsymbol{p}}_\perp} \propto \; x_1 f_{i/p}(x_1,\boldsymbol{p}_\perp^2)\int d^2\vec{\boldsymbol{r}}_\perp\; e^{i\vec{\boldsymbol{p}}_\perp\cdot\vec{\boldsymbol{r}}_\perp}\;\sigma_{\rm dip}(\vec{\boldsymbol{r}}_\perp,Y_2)$$

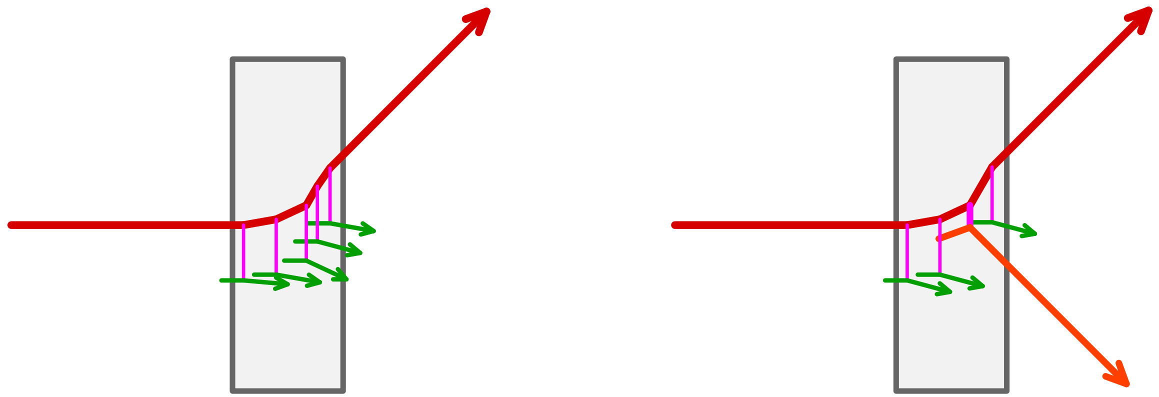
- $\boldsymbol{k}_\perp$-factorization :

$$\frac{d\sigma_{pA\to iX}}{d^2\vec{\boldsymbol{p}}_\perp} \propto \int d^2\vec{\boldsymbol{k}}_\perp\;\varphi_{i/p}(x_1,\vec{\boldsymbol{k}}_\perp)\underline{\int d^2\vec{\boldsymbol{r}}_\perp\; e^{i(\vec{\boldsymbol{p}}_\perp-\vec{\boldsymbol{k}}_\perp)\cdot\vec{\boldsymbol{r}}_\perp}\;\sigma_{\rm dip}(\vec{\boldsymbol{r}}_\perp,Y_2)}$$

- The first formula is obtained from the second one by :
  - Taking the limit $\boldsymbol{k}_\perp\to 0$ in the underlined factor
  - Integrating $\varphi_{i/p}$ over $\boldsymbol{k}_\perp$ up to $\boldsymbol{p}_\perp$ in order to recover the integrated parton distribution

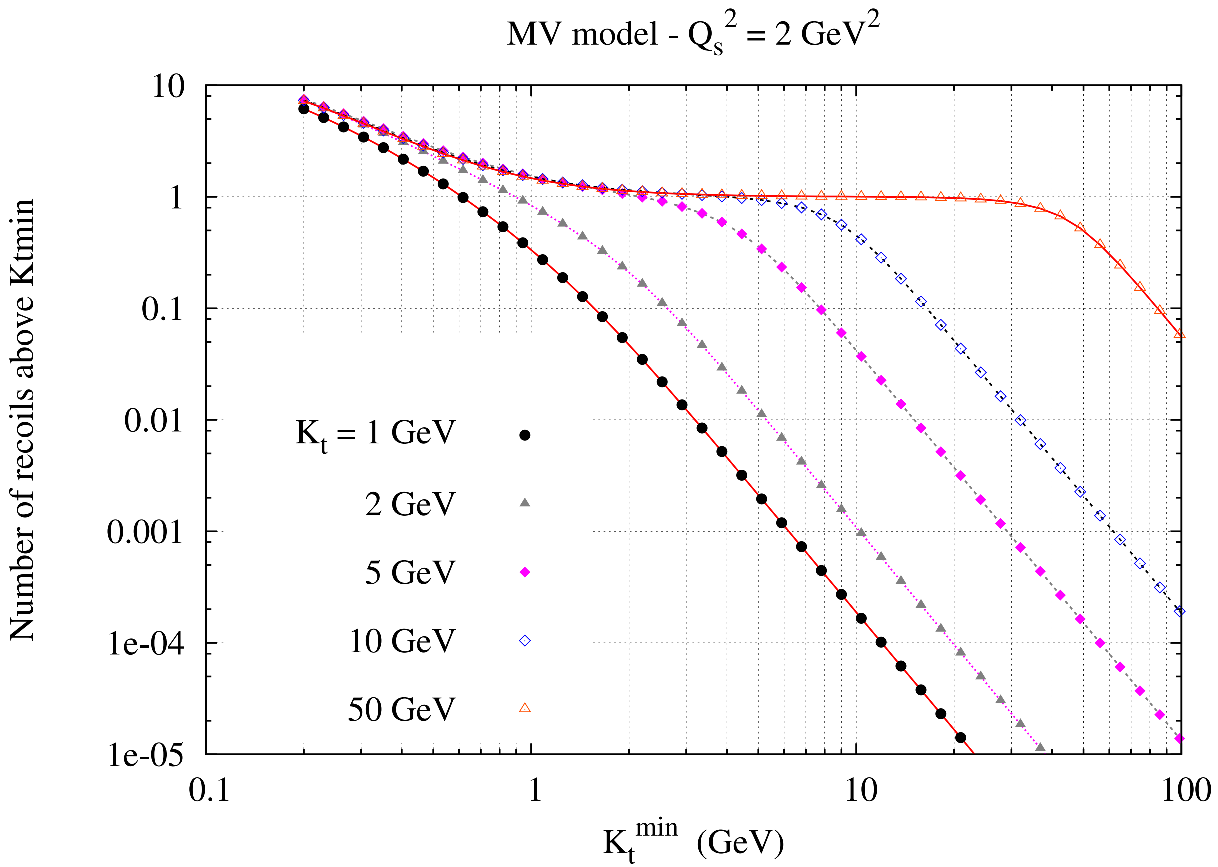
Monojets ?

- Since in this description a pA collision amounts to multiple scatterings of a parton from the proton on those of the nucleus, an interesting issue is the distribution of the recoils when the incoming parton is scattered at a high $p_\perp$



  - If the recoil momentum is shared evenly between a large number of partons, the final state will look like a monojet
  - If a single parton takes most of the recoil, then the final state will look like the usual di-jet events

Number of recoils above Ktmin



- When $Q_s \lesssim k_\perp^{\min} \lesssim k_\perp$, there is only one recoil

  ▷ the momentum of the scattered parton is absorbed by a single source ▷ pair of jets rather than a monojet

Levy random walks

- Interpretation : the scattering of the incoming parton can be seen a a random walk in $\boldsymbol{p}_\perp$ space, with a probability $\mathcal{P}(k_\perp)$ to gain $\vec{\boldsymbol{k}}_\perp$ at each step of the random walk
- A crucial property of $\mathcal{P}(k_\perp)$ is whether its second moment,

$$\sigma \equiv \int d^2\vec{\boldsymbol{k}}_\perp \; k_\perp^2 \; \mathcal{P}(k_\perp)\;, \qquad \text{is finite or not}$$

Levy random walks

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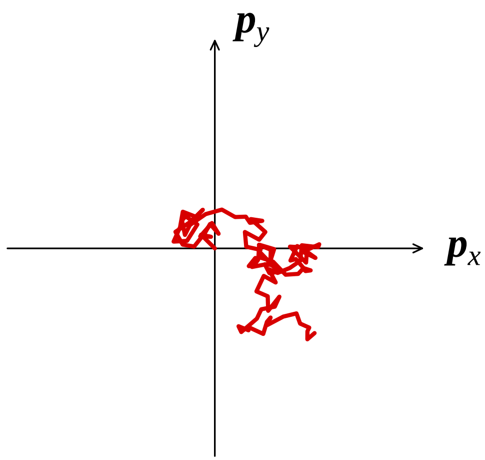
- If σ is finite, the random walk takes an exponentionally large number of steps to get far from the origin :



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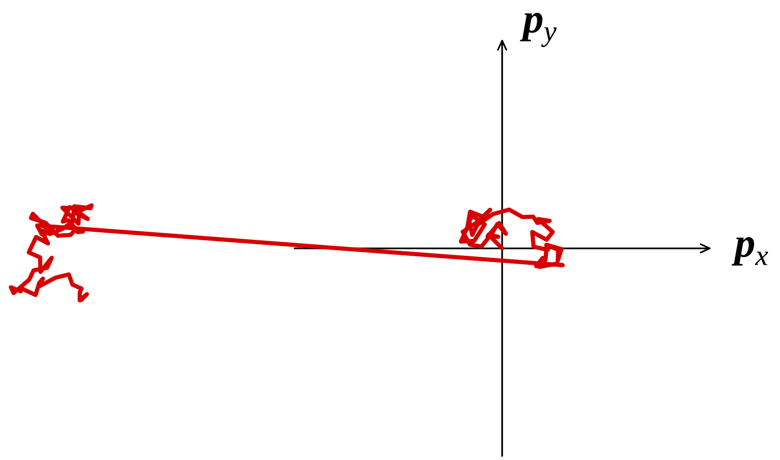
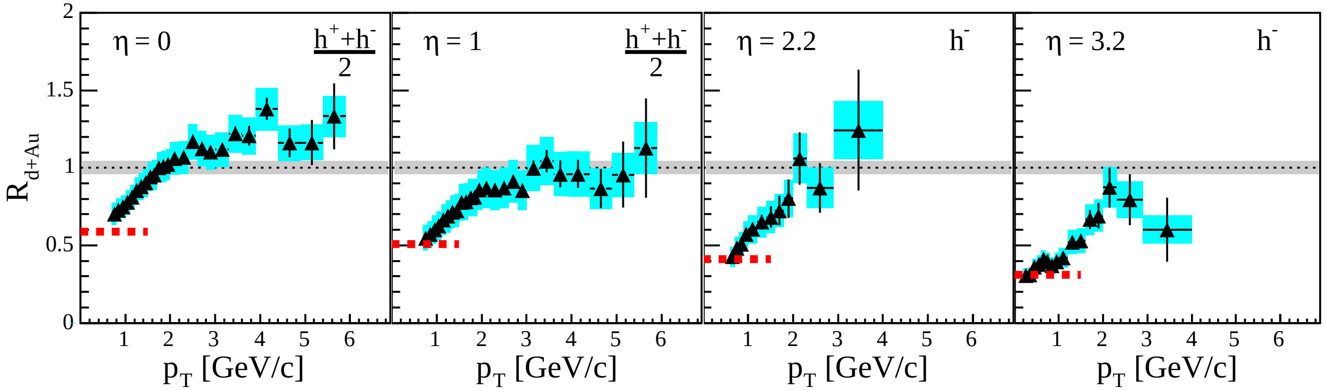
- If σ is infinite (true for the MV model), the random walk can go far from the origin in one big step and a few small ones :



High pt suppression at large Y

- Results of the BRAHMS experiment at RHIC for deuteron-gold collisions :

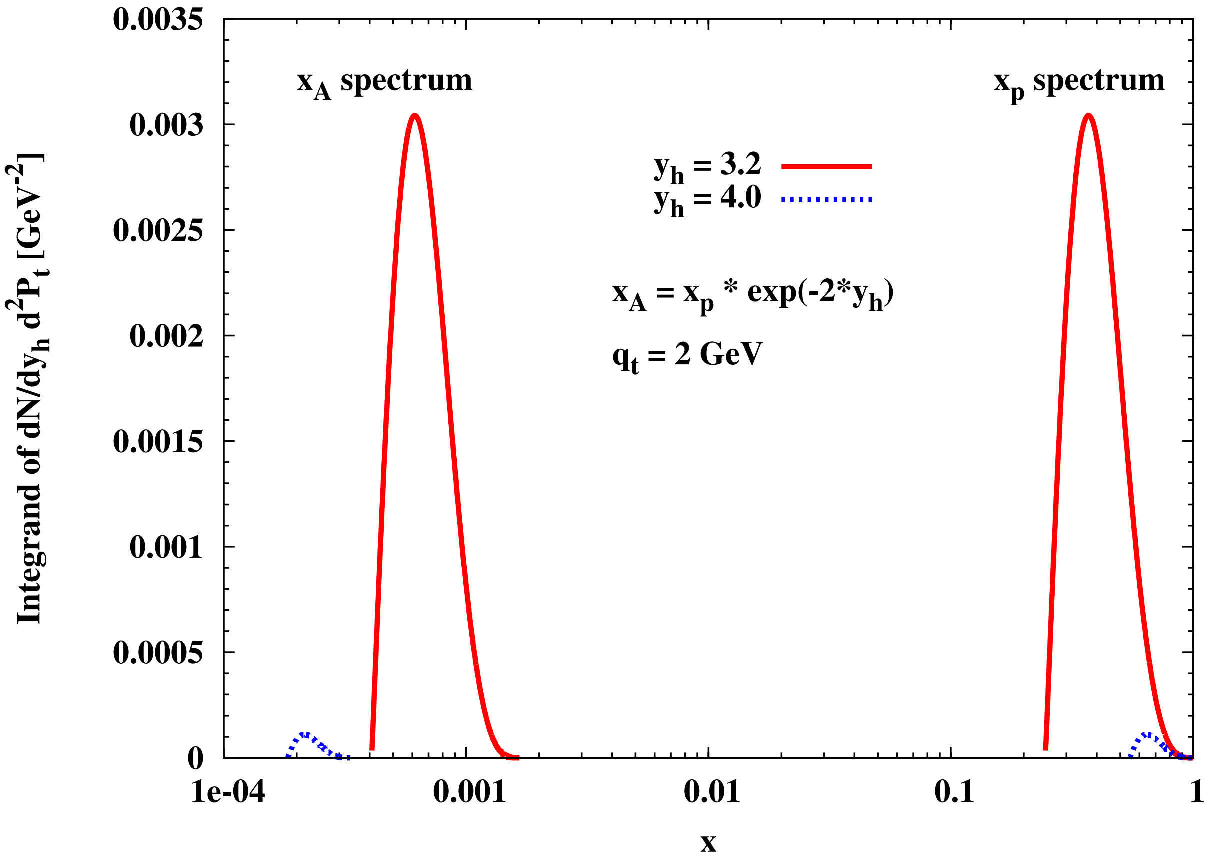
$$R_{\rm dAu} \equiv \frac{1}{N_{\rm coll}} \frac{\left.\frac{dN}{dp_\perp d\eta}\right|_{\rm dAu}}{\left.\frac{dN}{dp_\perp d\eta}\right|_{\rm pp}}$$



- At small rapidity, suppression at low $p_\perp$ and enhancement at high $p_\perp$ (multiple scatterings – Cronin effect)
- At large rapidity, suppression at all $p_\perp$'s (shadowing)

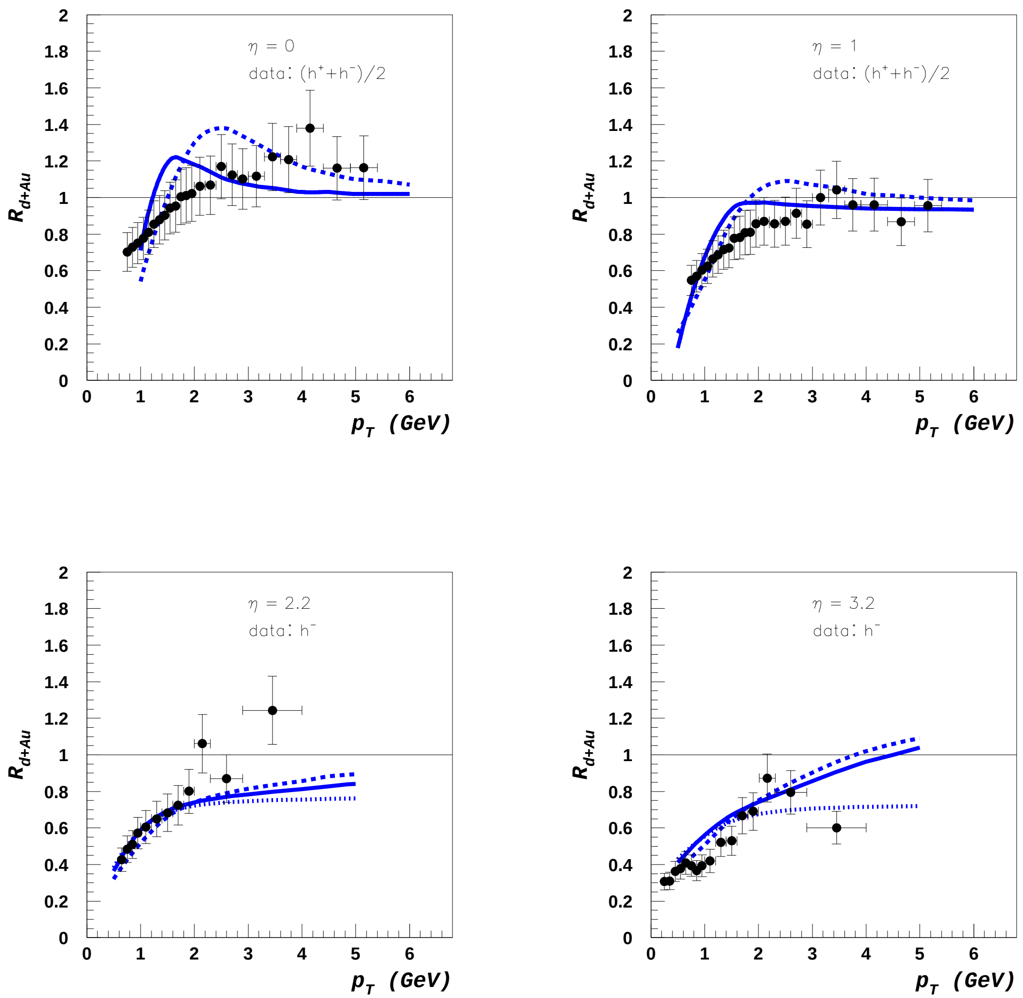
Kinematics

- Relevant values of $x_{1,2}$:



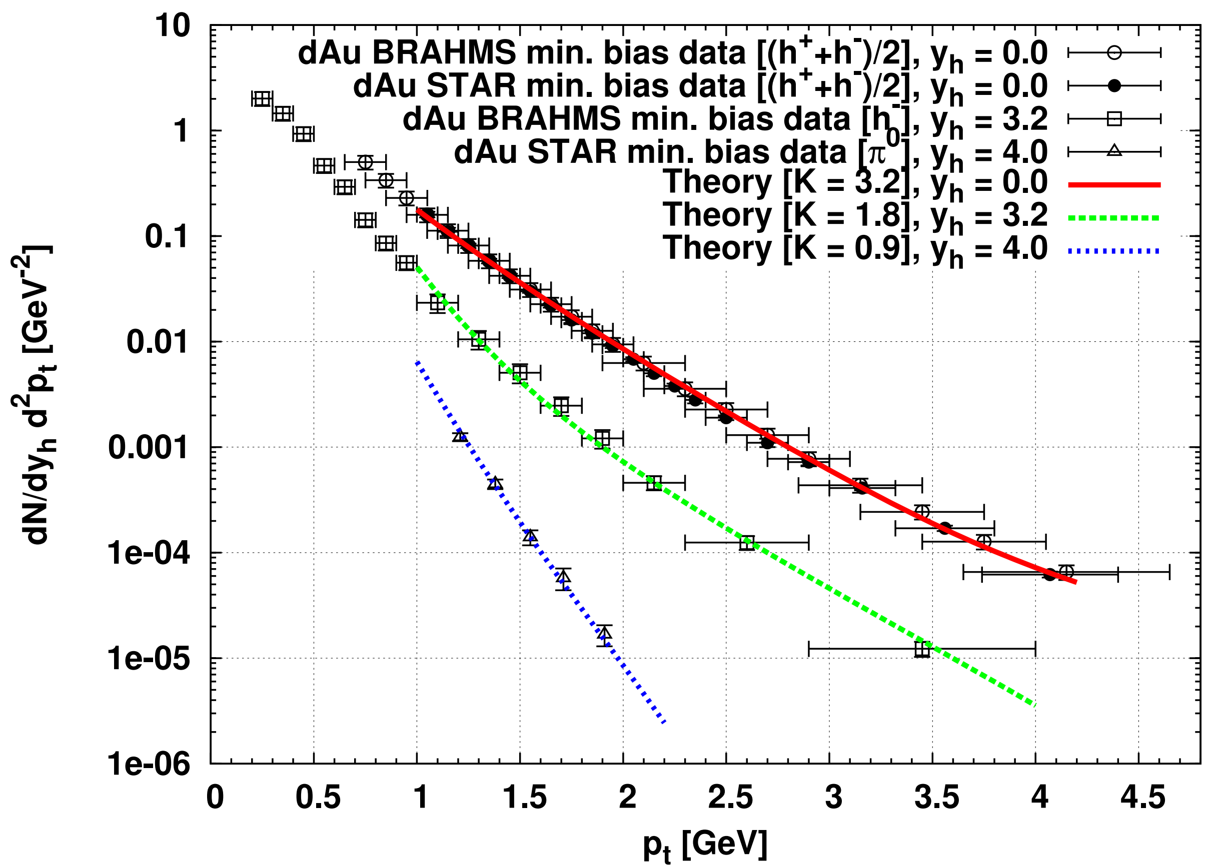
dA collisions at RHIC

- Kharzeev, Kovchegov, Tuchin (2005)



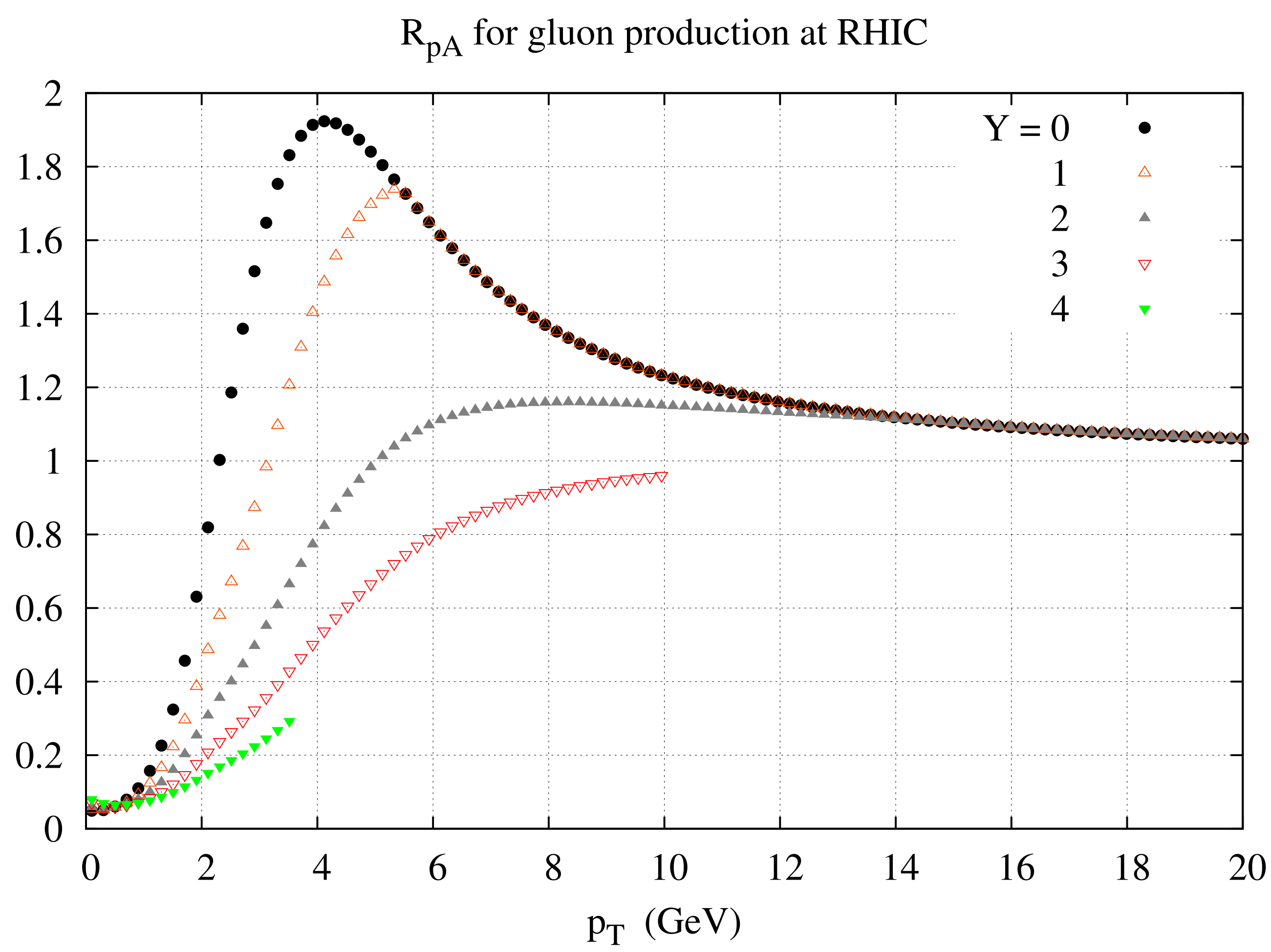
dA collisions at RHIC

- Dumitru, Hayashigaki, Jalilian-Marian (2005 – 2006)

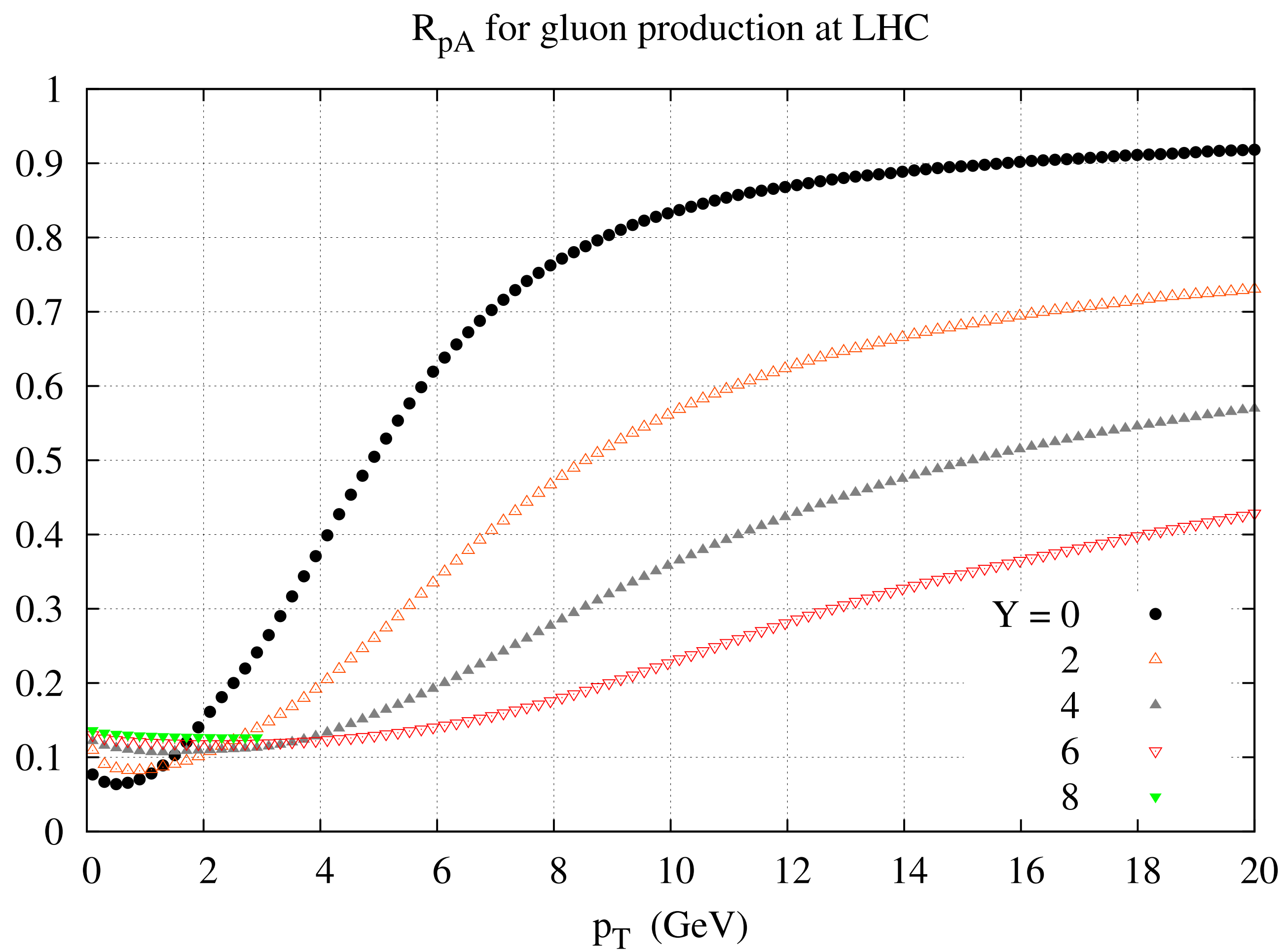


Note : the model predicts only the slope of the spectrum; its normalization is adjusted by a Y-dependent K-factor

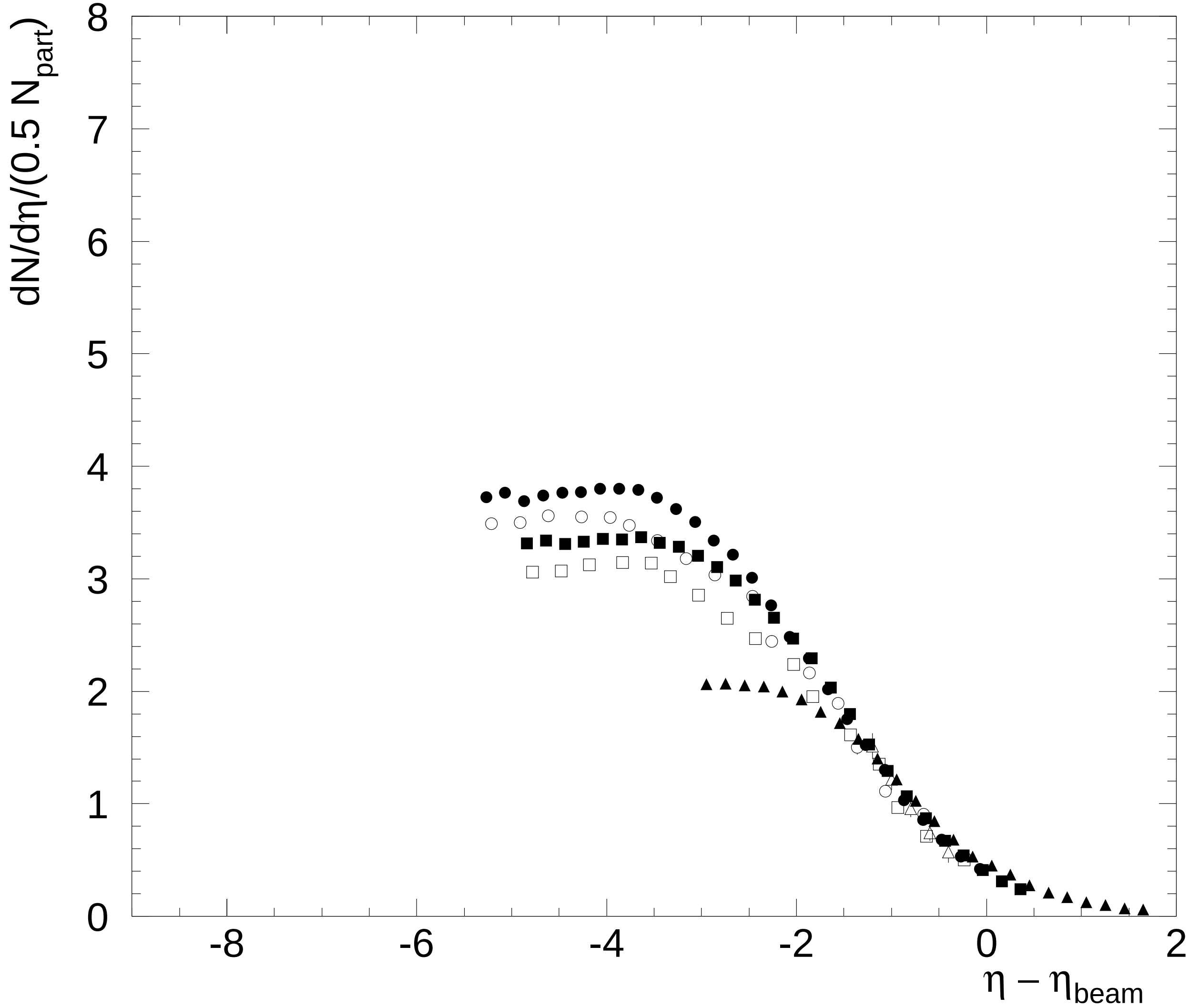
RdA at RHIC from the BK equation



RpA at LHC from the BK equation



Limiting fragmentation (RHIC)



Qualitative explanation

- The ratio of the two saturation scales is :
  $Q_s^2(x_2)/Q_s^2(x_1) \sim \exp(2\lambda Y) \sim 20$ with $\lambda \approx 0.3$ and $Y = 5$

  ▷ neglect the transverse momentum in the projectile at large x_1 compared to that in the projectile at x_2

  ▷ use collinear factorization for projectile 1

- The spectrum reads :

$$\frac{dN}{d^2\vec{\boldsymbol{P}}_\perp dY} \sim x_1 f(x_1, \boldsymbol{P}_\perp^2) \underline{\int d^2\vec{\boldsymbol{r}}_\perp \, e^{i\vec{\boldsymbol{P}}_\perp \cdot \vec{\boldsymbol{r}}_\perp} \left\langle \mathrm{tr}\left(U(0)U^\dagger(\vec{\boldsymbol{r}}_\perp)\right)\right\rangle_{x_2}}$$

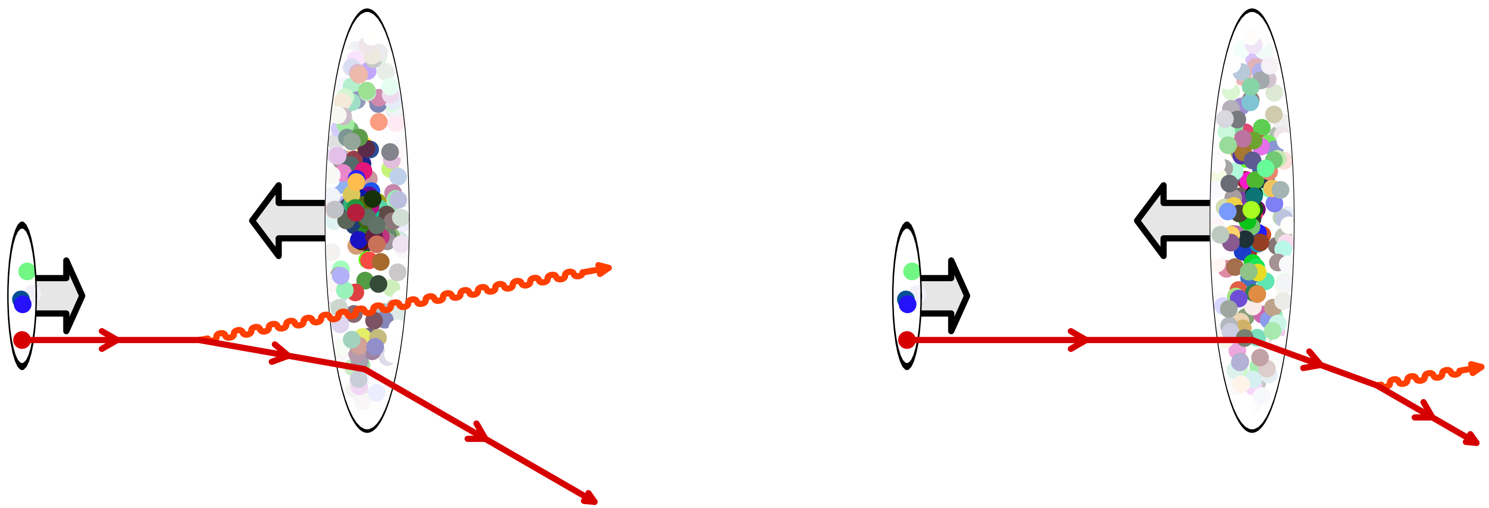
  Note : the underlined factor becomes independent of x_2 when integrated over $\vec{\boldsymbol{P}}_\perp$ because of the unitarity of the Wilson lines

- At large x_1, $x_1\, f(x_1, \boldsymbol{P}_\perp^2)$ is almost independent of $\boldsymbol{P}_\perp^2$ (Bjorken scaling), and the integration over $\vec{\boldsymbol{P}}_\perp$ leads to :

$$\frac{dN}{dY} \propto x_1\, f(x_1) \quad \Rightarrow \quad \text{depends only on } x_1 \sim \exp(Y - Y_{\mathrm{beam}})$$

Other processes

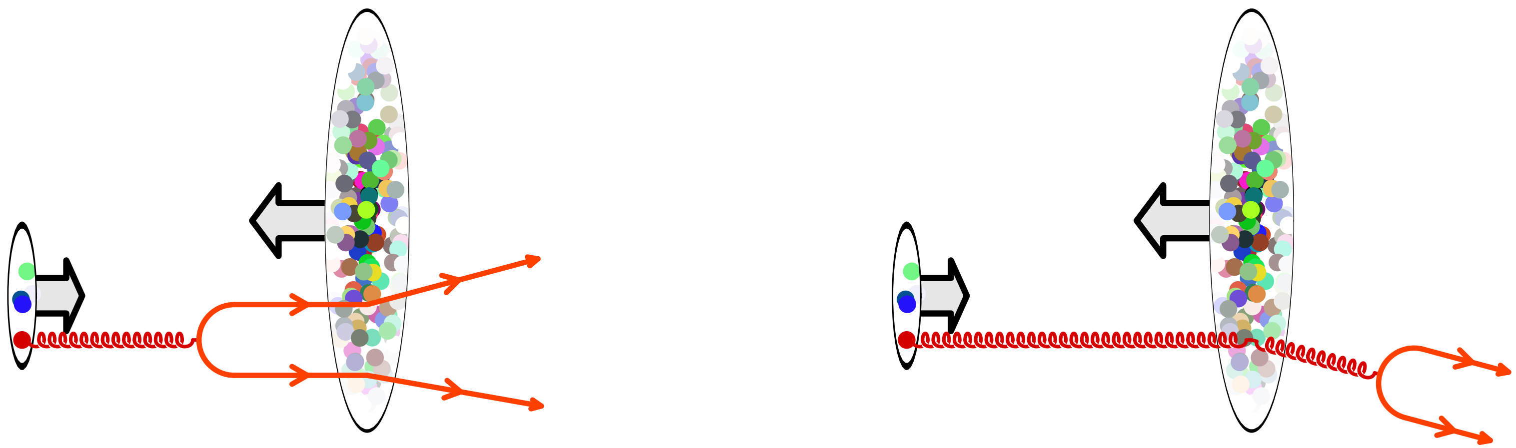
- Photon/dilepton production :



▷ the cross-section for this process can also be written in terms of the Fourier transformed dipole cross-section

Other processes

- Production of heavy quarks :



▷ the cross-section for this process involves new correlators, containing up to four Wilson lines

Lecture III : Nucleus-nucleus collisions

- Introduction to nucleus-nucleus collisions
- Power counting and bookeeping
- Inclusive gluon spectrum
- Loop corrections, factorization, unstable modes