Gluon saturation from DIS to AA collisions

III – Saturation in nucleus-nucleus collisions

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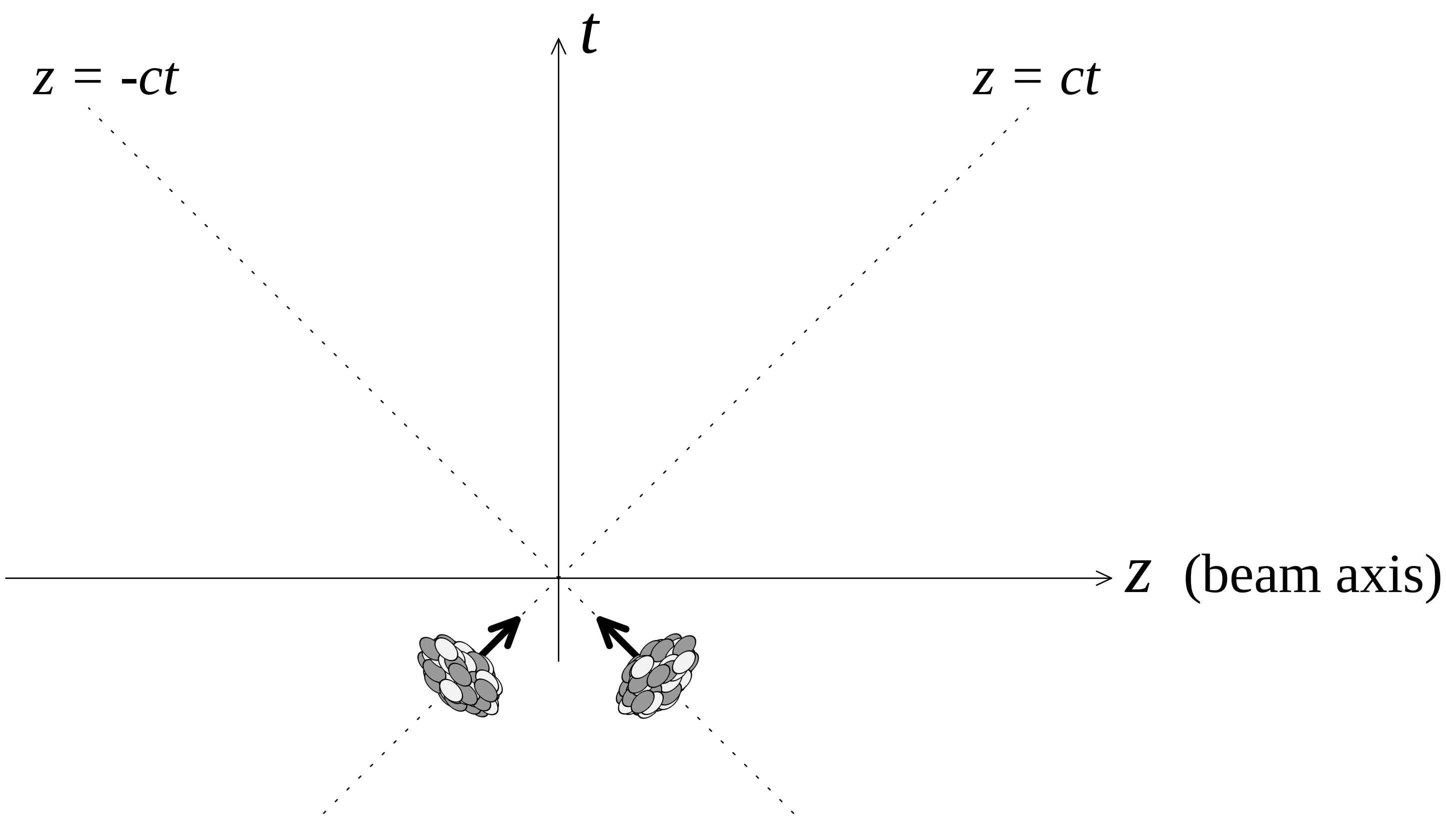
General outline

- Lecture I : Gluon saturation, Color Glass Condensate
- Lecture II : DIS and proton-nucleus collisions
- Lecture III : Saturation in nucleus-nucleus collisions

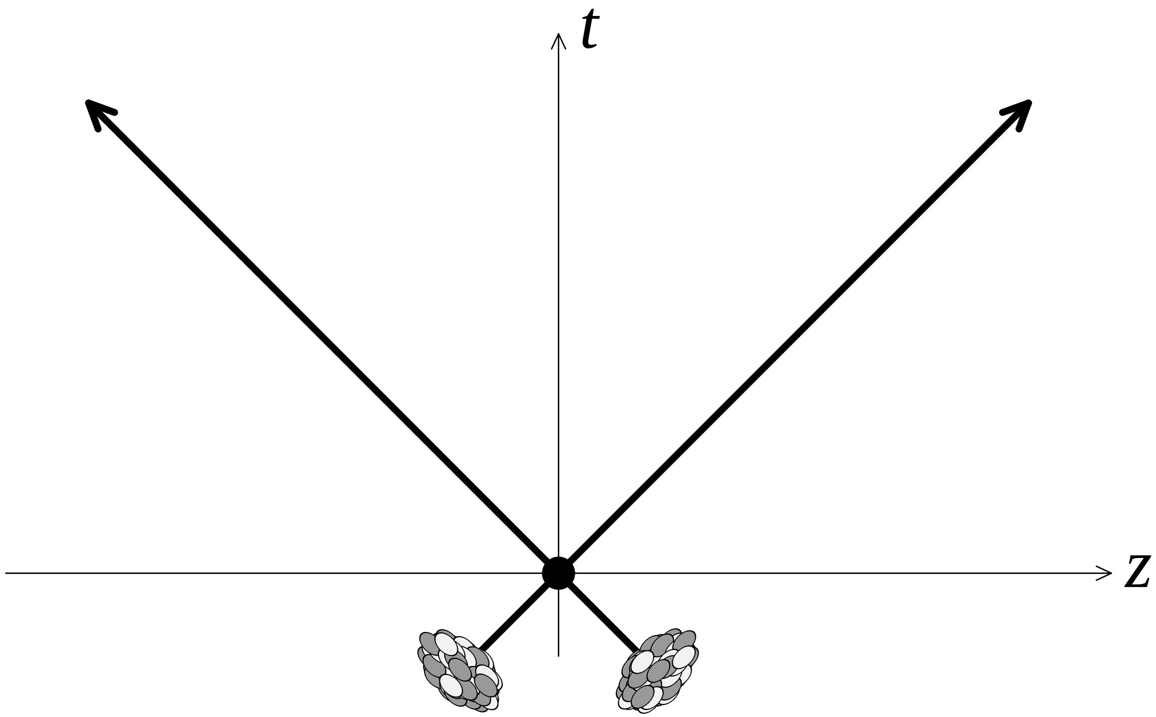
- Introduction to nucleus-nucleus collisions
- Power counting and bookeeping
- Inclusive gluon spectrum
- Loop corrections, factorization, unstable modes

Introduction to AA collisions

Stages of a nucleus-nucleus collision

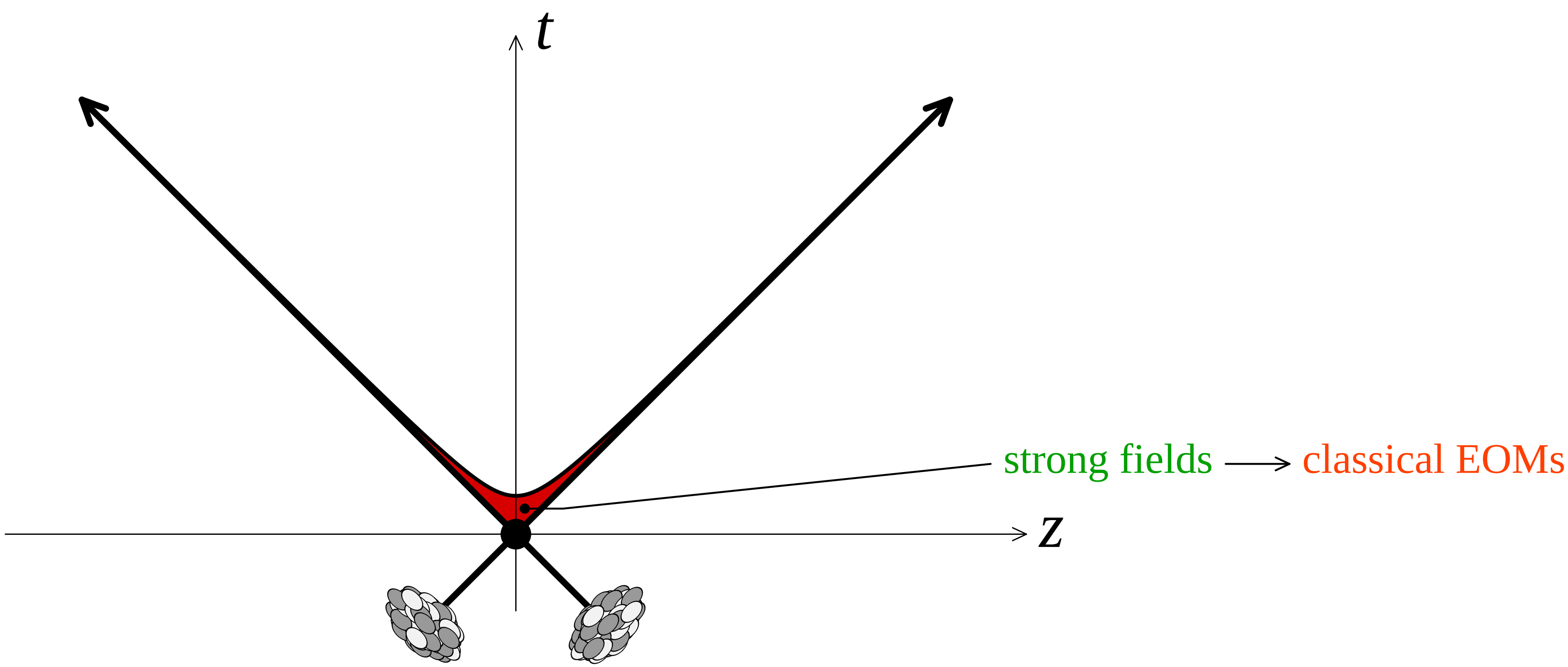


Stages of a nucleus-nucleus collision



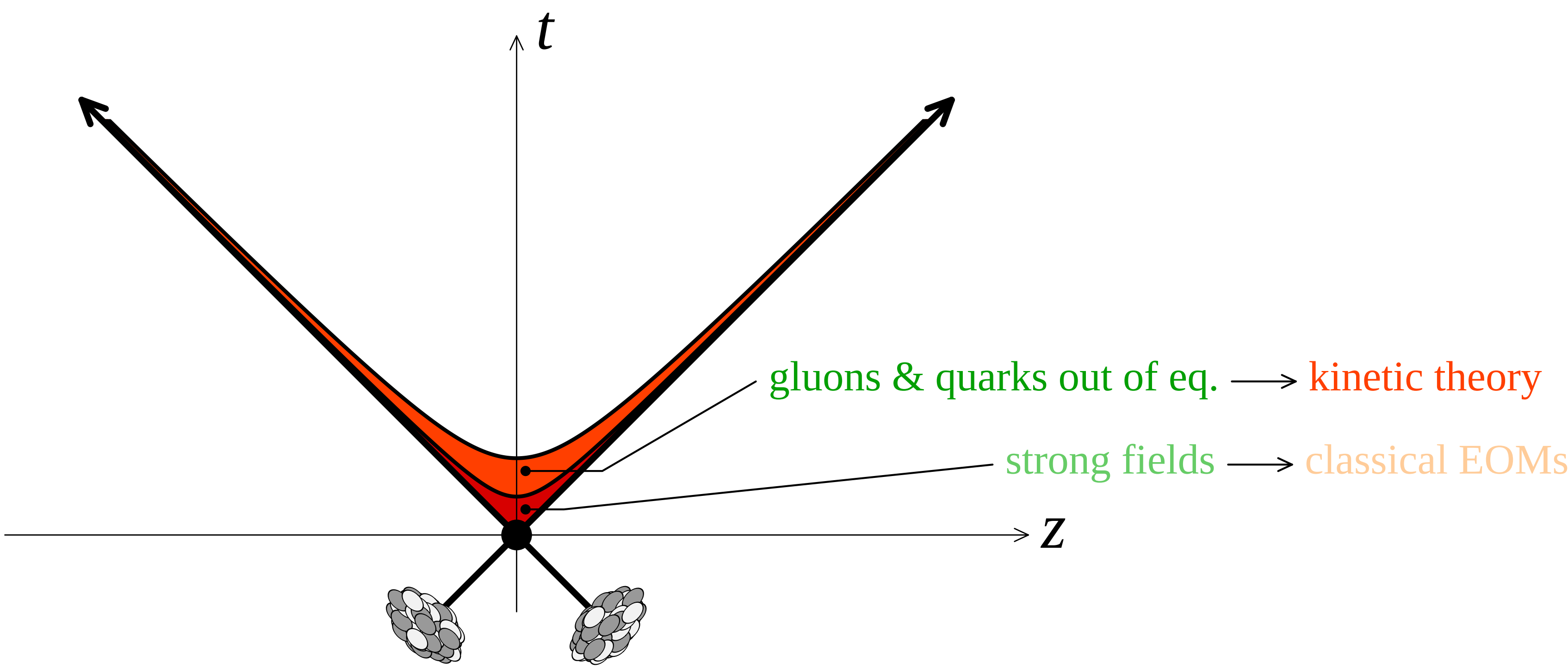
- $\tau \sim 0$ fm/c
- Production of hard particles :
  - jets, direct photons
  - heavy quarks
- calculable with perturbative QCD (collinear factorization)

Stages of a nucleus-nucleus collision



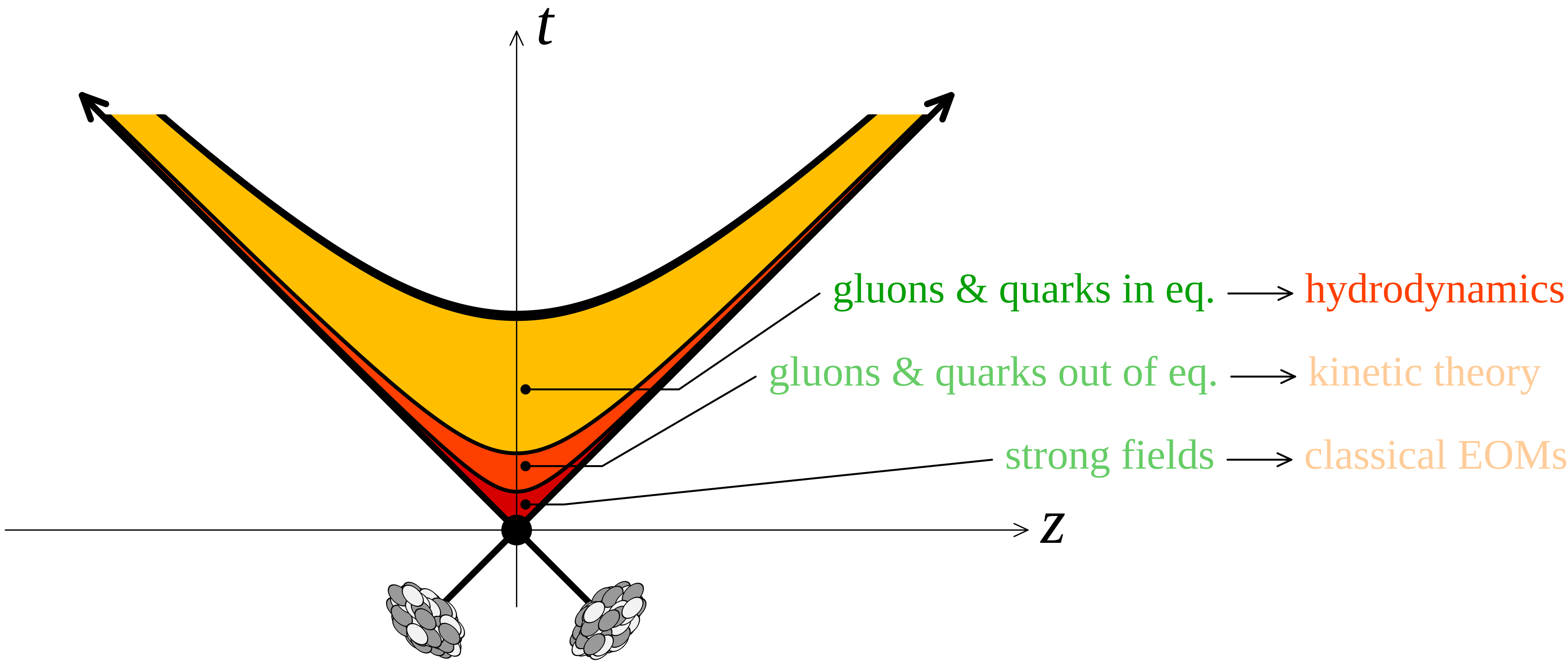
- $\tau \sim 0.2$ fm/c
- Production of semi-hard particles : gluons, light quarks
- relatively small momentum : $p_\perp \lesssim 2$–3 GeV
- make up for most of the multiplicity
- sensitive to the physics of saturation (multiple scatterings)

Stages of a nucleus-nucleus collision



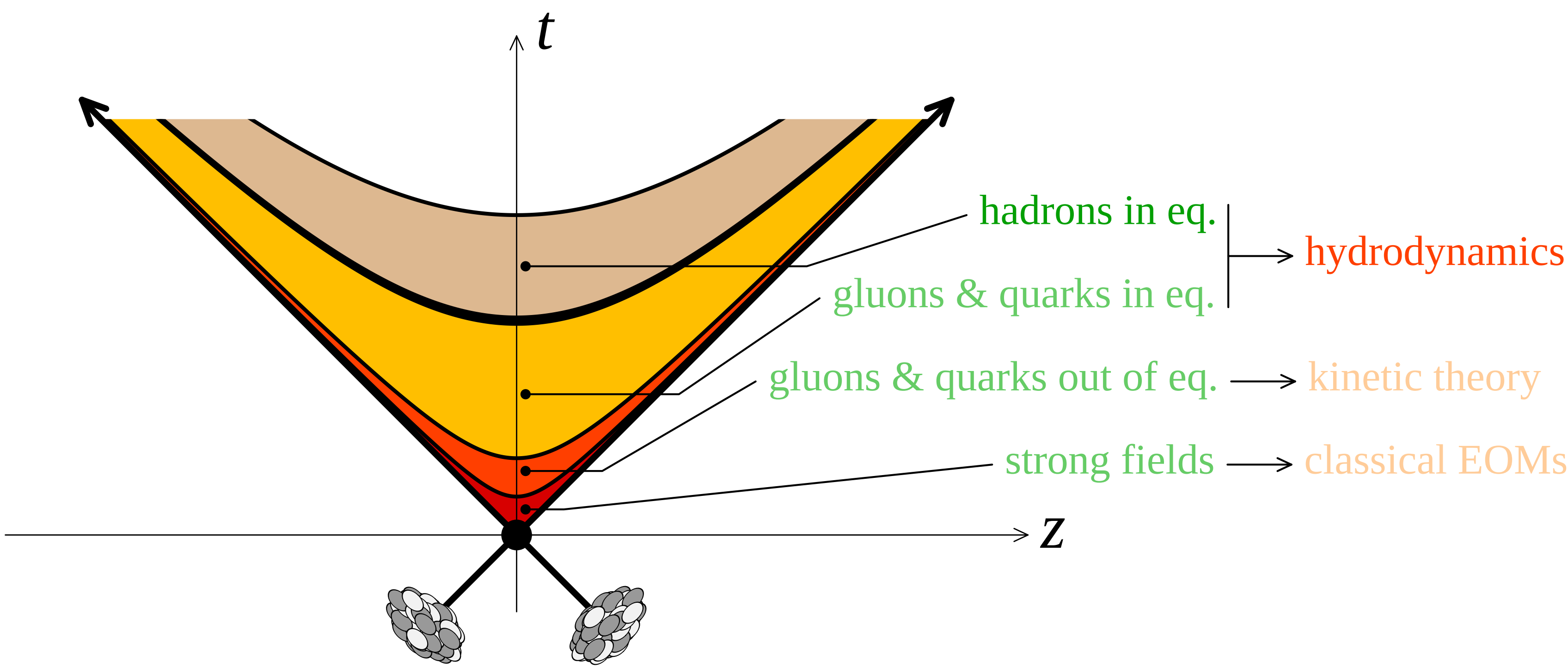
- $\tau \sim 1–2$ fm/c
- Thermalization
  - experiments suggest a fast thermalization
  - but this is still not well understood from QCD

Stages of a nucleus-nucleus collision



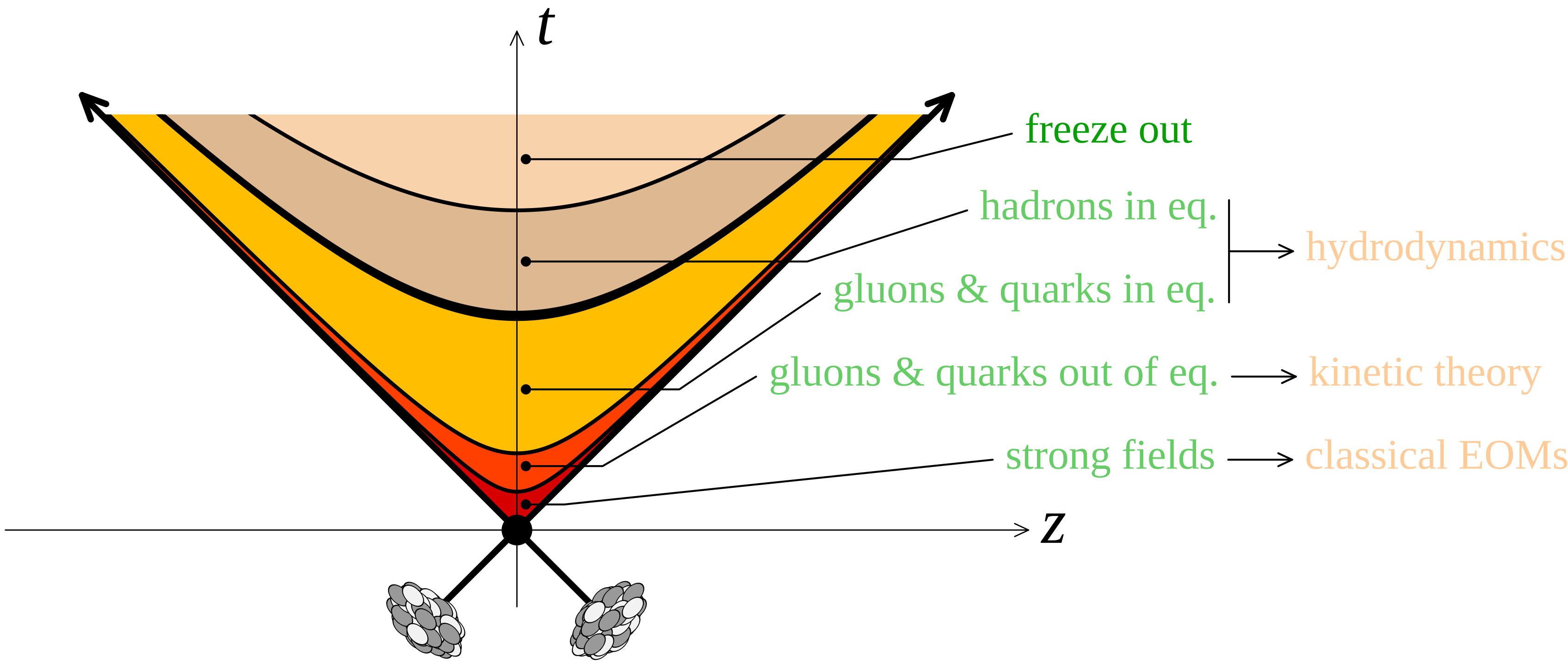
- $2 \le \tau \lesssim 10$ fm/c
- Quark gluon plasma

Stages of a nucleus-nucleus collision



- $10 \lesssim \tau \lesssim 20$ fm/c
- Hot hadron gas

Stages of a nucleus-nucleus collision

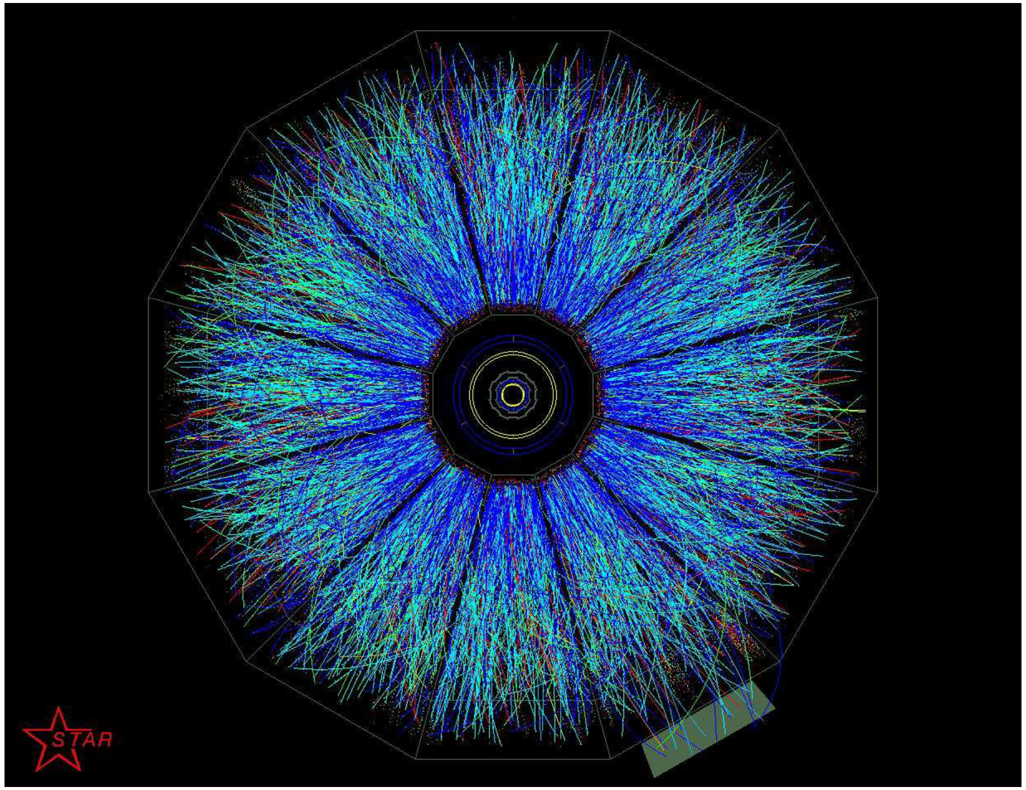


- $\tau \longrightarrow +\infty$
- Chemical freeze-out :
  density too small to have inelastic interactions
- Kinetic freeze-out :
  no more elastic interactions

Small x QCD in AA collisions

- Saturation affects the early stages of heavy ion collisions, up to a time $\tau \sim Q_s^{-1}$
- The dynamics that takes place afterwards blurs the physics of saturation (for instance, if the system reaches thermalization, it does not remember the details of the dynamics at early times)
  $\triangleright$ Saturation affects only inclusive observables, like the overall multiplicity and its energy dependence
  $\triangleright$ Nucleus-nucleus collisions are a limited framework in order to probe saturation
- The Color Glass Condensate provides a (consistent ?) framework in order to compute the spectrum of the particles that are produced initially, which is then used as an initial condition for the rest of the evolution

Small x QCD in AA collisions



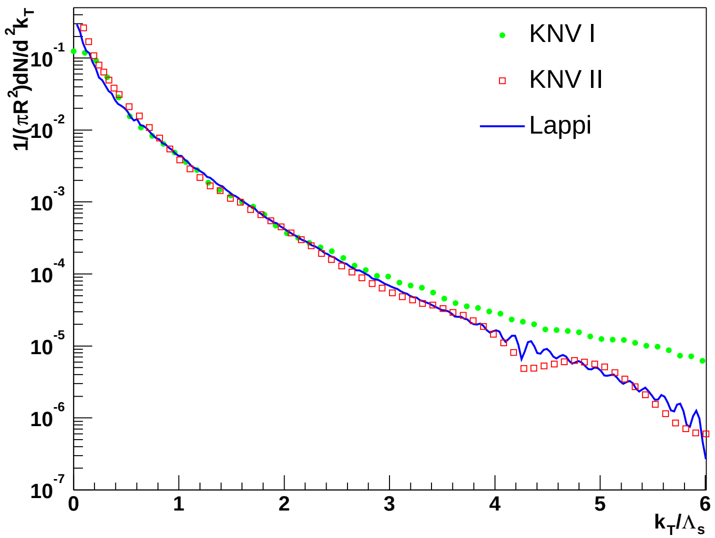
- 99% of the multiplicity below $p_\perp \sim 2$ GeV
- the bulk of particle production comes from (very) low x
  - ▷ high gluon density (even more so in nuclei : $G_A/G_{\rm p} \approx A$)

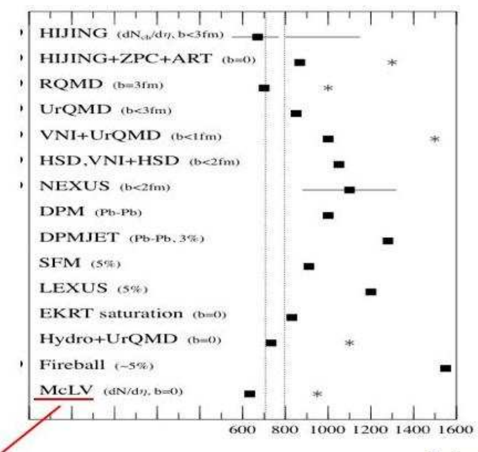
Krasnitz-Venugopalan computation

- Gluon spectrum from retarded classical solutions of Yang-Mills equations (Krasnitz, Venugopalan (1998)) :

$$\frac{d\overline{N}_{LO}}{dY\, d^2\vec{\boldsymbol{p}}_\perp} \propto \int_{x,y} e^{ip\cdot(x-y)} \;\cdots\; \left\langle \mathcal{A}_\mu(x)\mathcal{A}_\nu(y)\right\rangle_{\rho_1,\rho_2}$$

$$\left[\mathcal{D}_\mu, \mathcal{F}^{\mu\nu}\right] = \delta^{\nu+}\delta(x^-)\rho_1(\vec{\boldsymbol{x}}_\perp) + \delta^{\nu-}\delta(x^+)\rho_2(\vec{\boldsymbol{x}}_\perp) \quad \text{with} \lim_{x_0\to-\infty} \mathcal{A}_\mu(x) = 0$$





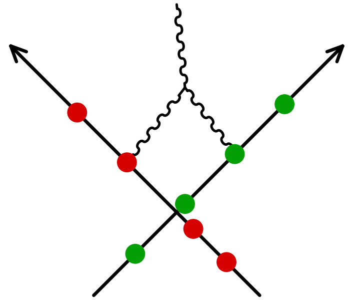
Eskola, QM 2001

Krasnitz, RV

Krasnitz-Venugopalan computation

- For symmetric collisions (e.g. nucleus-nucleus collisions), the two projectiles should be treated on the same footing
- For nucleus-nucleus collisions, there are two strong sources that contribute to the color current :

$$J^\mu \equiv \delta^{\mu+}\delta(x^-)\,\rho_1(\vec{\boldsymbol{x}}_\perp) + \delta^{\mu-}\delta(x^+)\,\rho_2(\vec{\boldsymbol{x}}_\perp)$$



- Average over the sources ρ_1, ρ_2

$$\left\langle \mathcal{O}_{\rm Y}\right\rangle_{\rho_1,\rho_2} = \int [D\rho_1]\,[D\rho_2]\; W_{{\rm Y}_{\rm beam}-{\rm Y}}\,[\rho_1]\; W_{{\rm Y}+{\rm Y}_{\rm beam}}\,[\rho_2]\;\mathcal{O}[\rho_1,\rho_2]$$

- How to compute $\mathcal{O}[\rho_1,\rho_2]$ in the saturation regime ?
- Can this factorization formula be justified ?

Krasnitz-Venugopalan computation

- The retarded nature of the classical fields involved in this approach makes the numerical resolution straightforward :
  - Discretize the spatial coordinates and put the fields on a lattice :

$$\mathcal{A}_\mu(x_0, x, y, z) \longrightarrow \mathcal{A}_\mu(x_0, i, j, k)$$

  - Write the Yang-Mills equations as

$$\frac{\partial}{\partial x^0}\mathcal{A} = F(\mathcal{A}, \vec{\boldsymbol{\nabla}}\mathcal{A})$$

  - Start at some large negative time $x^0_{\rm ini}$ with $\mathcal{A} = 0$
  - To update from x^0 to $x^0 + \Delta x^0$, do :

$$\mathcal{A}(x^0 + \Delta x^0) = \mathcal{A}(x^0) + \Delta x^0 \; F(\mathcal{A}(x^0), \vec{\boldsymbol{\nabla}}\mathcal{A}(x^0))$$

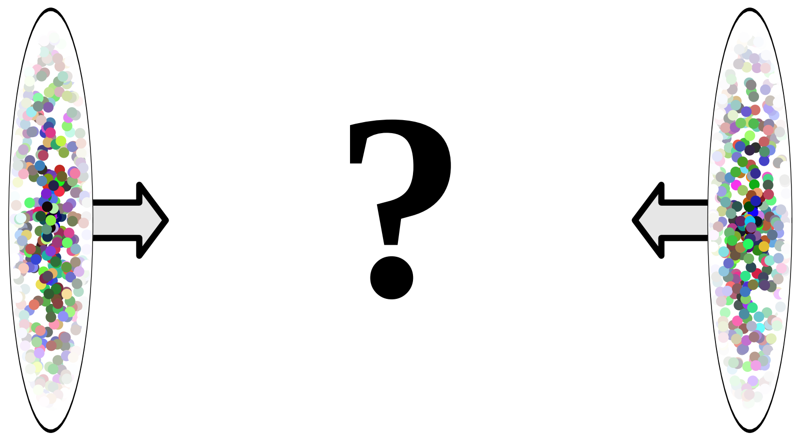
  - At a large positive time, perform a Fourier decomposition of the field $\mathcal{A}$, and compute the gluon spectrum

Goals of this lecture

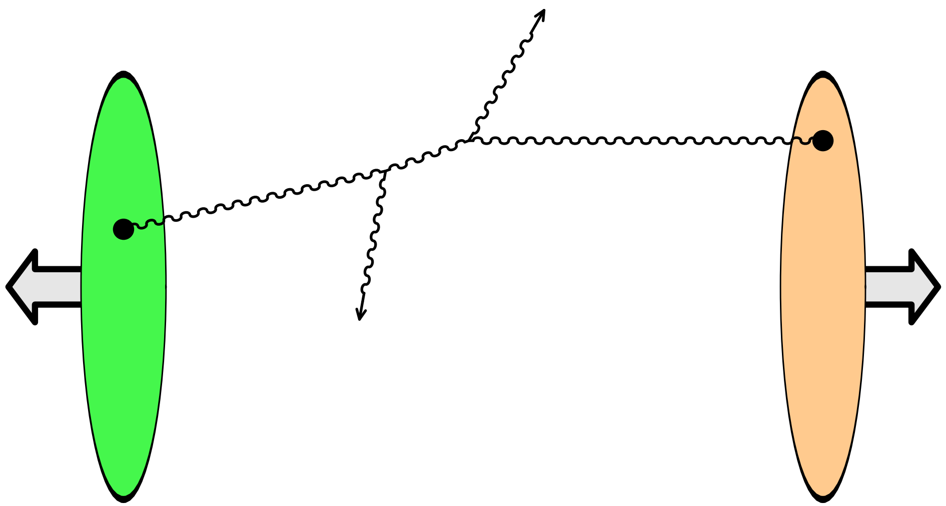
- Why can the gluon yield be obtained from classical solutions of Yang-Mills equations ?
- Why are the boundary conditions retarded ? What would it mean to chose different boundary conditions ?
- Is this a controlled approximation, i.e. the first term in a more systematic expansion ?
- Is it possible to go beyond this computation, and study the 1-loop corrections ? Logs(1/x) and factorization ?

Initial particle production

- Main difficulty : studying the collision of two densely occupied projectiles is much more complicated than the asymmetric cases involving an elementary probe

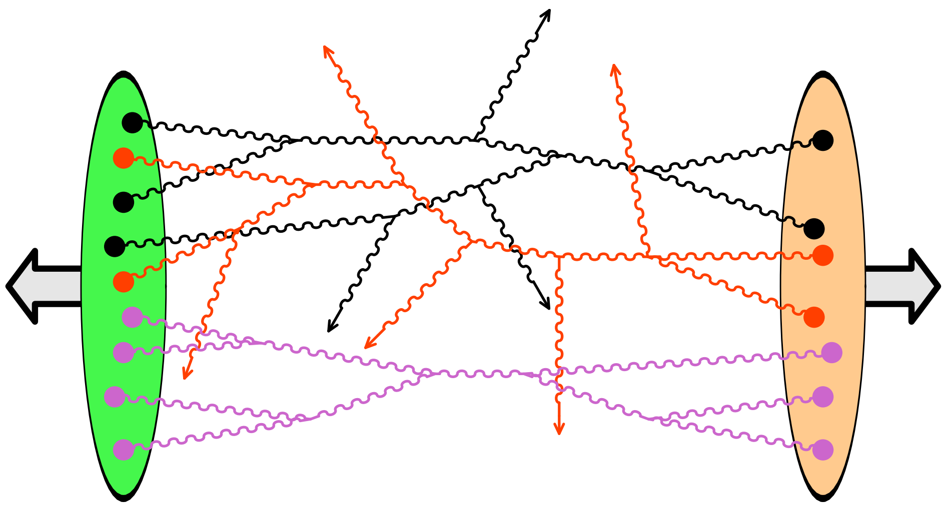


Initial particle production



- Dilute regime : one parton in each projectile interact

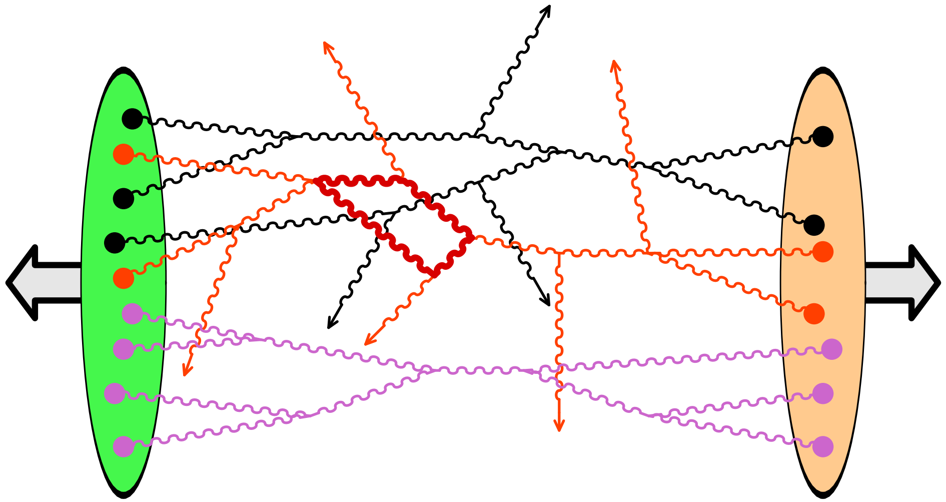
Initial particle production



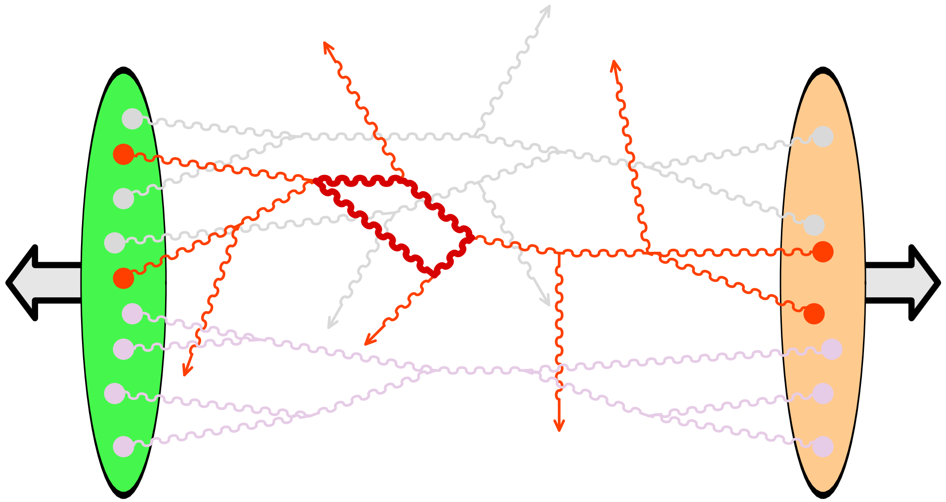
- Dilute regime : one parton in each projectile interact
- Dense regime : multiparton processes become crucial (+ pileup of many simultaneous scatterings)

Power counting and Bookkeeping

Power counting



Power counting

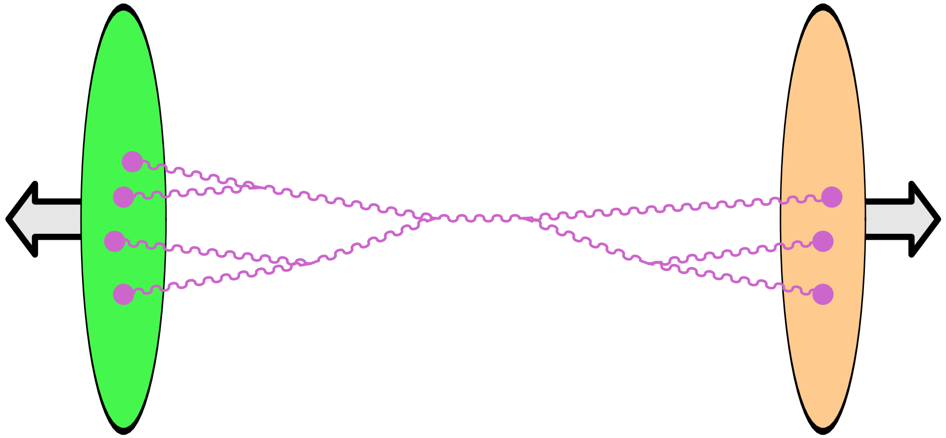


- In the saturated regime, the sources are of order $1/g$ (because $\langle\rho\rho\rangle \sim$ occupation number $\sim 1/\alpha_s$)
- The order of each connected diagram is given by :

$$\frac{1}{g^2}\; g^{\#\text{ produced gluons}}\; g^{2(\#\text{ loops})}$$

- The total order of a graph is the product of the orders of its disconnected subdiagrams $\triangleright$ somewhat messy...

Vacuum diagrams



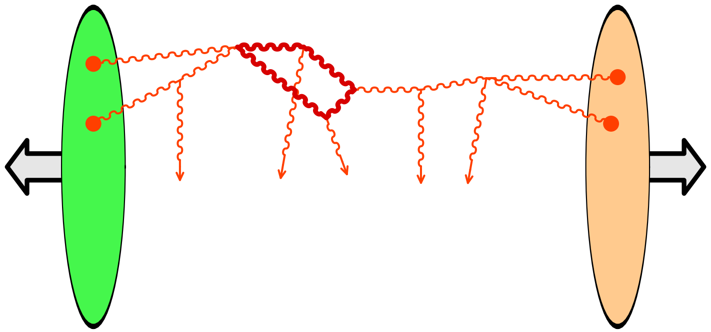
- Vacuum diagrams do not produce any gluon. They are contributions to the vacuum to vacuum amplitude $\langle 0_{\rm out} | 0_{\rm in} \rangle$
- The order of a connected vacuum diagram is given by :

$$g^{-2} \quad g^{2(\#\ \text{loops})}$$

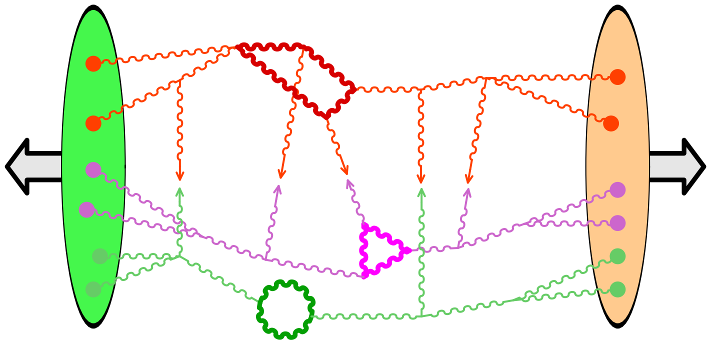
- Relation between connected and non connected vacuum diagrams :

$$\sum \begin{pmatrix} \text{all the vacuum} \\ \text{diagrams} \end{pmatrix} = \exp\left\{ \sum \begin{pmatrix} \text{simply connected} \\ \text{vacuum diagrams} \end{pmatrix} \right\} = e^{iV[j]}$$

Bookkeeping

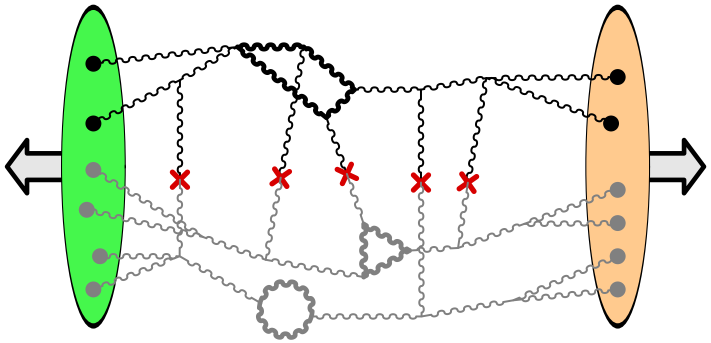


Bookkeeping



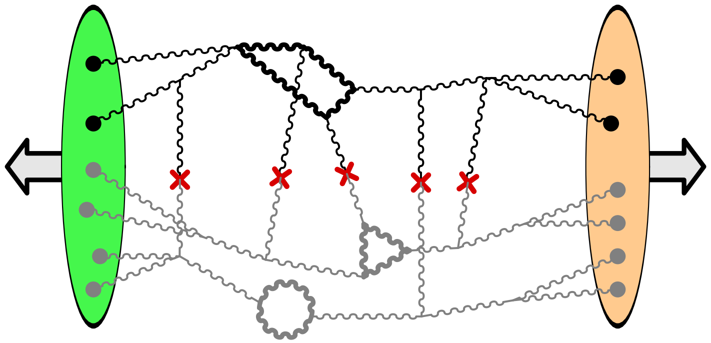
- Consider squared amplitudes (including interference terms) rather than the amplitudes themselves

Bookkeeping



- Consider squared amplitudes (including interference terms) rather than the amplitudes themselves
- See them as cuts through vacuum diagrams
  cut propagator : $2\pi\theta(-p^0)\delta(p^2)$

Bookkeeping



- Consider squared amplitudes (including interference terms) rather than the amplitudes themselves
- See them as cuts through vacuum diagrams
  cut propagator : $2\pi\theta(-p^0)\delta(p^2)$
- The sum of the vacuum diagrams, $\exp(iV[j])$, is the generating functional for time-ordered products of fields :

$$\big\langle 0_{\rm out}\big|T A(x_1)\cdots A(x_n)\big|0_{\rm in}\big\rangle = \frac{\delta}{\delta j(x_1)}\cdots\frac{\delta}{\delta j(x_n)}\; e^{iV[j]}$$

Bookkeeping

- The probability of producing exactly n particles is :

$$P_n = \frac{1}{n!}\int \frac{d^3\vec{\boldsymbol{p}}_1}{(2\pi)^3 2E_1}\cdots\frac{d^3\vec{\boldsymbol{p}}_n}{(2\pi)^3 2E_n}\left|\left\langle \vec{\boldsymbol{p}}_1\cdots\vec{\boldsymbol{p}}_{n\,\mathrm{out}}\middle|0_{\mathrm{in}}\right\rangle\right|^2$$

- Exercise :
  - Show that : $P_n = \frac{1}{n!}\,\mathcal{C}^n\, e^{iV[j_+]}\, e^{-iV^*[j_-]}\Big|_{j_+=j_-=j}$

with
$$\begin{cases} \mathcal{C} \equiv \int_{x,y} G^0_{+-}(x,y)\,\square_x\square_y\,\frac{\delta}{\delta j_+(x)}\,\frac{\delta}{\delta j_-(y)} \\ G^0_{+-}(x,y) \equiv \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\, 2\pi\theta(-p^0)\delta(p^2) \end{cases}$$

  - The sum of all the cut vacuum diagrams, with sources j_+ on one side of the cut and j_- on the other side, can be written as :

$$\sum\left(\begin{matrix}\text{all the cut}\\ \text{vacuum diagrams}\end{matrix}\right) = e^{\mathcal{C}}\, e^{iV[j_+]}\, e^{-iV^*[j_-]}$$

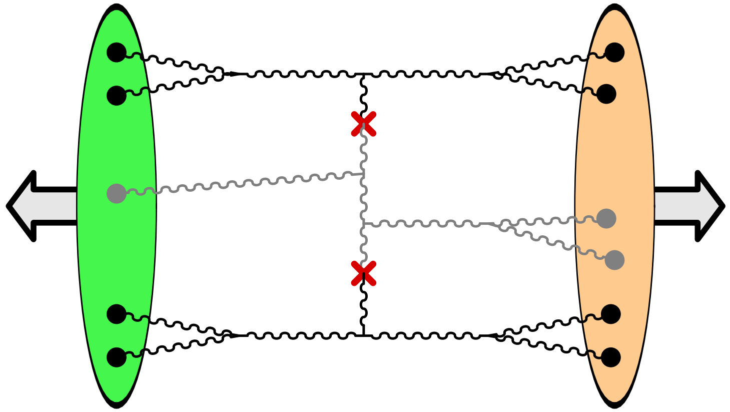
▷ Note : if we set $j_+ = j_- = j$, then this is $\sum_n P_n = 1$

Bookkeeping

- Reminder :

$$\mathcal{C} \equiv \int_{x,y} G^0_{+-}(x,y)\,\square_x \square_y \,\frac{\delta}{\delta j_+(x)}\frac{\delta}{\delta j_-(y)}$$

- Consider a generic cut vacuum diagram :



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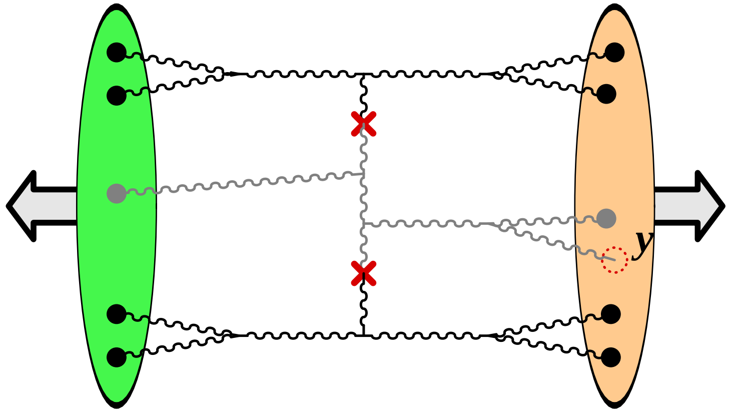

Bookkeeping

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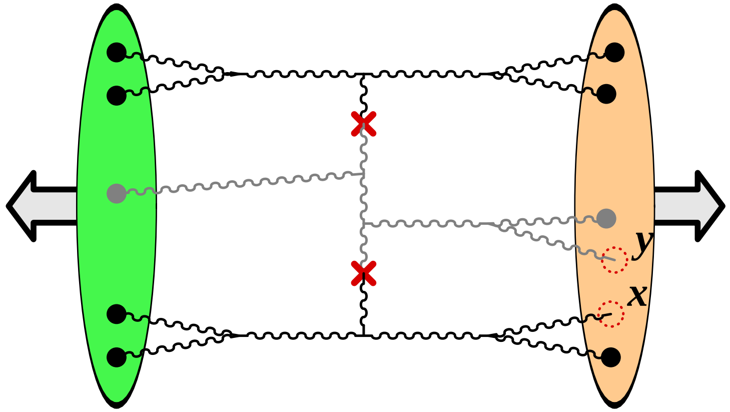

Bookkeeping

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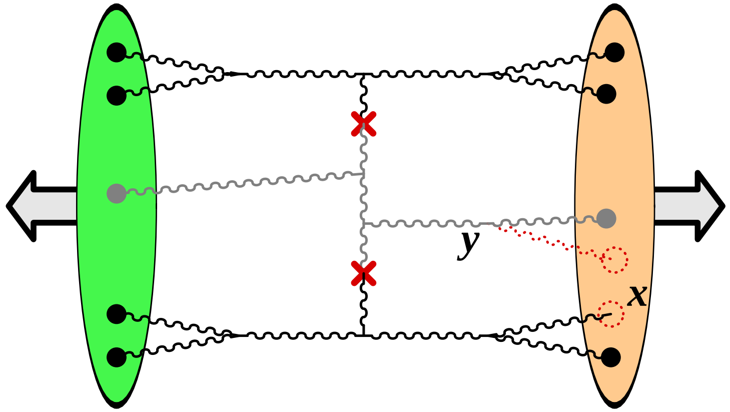
- Consider a generic cut vacuum diagram :

$$\square_y \quad \longrightarrow$$



Bookkeeping

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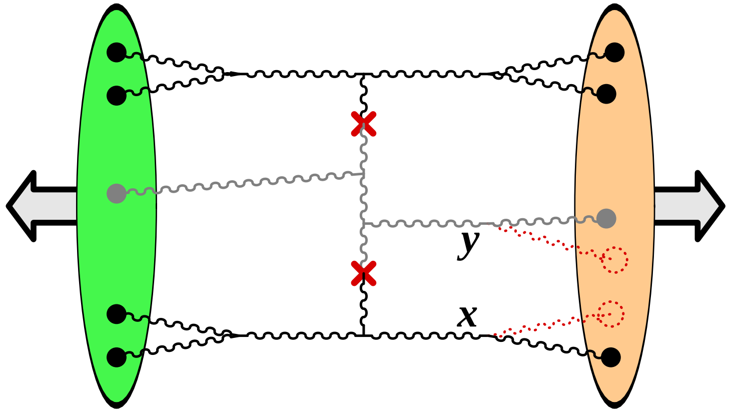
- Consider a generic cut vacuum diagram :

$$\square_x \quad \longrightarrow$$



Bookkeeping

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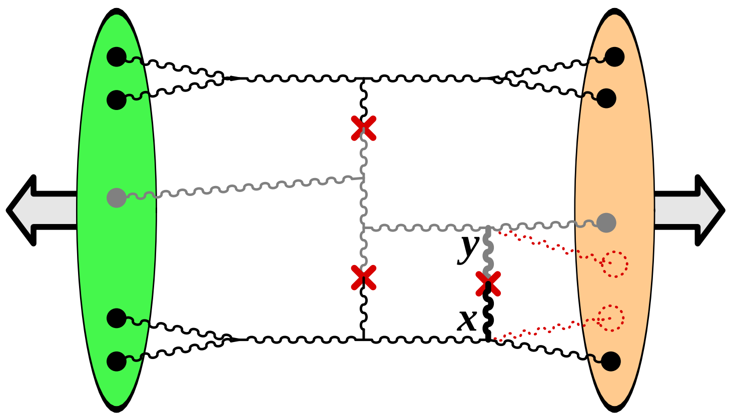


▷ the operator $\mathcal{C}$ removes two sources (one in the amplitude and one in the complex conjugated amplitude), and creates a new cut propagator

Bookkeeping

- The operator $\mathcal{C}$ can be used to derive many useful formulas :

$$F(z) = \sum_{n=0}^{+\infty} z^n \, P_n = e^{z\mathcal{C}} \, e^{iV[j_+]} \, e^{-iV^*[j_-]}\Big|_{j_+=j_-=j}$$

▷ sum of all cut vacuum graphs, where each cut is weighted by z

$$\overline{N} = F'(1) = \mathcal{C} \, e^{\mathcal{C}} \, e^{iV[j_+]} \, e^{-iV^*[j_-]}\Big|_{j_+=j_-=j}$$

$$\overline{N(N-1)} = F''(1) = \mathcal{C}^2 \, e^{\mathcal{C}} \, e^{iV[j_+]} \, e^{-iV^*[j_-]}\Big|_{j_+=j_-=j}$$

- Benefits :
  - The tracking of infinite sets of Feynman diagrams has been replaced by simple algebraic manipulations
  - The use of the identity $e^{\mathcal{C}} \, e^{iV[j_+]} \, e^{-iV^*[j_-]}\Big|_{j_+=j_-} = 1$ renders automatic an important cancellation that would be hard to see at the level of diagrams (somewhat related to AGK)

Inclusive gluon spectrum

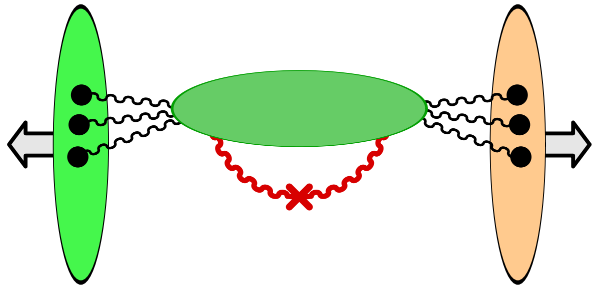
CERN

First moment of the distribution

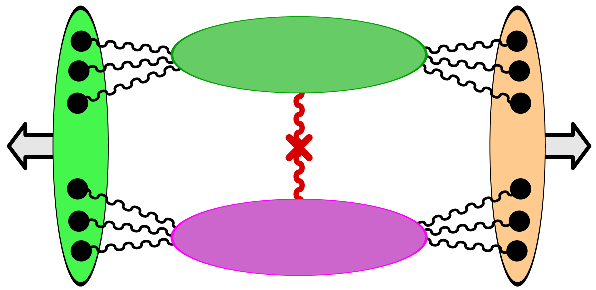
- It is easy to express the average multiplicity as :

$$\overline{N} = \sum_n n\, P_n = \mathcal{C}\left\{ e^{\mathcal{C}}\, e^{iV[j_+]}\, e^{-iV^*[j_-]} \right\}_{j_+=j_-=j}$$

- $\overline{N}$ is obtained by the action of $\mathcal{C}$ on the sum of all the cut vacuum diagrams. There are two kind of terms :
  - $\mathcal{C}$ picks two sources in the same connected cut diagram



  - $\mathcal{C}$ picks two sources in two distinct connected cut diagrams

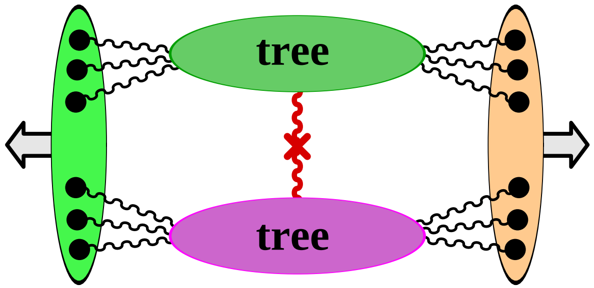


Gluon multiplicity at LO

- At LO, only tree diagrams contribute
  - ▷ the first type of topologies can be neglected (they have at least one loop)

- In each blob, we must sum over all the tree diagrams, and over all the possible cuts :

$$\overline{N}_{LO} = \sum_{\text{trees}} \sum_{\text{cuts}}$$



- Reminder : at the end, the sources on both sides of the cut must be set equal :

$$j_+ = j_-$$

Sum over the cuts

- In the previous diagrams, one must sum over all the possible ways of cutting lines inside the blobs
- This can be achieved via Cutkosky's cutting rules :
  - A vertex is $-ig$ on one side of the cut, and $+ig$ on the other side
  - There are four propagators, depending on the location w.r.t. the cut of the vertices they connect :

$$G^0_{++}(p) = i/(p^2 - m^2 + i\epsilon) \qquad \text{(standard Feynman propagator)}$$
$$G^0_{--}(p) = -i/(p^2 - m^2 - i\epsilon) \qquad \text{(complex conjugate of } G^0_{++}(p))$$
$$G^0_{+-}(p) = 2\pi\theta(-p^0)\delta(p^2 - m^2)$$
$$G^0_{-+}(p) = 2\pi\theta(p^0)\delta(p^2 - m^2)$$

  - At each vertex of a given diagram, sum over the types $+$ and $-$ (2^n terms for a diagram with n vertices)

Sum over the cuts

- When summing over the cuts with $j_+ = j_-$, we only get combinations of propagators such as :

$$G^0_{++}(p) - G^0_{+-}(p) = \boldsymbol{\frac{i}{p^2 - m^2 + i\epsilon}} - 2\pi\theta(-p^0)\delta(p^2 - m^2)$$

Sum over the cuts

- When summing over the cuts with $j_+ = j_-$, we only get combinations of propagators such as :

$$G^0_{++}(p) - G^0_{+-}(p) = \mathrm{PP}\left[\frac{i}{p^2-m^2}\right] + \pi\delta(p^2-m^2) - 2\pi\theta(-p^0)\delta(p^2-m^2)$$

$$\overbrace{\mathbf{1 = \theta(p^0) + \theta(-p^0)}}$$

Sum over the cuts

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$$G^0_{++}(p) - G^0_{+-}(p) = \mathrm{PP}\left[\frac{i}{p^2 - m^2}\right] + \pi\,\underbrace{\left[\theta(p^0) - \theta(-p^0)\right]}_{\textbf{sign}\,(\boldsymbol{p^0})}\,\delta(p^2 - m^2)$$

Sum over the cuts

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$$G^0_{++}(p) - G^0_{+-}(p) = \frac{\boldsymbol{i}}{\boldsymbol{p^2 - m^2 + i\,\mathrm{sign}(p^0)\epsilon}}$$

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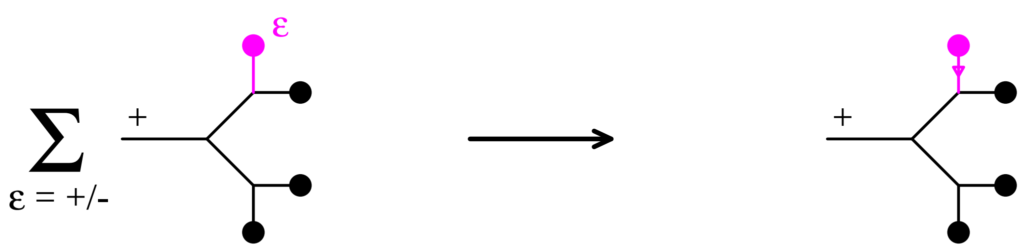
- When summing over the cuts with $j_+ = j_-$, we only get combinations of propagators such as :

$$G^0_{++}(p) - G^0_{+-}(p) = \boldsymbol{G^0_R(p)}$$

- Similarly : $G^0_{-+}(p) - G^0_{--}(p) = G^0_R(p)$

- Starting from the "leaves" of the trees, use these formulas recursively in order to replace all the $G^0_{\pm\pm}$ propagators by retarded propagators. Example :



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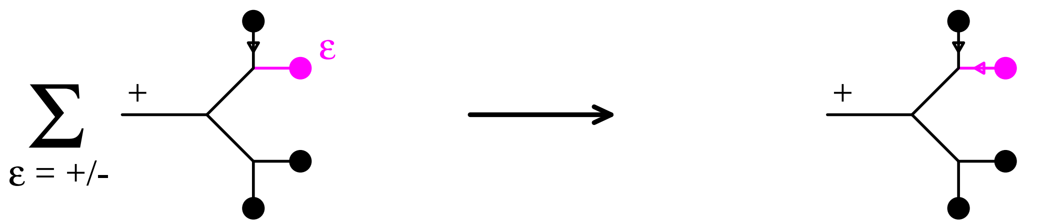

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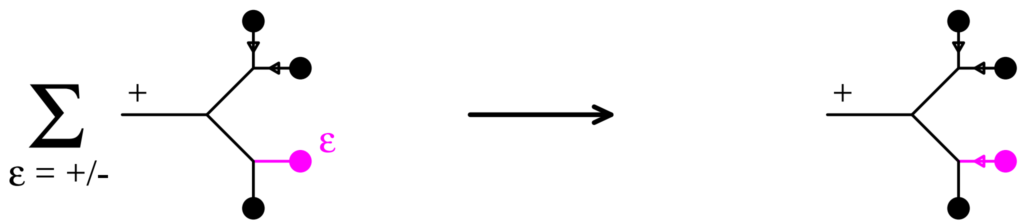
- Starting from the “leaves” of the trees, use these formulas recursively in order to replace all the $G^0_{\pm\pm}$ propagators by retarded propagators. Example :



Sum over the cuts

- When summing over the cuts with $j_+ = j_-$, we only get combinations of propagators such as :

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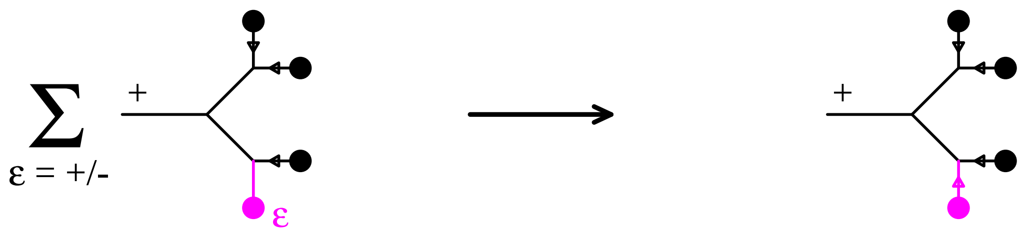

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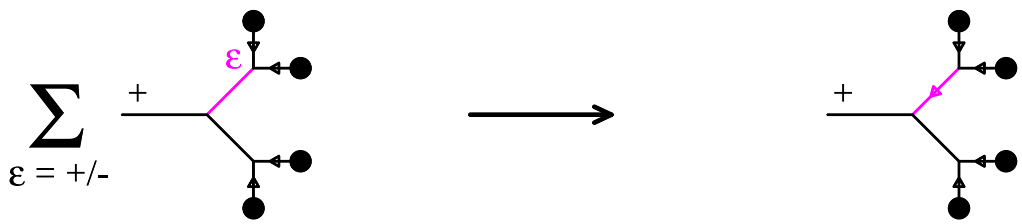

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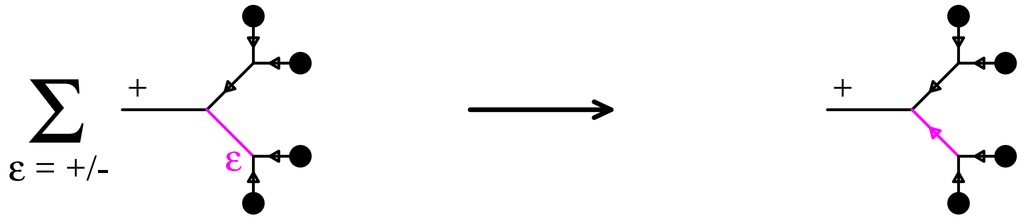

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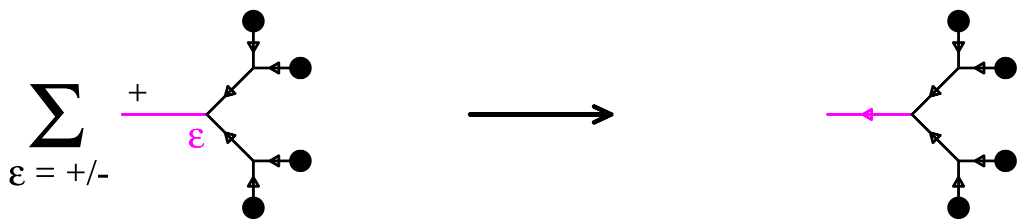


$\triangleright$ we have a sum of tree diagrams with retarded propagators

Retarded classical solution

- The sum of all the tree diagrams constructed with retarded propagators is the solution of Yang-Mills equations,

$$[D_\mu, F^{\mu\nu}] = J^\nu \ , \quad \text{with retarded boundary condition} \quad A^\mu(x_0 = -\infty) = 0$$

- Proof (for a scalar theory). The classical EOM reads

$$(\square + m^2)\, \varphi(x) + \frac{g}{2}\varphi^2(x) = j(x)$$

- Write the Green's formula for the retarded solution that obeys $\varphi(x) = 0$ at $x^0 = -\infty$:

$$\varphi(x) = \int d^4y\, G_R^0(x-y)\left[-i\,\frac{g}{2}\varphi^2(y) + i\, j(y)\right]$$

where $G_R^0(x-y)$ is the free retarded propagator

Retarded classical solution

- One can construct the solution iteratively, by using in the r.h.s. the solution found in the previous orders

- Order g^0 :

$$\varphi_{(0)}(x) = \int d^4y\, G_R^0(x-y)\, i\, j(y)$$

- Order g^1 :

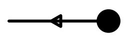
$$\varphi_{(0)}(x) + \varphi_{(1)}(x) = \int d^4y\, G_R^0(x-y) \left[-i\,\frac{g}{2}\varphi_{(0)}^2(y) + i\, j(y)\right]$$

i.e.

$$\varphi_{(1)}(x) = -i\,\frac{g}{2} \int d^4y\, G_R^0(x-y) \left[\int d^4z\, G_R^0(y-z)\, i\, j(z)\right]^2$$

Retarded classical solution

- The diagrammatic expansion of this classical solution is :



Retarded classical solution

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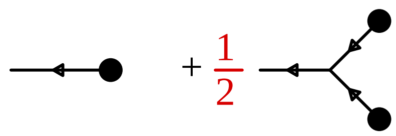

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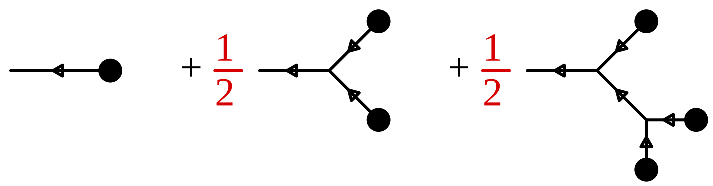

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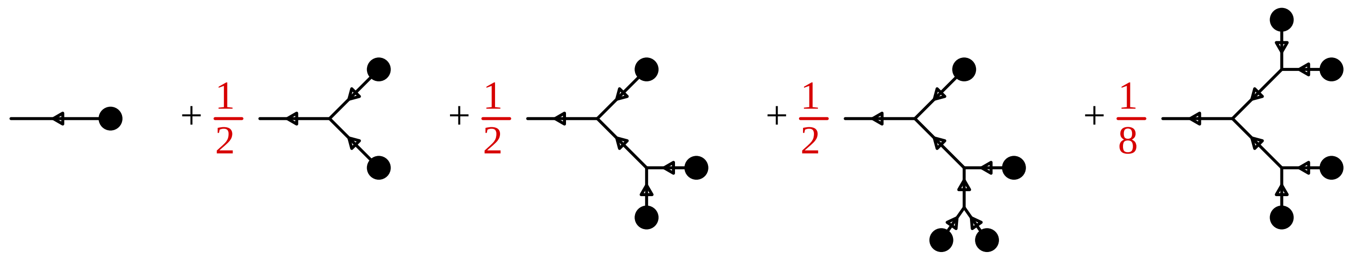

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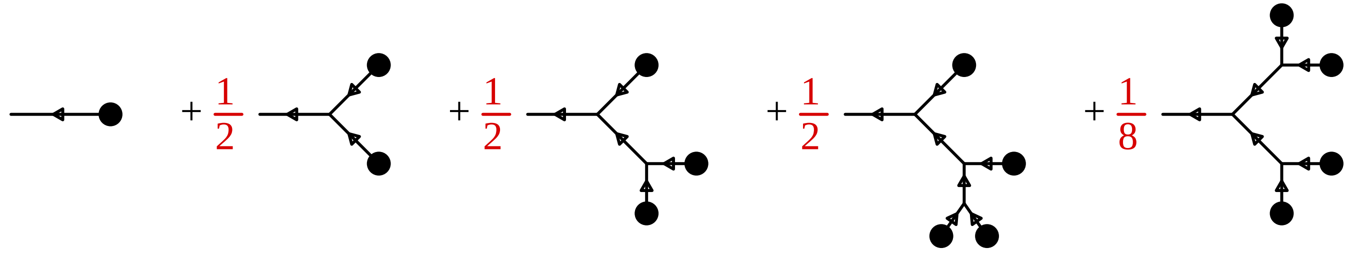


- The classical solution is given by the sum of all the tree diagrams with retarded propagators

Gluon spectrum at LO

Krasnitz, Nara, Venugopalan (1999 – 2001), Lappi (2003)

- The gluon spectrum at LO is given by :

$$\frac{d\overline{N}_{LO}}{dY\,d^2\vec{\boldsymbol{p}}_\perp} = \frac{1}{16\pi^3}\int_{x,y} e^{ip\cdot(x-y)}\,\Box_x\Box_y \sum_\lambda \epsilon^\mu_\lambda \epsilon^\nu_\lambda \;\; \mathcal{A}_\mu(x)\mathcal{A}_\nu(y)$$

where $\mathcal{A}_\mu(x)$ is the retarded solution of Yang-Mills equations such that

$$\lim_{x^0\to-\infty} \mathcal{A}_\mu(x) = 0$$

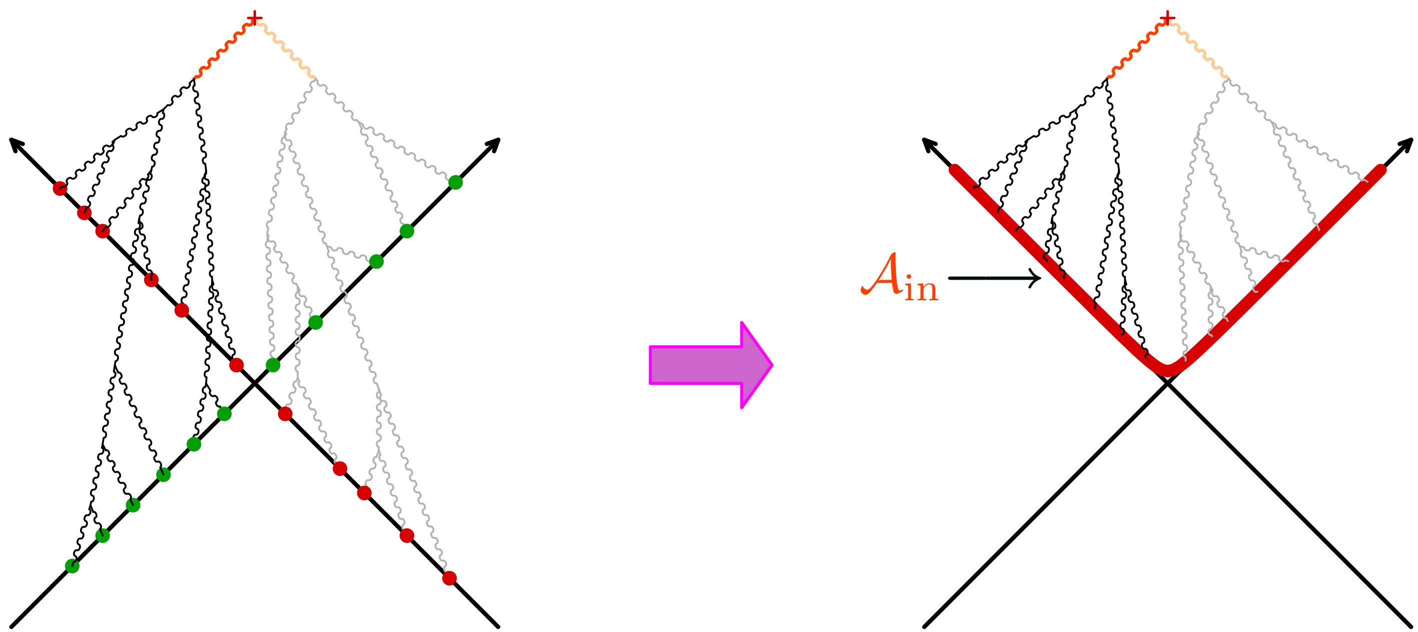
- Note :

$$\int d^4x\; e^{ip\cdot x}\;\Box_x\,\mathcal{A}_\mu(x) = \lim_{x^0\to+\infty}\int d^3\vec{\boldsymbol{x}}\; e^{ip\cdot x}\;\left[\partial_0 - iE_p\right]\mathcal{A}_\mu(x)$$

▷ the gluon spectrum depends only of the fields at late time

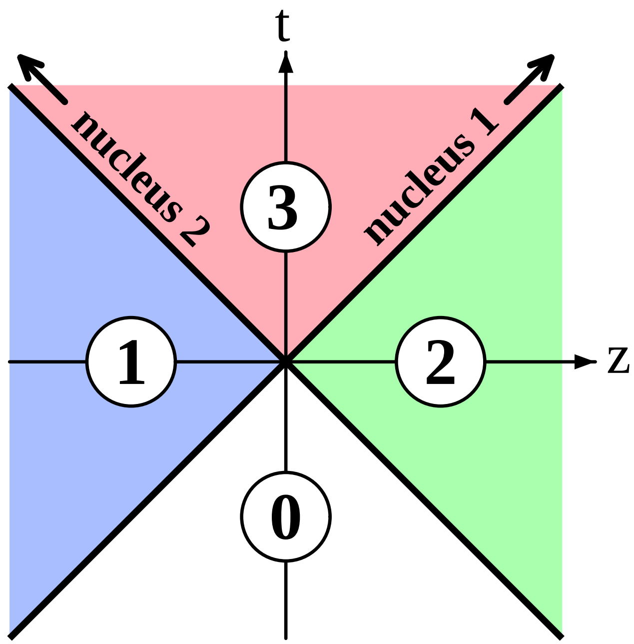
Gluon spectrum at LO

- The calculation is done in the gauge : $x^+\mathcal{A}^- + x^-\mathcal{A}^+ = 0$
  - ▷ $\mathcal{A}^- = 0$ at $z = t$ and $\mathcal{A}^+ = 0$ at $z = -t$
  - ▷ the produced gauge field does not modify the currents J^+, J^-
- In this gauge, one can find analytically the field at $\tau = 0^+$, and then let it evolve according to the vacuum Yang-Mills equations (because there are no sources at $\tau > 0$)



Gluon spectrum at LO

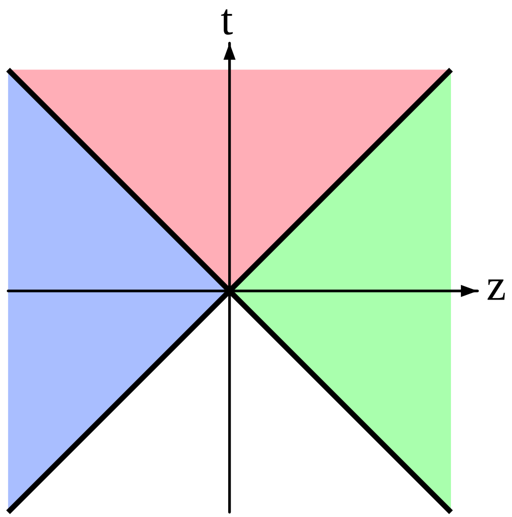
- Space-time structure of the classical color field:



- Region 0 : no causal relation to either nuclei
- Region 1 : causal relation to the 1st nucleus only
- Region 2 : causal relation to the 2nd nucleus only
- Region 3 : causal relation to both nuclei

Gluon spectrum at LO

- Propagation through region 0:

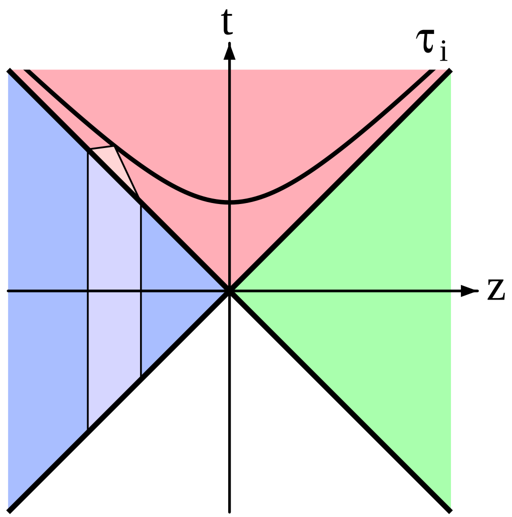


▷ trivial : the classical field is entirely determined by the initial condition, i.e.

$$\mathcal{A}^\mu = 0$$

Gluon spectrum at LO

- Propagation through region 1:



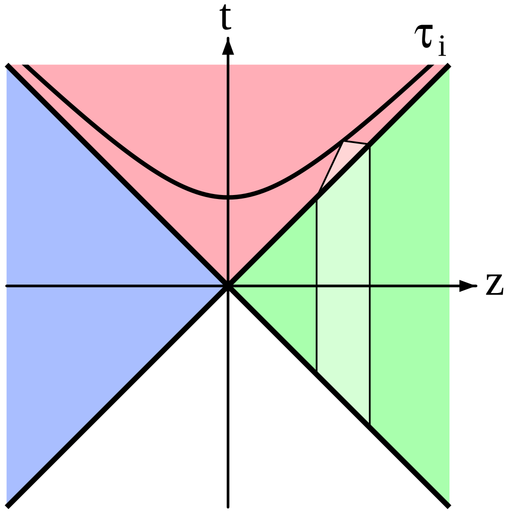
▷ the Yang-Mills equation can be solved analytically when there is only one nucleus :

$$\mathcal{A}_1^+ = \mathcal{A}_1^- = 0 \quad , \quad \mathcal{A}_1^i = \frac{i}{g} U_1(\vec{\boldsymbol{x}}_\perp)\partial^i U_1^\dagger(\vec{\boldsymbol{x}}_\perp)$$

with $\quad U_1(\vec{\boldsymbol{x}}_\perp) = T_+ \exp\, ig \int dx^+ T^a \frac{1}{\boldsymbol{\nabla}_\perp^2} \rho_1^a(x^+, \vec{\boldsymbol{x}}_\perp)$

Gluon spectrum at LO

- Propagation through region 2:



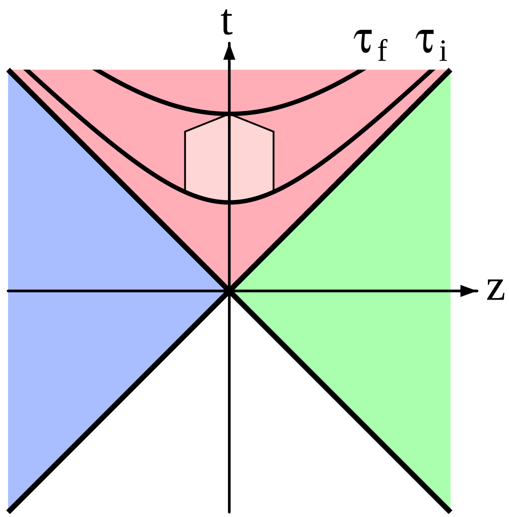
▷ the Yang-Mills equation can be solved analytically when there is only one nucleus :

$$\mathcal{A}_2^+ = \mathcal{A}_2^- = 0 \quad , \quad \mathcal{A}_2^i = \frac{i}{g} U_2(\vec{\boldsymbol{x}}_\perp) \partial^i U_2^\dagger(\vec{\boldsymbol{x}}_\perp)$$

with $\quad U_2(\vec{\boldsymbol{x}}_\perp) = T_- \exp\; ig \int dx^- T^a \frac{1}{\boldsymbol{\nabla}_\perp^2} \rho_2^a(x^-, \vec{\boldsymbol{x}}_\perp)$

Gluon spectrum at LO

- Propagation through region 3:



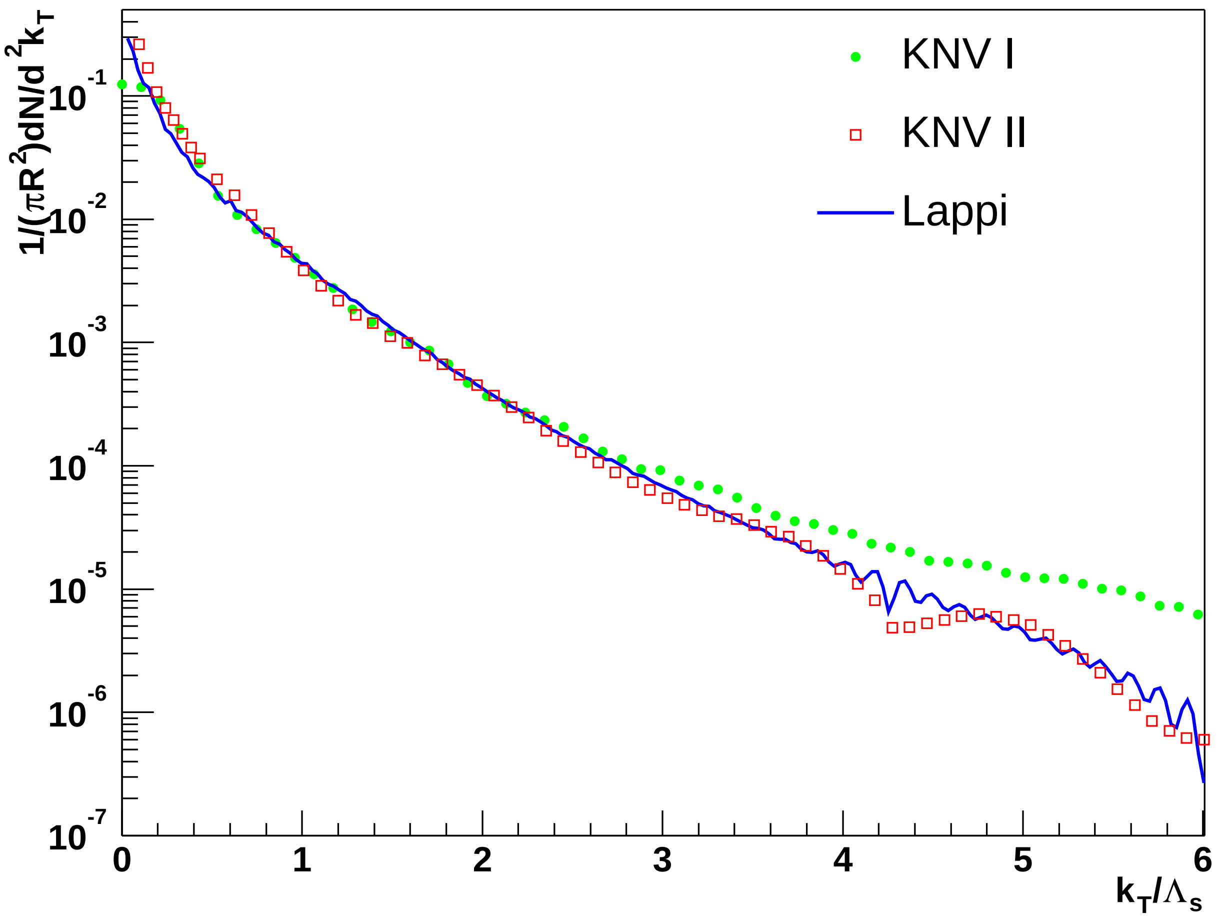
▷ one must solve numerically the Yang-Mills equations with the following initial condition at $\tau_i = 0^+$:

$$\mathcal{A}^i(\tau = 0, \vec{\boldsymbol{x}}_\perp) = \mathcal{A}_1^i(\vec{\boldsymbol{x}}_\perp) + \mathcal{A}_2^i(\vec{\boldsymbol{x}}_\perp)$$

$$\mathcal{A}^\eta(\tau = 0, \vec{\boldsymbol{x}}_\perp) = \frac{ig}{2}\left[\mathcal{A}_1^i(\vec{\boldsymbol{x}}_\perp)\ ,\ \mathcal{A}_2^i(\vec{\boldsymbol{x}}_\perp)\right]$$

$\mathcal{A}^\tau = 0$ (gauge condition)

Gluon spectrum at LO



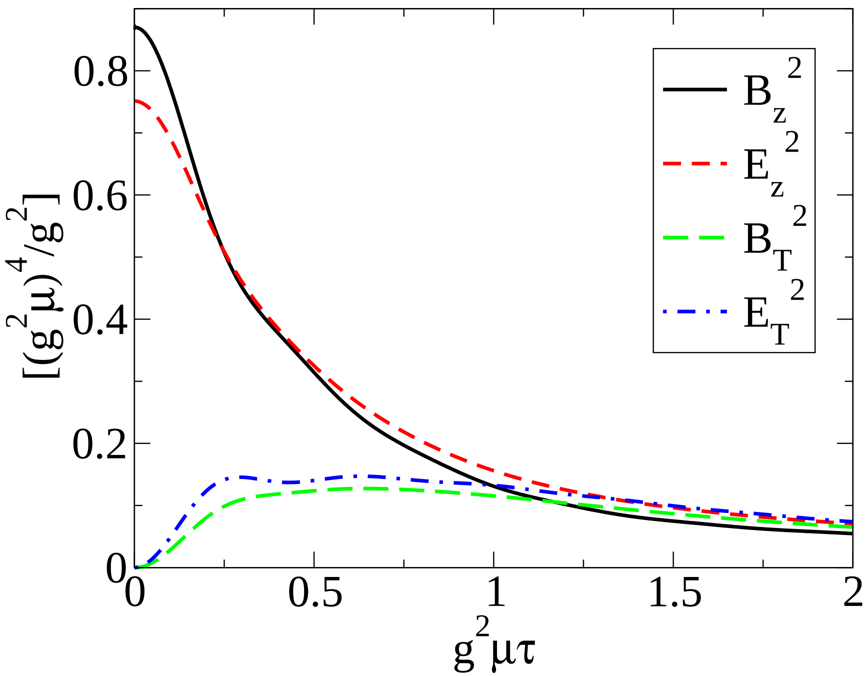
- Lattice artifacts at large momentum
  (they do not affect much the overall number of gluons)
- Important softening at small $k_\perp$ compared to pQCD (saturation)

Initial Glasma fields

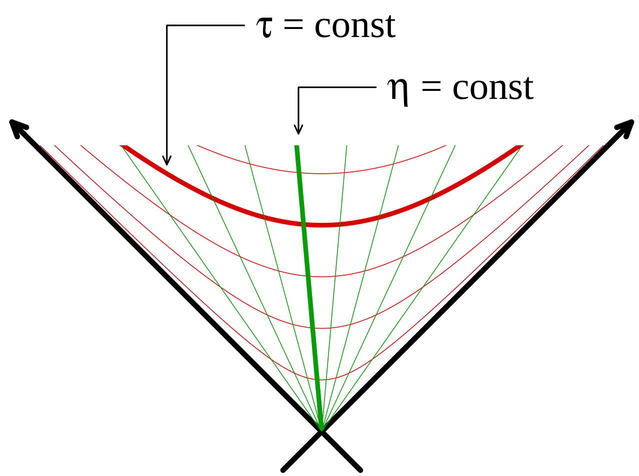
Lappi, McLerran (2006)   (Semantics : Glasma $\equiv$ Glas[s - plas]ma)

- Before the collision, the chromo-$\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ fields are localized in two sheets transverse to the beam axis
- Immediately after the collision, the chromo-$\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ fields have become longitudinal :

$$\boldsymbol{E}^z = ig\left[\mathcal{A}_1^i, \mathcal{A}_2^i\right] \quad , \qquad \boldsymbol{B}^z = ig\epsilon^{ij}\left[\mathcal{A}_1^i, \mathcal{A}_2^i\right]$$



Initial conditions and boost invariance



- Gauge condition : $x^+\mathcal{A}^- + x^-\mathcal{A}^+ = 0$

$$\Rightarrow \quad \mathcal{A}^\pm(x) \quad = \quad \pm\, x^\pm\, \beta(\tau,\eta,\vec{\boldsymbol{x}}_\perp)$$

- Initial values at $\tau = 0^+$: $\mathcal{A}^i(0^+,\eta,\vec{\boldsymbol{x}}_\perp)$ and $\beta(0^+,\eta,\vec{\boldsymbol{x}}_\perp)$ do not depend on the rapidity η

  ▷ $\mathcal{A}^i$ and β remain independent of η at all times (invariance under boosts in the z direction)

  ▷ numerical resolution performed in $1+2$ dimensions

Less inclusive quantities

- One can also express less inclusive quantities at Leading Order in terms of classical solutions of Yang-Mills equations, but with complicated boundary conditions in general

- Example: Generating function
  - Definition : $F(z) \equiv \sum_{n=0}^{\infty} P_n \; z^n$
  - $F'(z)/F(z)$ has the same diagrammatic expansion as $\overline{N}$, but with each cut propagator multiplied by z
  - Now, performing the sum over the cuts does not give retarded propagators :

retarded propagator

  - $F'(z)/F(z)$ can be written in terms of classical solutions of Yang-Mills, but they must obey boundary conditions both at $t = -\infty$ and $t = +\infty$ (these boundary conditions depend on z)

Loop corrections

WARNING : work in progress !!

Reasons for studying loop corrections

- See whether the retarded nature of the objects involved in the computation survives (this property is crucial in practice at tree level)
- Factorization. Proving "factorization at leading order" in fact requires to look at loop corrections :
  - Loop corrections in QCD have large logarithms of $1/x_{1,2}$
  - These large logs can compensate the smallness of the coupling constant α_s, i.e. $\alpha_s \log(1/x_{1,2}) \sim 1$ even though $\alpha_s \ll 1$
  - Factorization in this context means that all the powers $[\alpha_s \log(1/x_{1,2})]^m [Q_s/p_\perp]^n$ can be resummed by letting the generalized "parton distributions" $W[\rho_{1,2}]$ evolve according to JIMWLK
- The boost invariant solution found at tree level is a very peculiar configuration : it is known from numerical studies that it is unstable if perturbed by rapidity dependent fluctuations. Loop corrections generate such perturbations!

Reminder on the LO result

- The LO inclusive gluon spectrum reads :

$$\frac{d\overline{N}_{LO}}{dY\, d^2\vec{\boldsymbol{p}}_\perp} \propto \int_{x,y} e^{ip\cdot(x-y)} \cdots \mathcal{A}_\mu(x)\mathcal{A}_\nu(y)$$

where $\mathcal{A}_\mu(x)$ is the retarded solution of Yang-Mills equations that vanishes at $x^0 = -\infty$

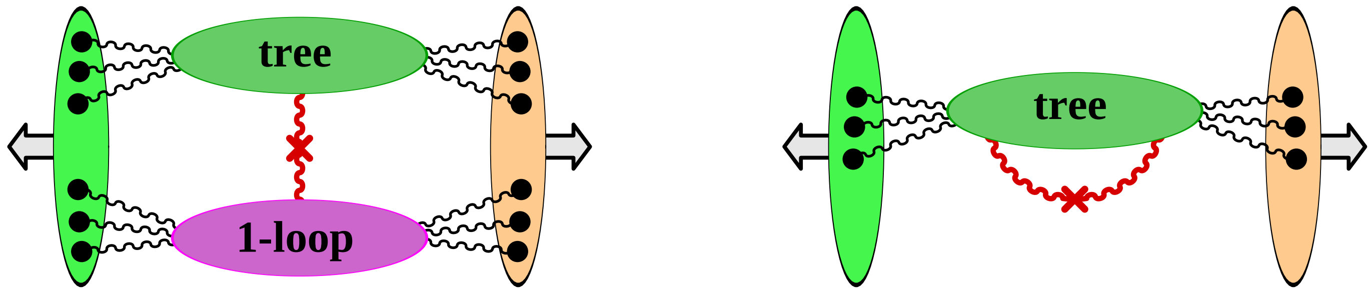
- It is useful to see $\overline{N}_{LO}$ as a functional of the classical gauge field $\mathcal{A}_{\rm in}$ on the light-cone
Note : the functional $\overline{N}_{LO}[\mathcal{A}_{\rm in}]$ has no explicit dependence on the sources $\rho_{1,2}$, because there are no sources above the light-cone

- The dependence on ρ_1 and ρ_2 is all contained in $\mathcal{A}_{\rm in}$. Hence, we can write

$$\overline{N}_{LO} \equiv \overline{N}_{LO}[\mathcal{A}_{\rm in}[\rho_1, \rho_2]]$$

1-loop corrections to $\overline{N}$

- 1-loop diagrams for $\overline{N}$



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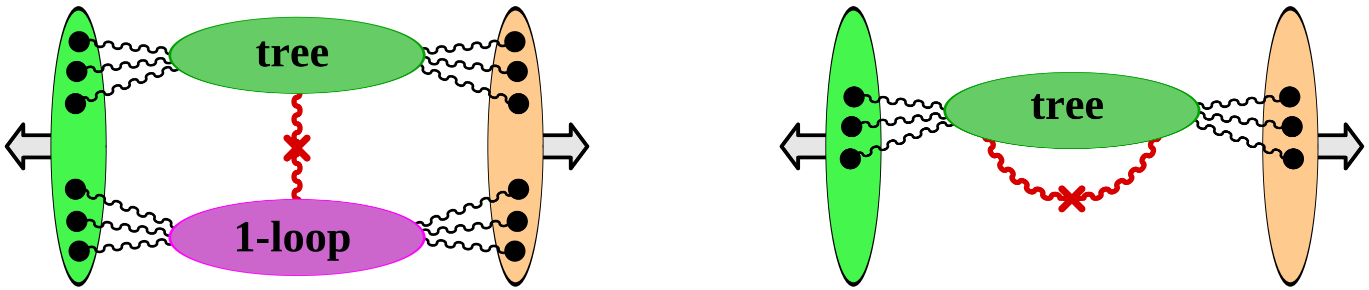
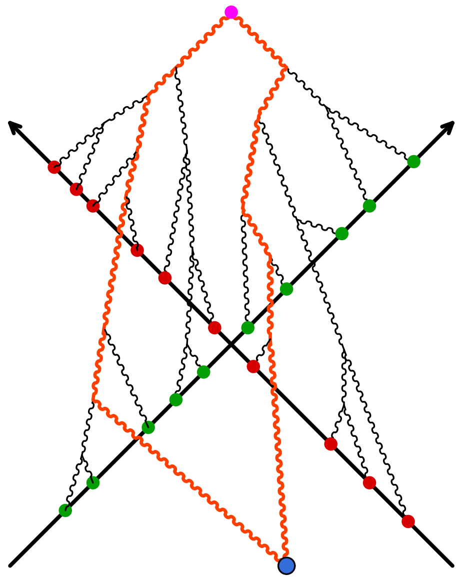
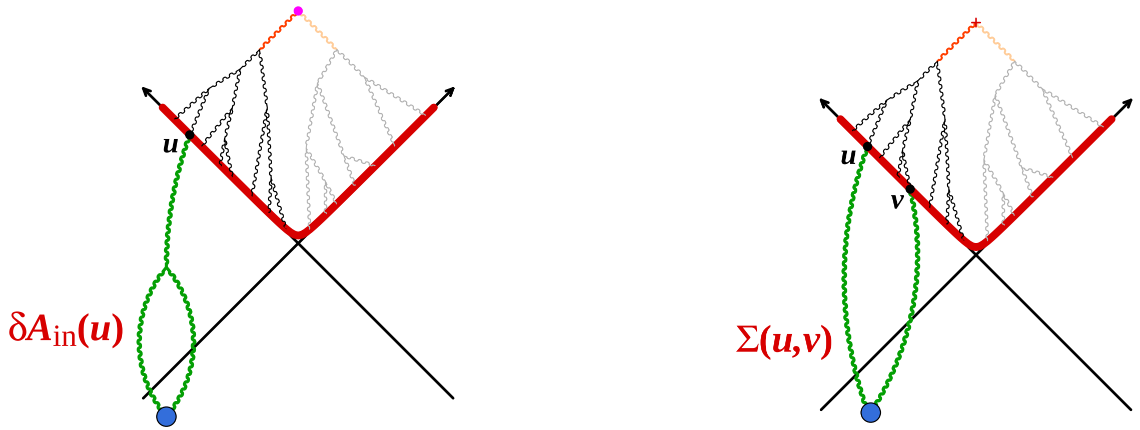


- This involves diagrams such as :



1-loop corrections to $\overline{N}$

- The 1-loop correction to $\overline{N}$ can be written as a perturbation of the initial value problem encountered at LO :



$$\delta\overline{N} = \left[\int_{\vec{\boldsymbol{u}}\,\in\,\text{light cone}} \delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{u}})\;\boldsymbol{T}_{\vec{\boldsymbol{u}}} + \int_{\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}\,\in\,\text{light cone}} \frac{1}{2}\,\boldsymbol{\Sigma}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\;\boldsymbol{T}_{\vec{\boldsymbol{u}}}\,\boldsymbol{T}_{\vec{\boldsymbol{v}}} \right] \overline{N}_{LO}$$

- $\boldsymbol{T}_{\vec{\boldsymbol{u}}}$ is the generator of shifts of the initial condition at the point $\vec{\boldsymbol{u}}$ on the light-cone, i.e. : $\boldsymbol{T}_{\vec{\boldsymbol{u}}} \sim \delta/\delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{u}})$
- $\delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{u}})$ and $\boldsymbol{\Sigma}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})$ are in principle calculable analytically

Divergences

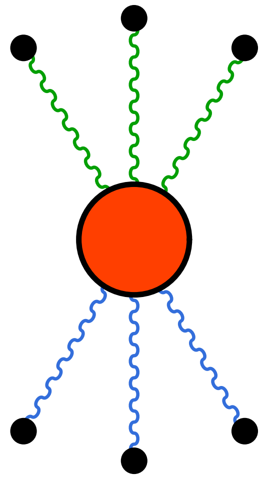
- If taken at face value, this 1-loop correction is plagued by several divergences :
  - The two coefficients $\delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{x}})$ and $\boldsymbol{\Sigma}(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})$ are infinite, because of an unbounded integration over a rapidity variable
  - At late times, $\boldsymbol{T}_{\vec{\boldsymbol{x}}}\mathcal{A}(\tau,\vec{\boldsymbol{y}})$ diverges exponentially,

$$\boldsymbol{T}_{\vec{\boldsymbol{x}}}\mathcal{A}(\tau,\vec{\boldsymbol{y}}) \sim \frac{\delta\mathcal{A}(\tau,\vec{\boldsymbol{y}})}{\delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{x}})} \underset{\tau\to+\infty}{\sim} e^{\sqrt{\mu\tau}}$$

    because of an instability of the classical solution of Yang-Mills equations under rapidity dependent perturbations (Romatschke, Venugopalan (2005))

Initial state factorization

- Anatomy of the full calculation :



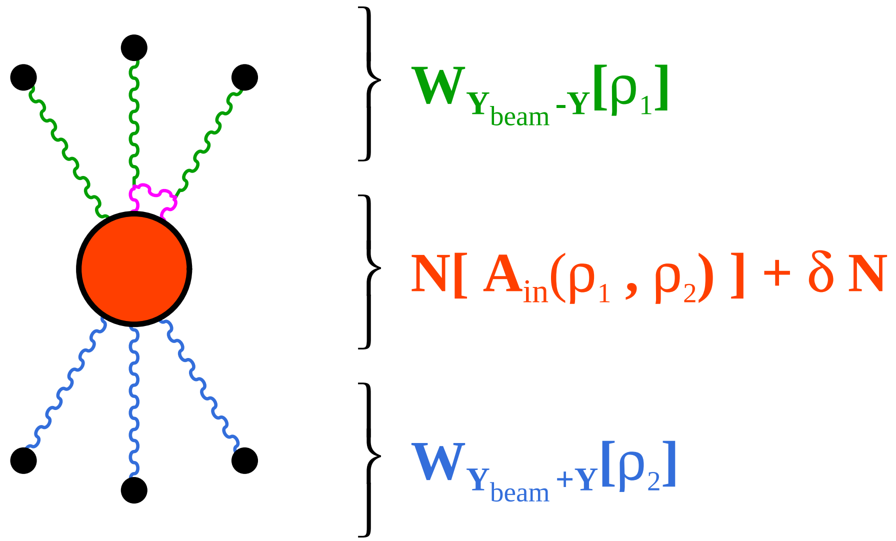
$\Big\}\ \mathbf{W}_{\mathbf{Y}_{\mathrm{beam}}\mathbf{-Y}}[\rho_1]$

$\Big\}\ \mathbf{N}[\,\mathbf{A}_{\mathrm{in}}(\rho_1\,\mathbf{,}\ \rho_2)\,]$

$\Big\}\ \mathbf{W}_{\mathbf{Y}_{\mathrm{beam}}\mathbf{+Y}}[\rho_2]$

Initial state factorization

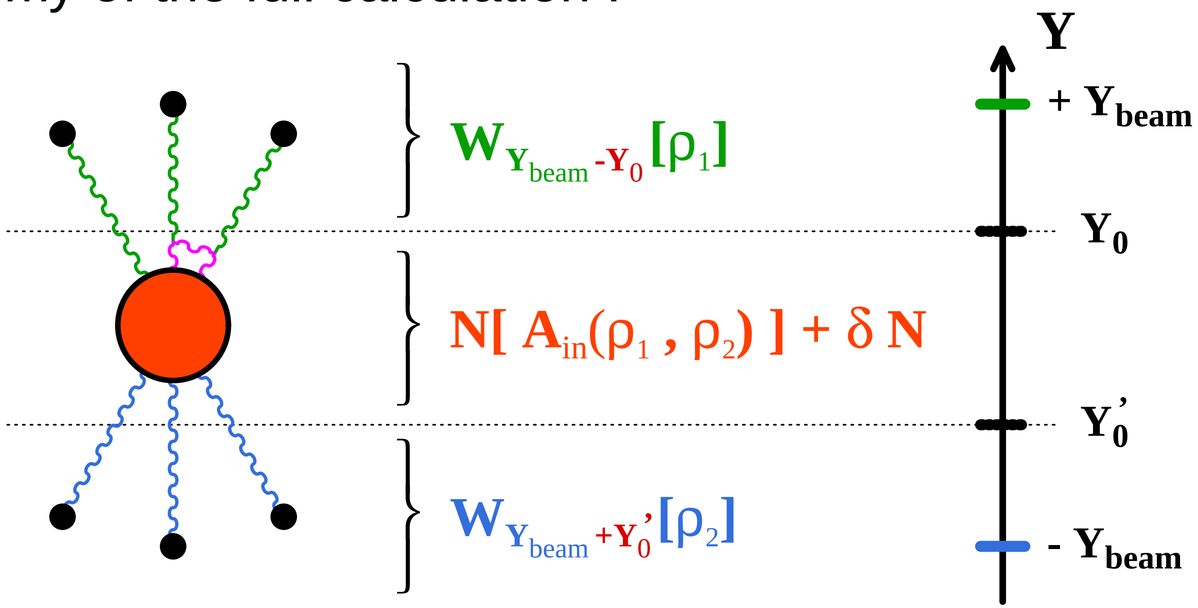
- Anatomy of the full calculation :



- When the observable $\overline{N}[\mathcal{A}_{\rm in}(\rho_1,\rho_2)]$ is corrected by an extra gluon, one gets divergences of the form $\alpha_s\int dY$ in $\delta\overline{N}$

Initial state factorization

- Anatomy of the full calculation :



- When the observable $\overline{N}[\mathcal{A}_{\rm in}(\rho_1,\rho_2)]$ is corrected by an extra gluon, one gets divergences of the form $\alpha_s\int dY$ in $\delta\overline{N}$
- Put some arbitrary cutoffs Y_0 and Y_0' between the "observable" and the "source distributions" : the dependence on Y_0, Y_0' should cancel between the various factors

Initial state factorization

- The two kind of divergences don't mix, because the divergent part of the coefficients is boost invariant.

- In order to prove the factorization of these divergences in the initial state distributions of sources, one needs to establish :

$$\Big[\delta\overline{N}\Big]_{\substack{\text{divergent}\\\text{coefficients}}} = \Big[(Y_0 - Y)\,\mathcal{H}^\dagger[\rho_1] + (Y - Y_0')\,\mathcal{H}^\dagger[\rho_2]\Big]\;\overline{N}_{LO}$$

where $\mathcal{H}[\rho]$ is the Hamiltonian that drives the rapidity dependence of the source distribution $W_Y[\rho]$:

$$\frac{\partial W_Y[\rho]}{\partial Y} = \mathcal{H}[\rho]\;W_Y[\rho]$$

Initial state factorization

- Why is this factorization plausible ?
  - Reminder :

$$\left[\delta\overline{N}\right]_{\substack{\text{divergent}\\\text{coefficients}}} = \left\{\int_{\vec{\boldsymbol{x}}} \left[\delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{x}})\right]_{\rm div} \boldsymbol{T}_{\vec{\boldsymbol{x}}} + \frac{1}{2}\int_{\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}} \left[\boldsymbol{\Sigma}(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})\right]_{\rm div} \boldsymbol{T}_{\vec{\boldsymbol{x}}}\boldsymbol{T}_{\vec{\boldsymbol{y}}}\right\} \overline{N}_{LO}$$

  - Compare with the structure of the evolution Hamiltonian :

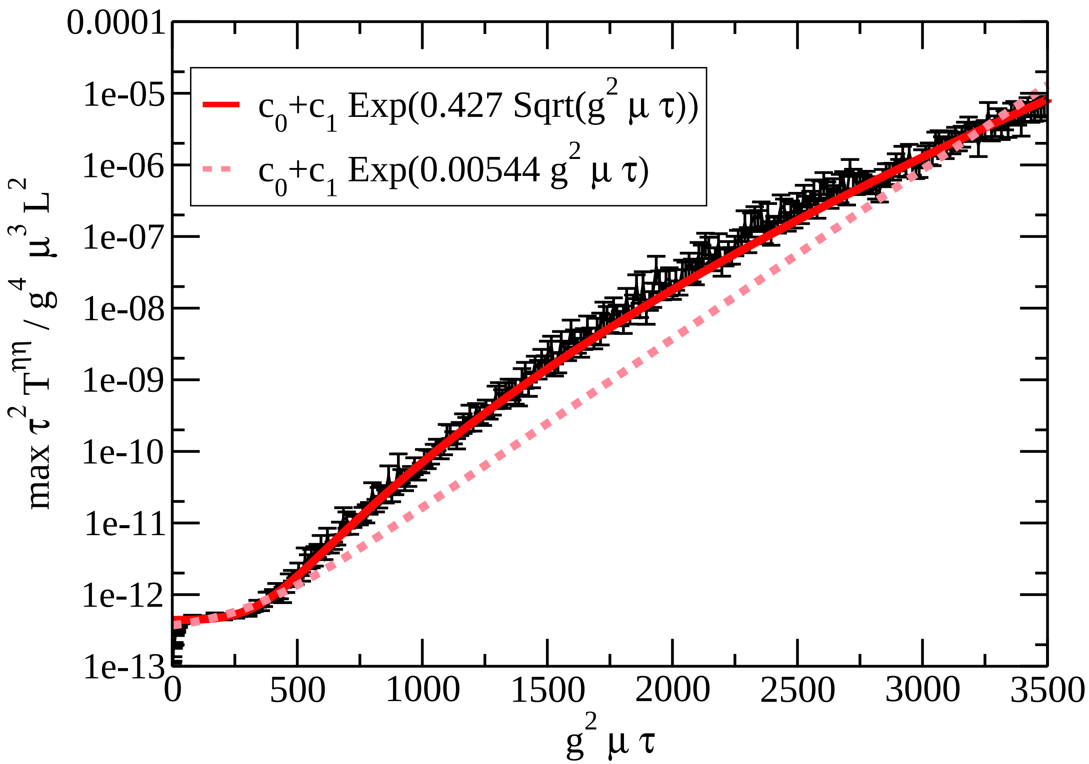
$$\mathcal{H}[\rho] = \int_{\vec{\boldsymbol{x}}_\perp} \sigma(\vec{\boldsymbol{x}}_\perp)\frac{\delta}{\delta\rho(\vec{\boldsymbol{x}}_\perp)} + \frac{1}{2}\int_{\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp} \chi(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\frac{\delta^2}{\delta\rho(\vec{\boldsymbol{x}}_\perp)\delta\rho(\vec{\boldsymbol{y}}_\perp)}$$

- The coefficients σ and χ in the Hamiltonian are well known
  ▷ one must compute analytically the divergent part of $\delta\mathcal{A}_{\rm in}$ and $\boldsymbol{\Sigma}$

Unstable modes

Romatschke, Venugopalan (2005)

- Rapidity dependent perturbations to the classical fields grow like $\exp(\#\sqrt{\tau})$ until the non-linearities become important :



Unstable modes

- The coefficient $\delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{x}})$ is boost invariant.
  Hence, the divergences due to the unstable modes all come from the quadratic term in $\delta\overline{N}$:

$$\Big[\delta\overline{N}\Big]_{\substack{\rm unstable\\ \rm modes}} = \left\{\frac{1}{2}\int_{\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}} \boldsymbol{\Sigma}(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})\; \boldsymbol{T}_{\vec{\boldsymbol{x}}}\boldsymbol{T}_{\vec{\boldsymbol{y}}}\right\}\; \overline{N}_{LO}[\mathcal{A}_{\rm in}(\rho_1,\rho_2)]$$

- When summed to all orders, this becomes a certain functional $Z[\boldsymbol{T}_{\vec{\boldsymbol{x}}}]$:

$$\Big[\delta\overline{N}\Big]_{\substack{\rm unstable\\ \rm modes}} = Z[\boldsymbol{T}_{\vec{\boldsymbol{x}}}]\;\; \overline{N}_{LO}[\mathcal{A}_{\rm in}(\rho_1,\rho_2)]$$

Unstable modes

- This can be arranged in a more intuitive way :

$$\left[\delta\overline{N}\right]_{\substack{\text{unstable}\\\text{modes}}} = \int [Da]\ \widetilde{Z}[a(\vec{x})]\ e^{i\int_{\vec{x}} a(\vec{x})\ \boldsymbol{T}_{\vec{x}}}\ \overline{N}_{LO}[\mathcal{A}_{\rm in}(\rho_1,\rho_2)]$$

$$= \int [Da]\ \ \widetilde{Z}[a(\vec{x})]\ \ \overline{N}_{LO}[\mathcal{A}_{\rm in}(\rho_1,\rho_2)+a]$$

▷ summing these divergences simply requires to add fluctuations to the initial condition for the classical EOM
▷ the fact that $\delta\mathcal{A}_{\rm in}(\vec{x})$ does not contribute implies that the distribution of fluctuations is real

- Interpretation :

Despite the fact that the fields are coupled to strong sources, the classical approximation alone is not good enough, because the classical solution has unstable modes that can be triggered by the quantum fluctuations

Unstable modes

- If everything works as expected, one should write

$$\frac{d\overline{N}}{dY\,d^2\vec{\boldsymbol{p}}_\perp} = \int [D\rho_1]\,[D\rho_2]\;W_{Y_{\rm beam}-Y}[\rho_1]\,W_{Y_{\rm beam}+Y}[\rho_2] \times \int [Da]\;\;\widetilde{Z}[a]\;\frac{d\overline{N}[\mathcal{A}_{\rm in}(\rho_1,\rho_2)+a]}{dY\,d^2\vec{\boldsymbol{p}}_\perp}$$

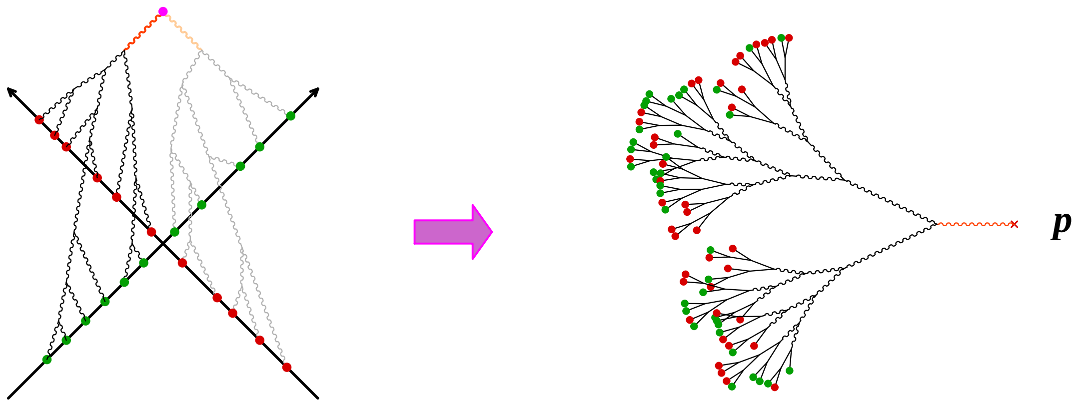
$\triangleright$ This formula resums (all?) the divergences that occur at one loop

- Somewhat analogous to factorization in conventional pQCD :

$W_Y[\rho]$	$\longleftrightarrow$	parton distribution
$\widetilde{Z}[a]$	$\longleftrightarrow$	fragmentation function

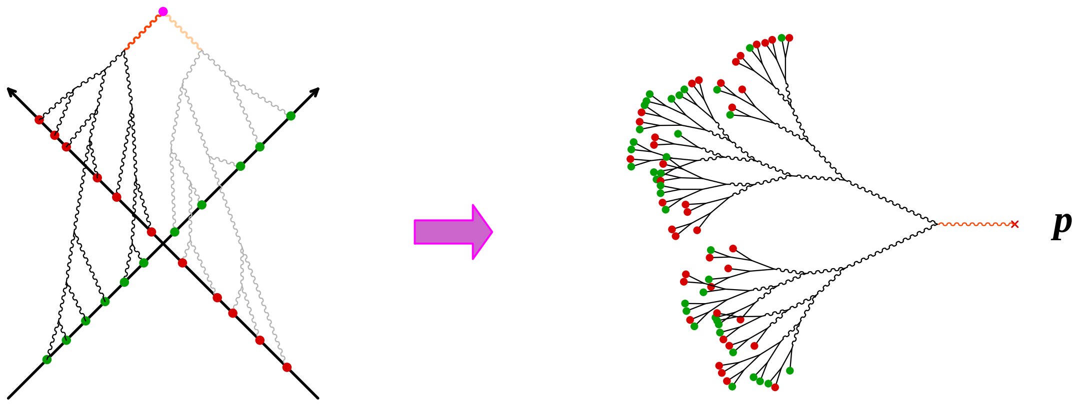
Unstable modes

- Tree level :

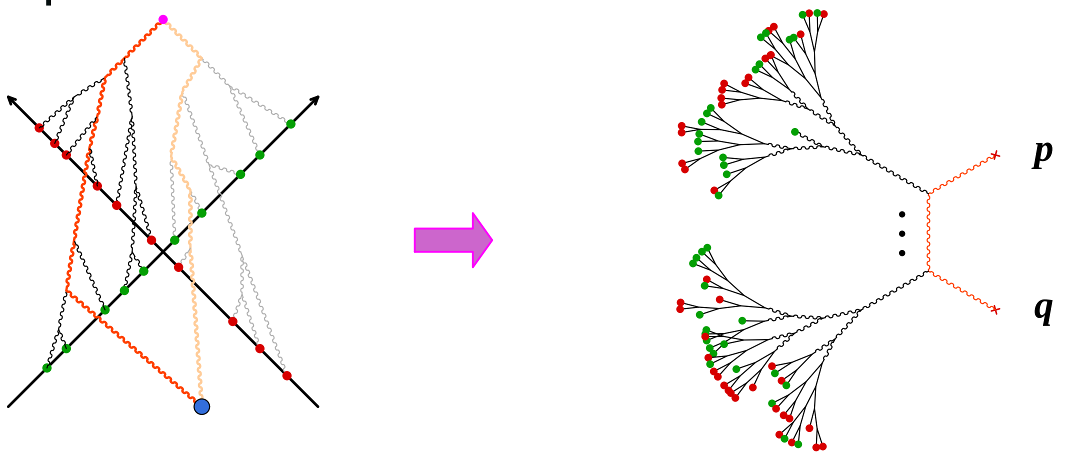


Unstable modes

- Tree level :



- One loop :



▷ The momentum $\vec{\boldsymbol{q}}$ is integrated out

▷ If $\alpha_s^{-1} \lesssim \left|y_{\boldsymbol{p}} - y_{\boldsymbol{q}}\right|$, the correction is absorbed in $W[\rho_{1,2}]$

▷ If $\left|y_{\boldsymbol{p}} - y_{\boldsymbol{q}}\right| \lesssim \alpha_s^{-1}$: gluon splitting in the final state

Summary

Summary

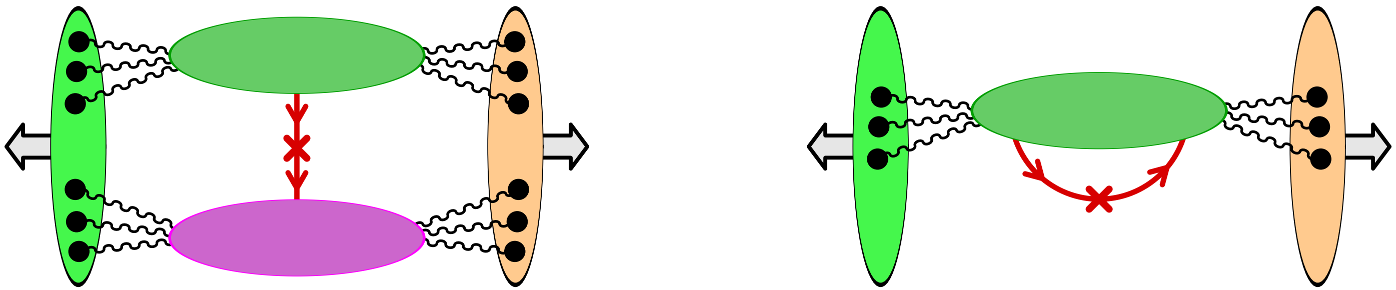
- The physics of saturation can be probed in a number of ways in Deep Inelastic Scattering (especially eA collisions) and asymmetric collisions like proton-nucleus collisions
  - Scaling properties
  - Multiple scatterings
  - Shadowing
- Nucleus-nucleus collisions are not a good framework in order to probe saturation, but the physics of saturation is crucial in order to correctly assess what happens in the early stages of AA collisions
  - Leading order $\triangleright$ classical fields
    (retarded in the case of inclusive observables)
  - Factorization likely (at LO) for inclusive quantities
  - Instabilities affect the dynamics after the collision

Extra bits

Inclusive quark spectrum

FG, Kajantie, Lappi (2004, 2005)

- One can construct for quarks an operator $\mathcal{C}_{\mathrm{q}}$ that plays the same role as $\mathcal{C}$ for the gluons
- By repeating the same arguments, we find two generic topologies contributing to the inclusive quark spectrum :



(the blobs are sums of cut diagrams)

- The first topology cannot exist because the quark line is not closed on itself
  - ▷ the quark spectrum starts at one loop

Quark production at one loop

- At lowest order (one loop), the quark spectrum reads :

$$\frac{d\overline{N}_{\rm q}}{dY\,d^2\vec{\boldsymbol{p}}_\perp}=\frac{1}{16\pi^3}\int_{x,y}e^{ip\cdot x}\,\overline{u}(\vec{\boldsymbol{p}})\,(i\,\overrightarrow{\not\partial}_x-m)\,S_{+-}(x,y)\,(i\,\overleftarrow{\not\partial}_y+m)\,u(\vec{\boldsymbol{p}})\,e^{-ip\cdot y}$$

where S_{+-} is the quark propagator (with one endpoint on each side of the cut) to which are attached tree graphs in all the possible ways

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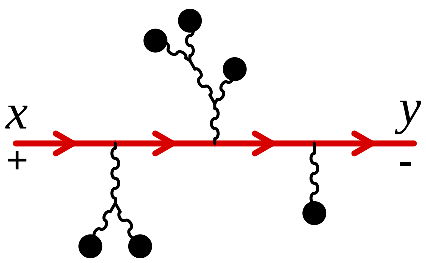
where S_{+-} is the quark propagator (with one endpoint on each side of the cut) to which are attached tree graphs in all the possible ways

- We need to calculate the sum of the following tree diagrams :



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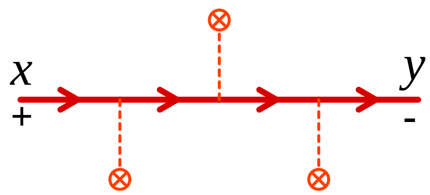
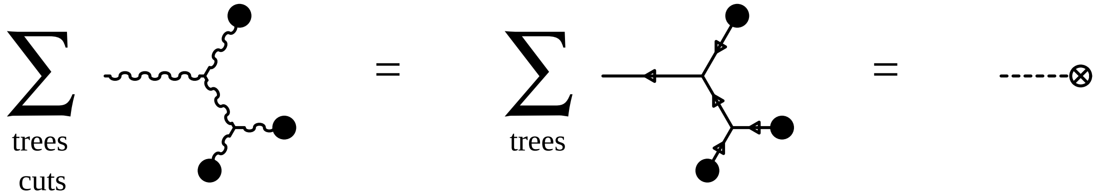


- Perform a resummation of all the sub-diagrams that correspond to the retarded classical solution :



Quark propagator

- The summation of all the classical field insertions can be done by solving a Lippmann-Schwinger equation :

$$S_{\epsilon\epsilon'}(x,y) = S^0_{\epsilon\epsilon'}(x,y) - ig \sum_{\eta=\pm} (-1)^\eta \int d^4z\, S^0_{\epsilon\eta}(x,z)\mathcal{A}_\mu(z)\gamma^\mu S_{\eta\epsilon'}(z,y)$$

- This equation is rather non-trivial to solve in this form, because of the mixing of the 4 components of the propagator. Perform a rotation on the $\pm$ indices :

$$S_{\epsilon\epsilon'} \quad\rightarrow\quad \boldsymbol{S}_{\alpha\beta} \equiv \sum_{\epsilon,\epsilon'=\pm} U_{\alpha\epsilon} U_{\beta\epsilon'} S_{\epsilon\epsilon'}$$

$$(-1)^\epsilon \delta_{\epsilon\epsilon'} \quad\rightarrow\quad \boldsymbol{\Sigma}_{\alpha\beta} \equiv \sum_{\epsilon=\pm} U_{\alpha\epsilon} U_{\beta\epsilon} (-1)^\epsilon$$

- A useful choice for the rotation matrix U is $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

Quark propagator

- Under this rotation, the matrix propagator and field insertion become :

$$\boldsymbol{S}_{\alpha\beta} = \begin{pmatrix} 0 & S_A \\ S_R & S_D \end{pmatrix} \quad , \qquad \boldsymbol{\Sigma}_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $S_D^0(p) = 2\pi(\not{p} + m)\delta(p^2 - m^2)$

- The main simplification comes from the fact that $\boldsymbol{S}^0\boldsymbol{\Sigma}$ is the sum of a diagonal matrix and a nilpotent matrix
- One finds that S_R and S_A do not mix, i.e. they obey equations such as :

$$S_R(x,y) = S_R^0(x,y) - i\,g \int d^4z\, S_R^0(x,z)\mathcal{A}_\mu(z)\gamma^\mu S_R(z,y)$$

- One can solve S_D in terms of S_R and S_A :

$$S_D = S_R * S_R^{0\,-1} * S_D^0 * S_A^{0\,-1} * S_A$$

Quark propagator

- In order to go back to S_{+-}, invert the rotation :

$$S_{+-} = \frac{1}{2}\left[S_A - S_R - S_D\right]$$

- At this point, we can rewrite the quark spectrum in terms of retarded and advanced quark propagators in the classical background

- Finally, one can rewrite it in terms of retarded solutions of the Dirac equation on top of the background $\mathcal{A}_\mu(x)$

$$\frac{d\overline{N}_{\rm q}}{dY d^2\vec{\boldsymbol{p}}_\perp} = \frac{1}{16\pi^3}\int \frac{d^3\vec{\boldsymbol{q}}}{(2\pi)^3 2E_q}\left|\mathcal{M}(\vec{\boldsymbol{p}},\vec{\boldsymbol{q}})\right|^2$$

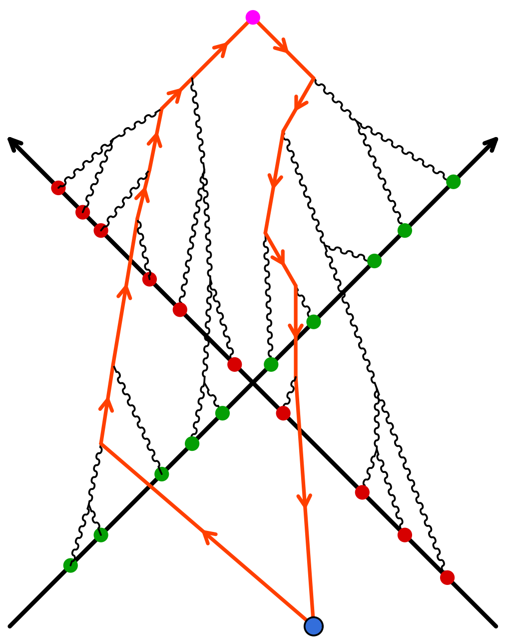
with

$$\mathcal{M}(\vec{\boldsymbol{p}},\vec{\boldsymbol{q}}) = \lim_{x^0\to+\infty}\int d^3\vec{\boldsymbol{x}}\; e^{ip\cdot x}\; u^\dagger(\vec{\boldsymbol{p}})\psi_{\boldsymbol{q}}(x)$$

$$(i\partial\!\!\!/_x - g\mathcal{A}\!\!\!/(x) - m)\psi_{\boldsymbol{q}}(x) = 0\;,\quad \psi_{\boldsymbol{q}}(x^0,\vec{\boldsymbol{x}}) \underset{x^0\to-\infty}{=} v(\vec{\boldsymbol{q}})e^{iq\cdot x}$$

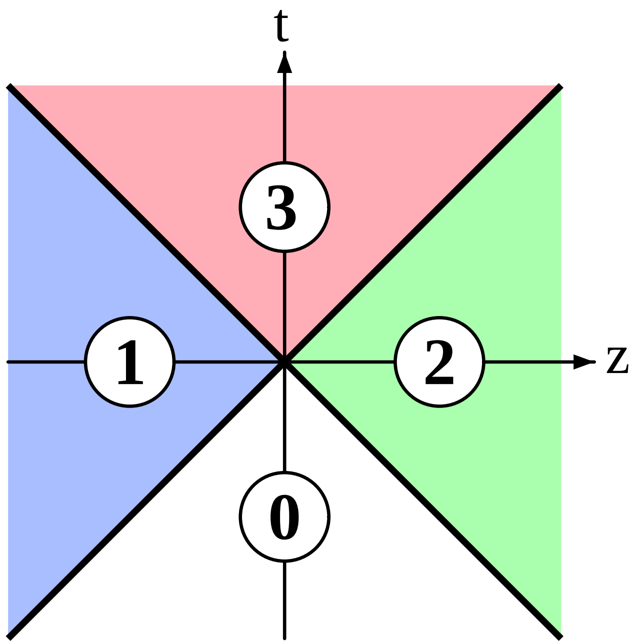
Quark propagator

- This calculation amounts to summing the following diagrams :



Background field

- Space-time structure of the classical color field:

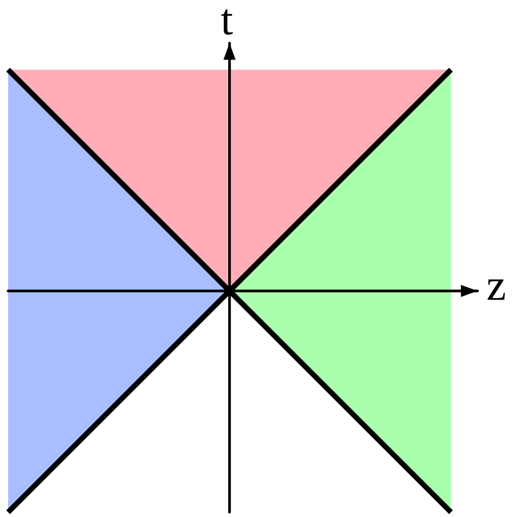


- Region 0: $\mathcal{A}^\mu = 0$
- Region 1: $\mathcal{A}^\pm = 0$, $\mathcal{A}^i = \frac{i}{g} U_1 \nabla_\perp^i U_1^\dagger$
- Region 2: $\mathcal{A}^\pm = 0$, $\mathcal{A}^i = \frac{i}{g} U_2 \nabla_\perp^i U_2^\dagger$
- Region 3: $\mathcal{A}^\mu \neq 0$

- **Notes:**
  - In the region 3, $\mathcal{A}^\mu$ is known only numerically
  - We must solve the Dirac equation numerically as well

Quark propagation

- Propagation through region 0:

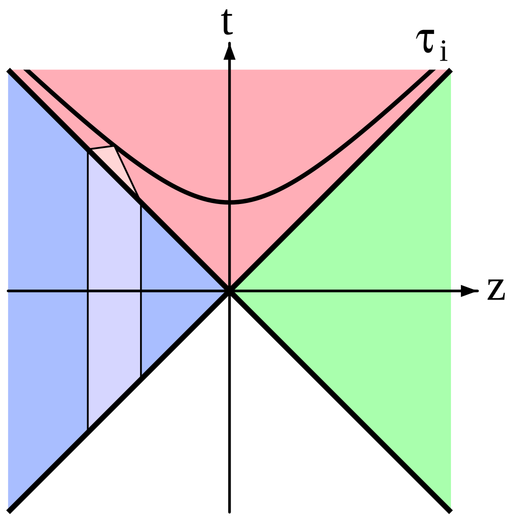


▷ trivial because there is no background field

$$\psi_{\boldsymbol{q}}(x) = v(\vec{\boldsymbol{q}})e^{iq\cdot x}$$

Quark propagation

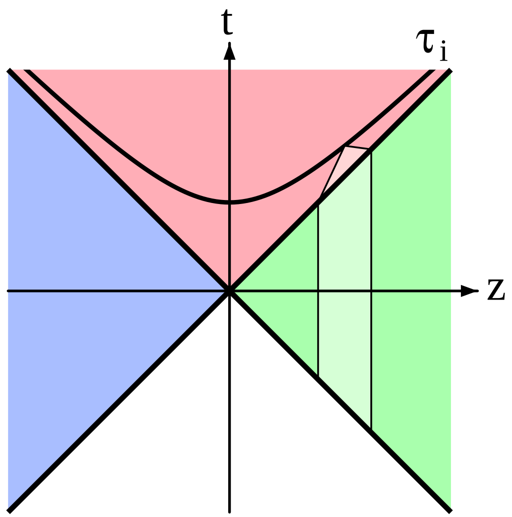
- Propagation through region 1:



- ▷ Pure gauge background field
  - ▷ $\psi_{\boldsymbol{q},1}(\tau_i)$ can be obtained analytically

Quark propagation

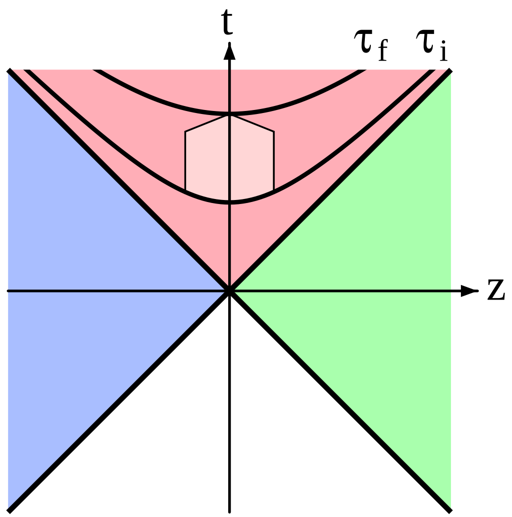
- Propagation through region 2:



  - ▷ Pure gauge background field
    - ▷ $\psi_{\boldsymbol{q},2}(\tau_i)$ can be obtained analytically

Quark propagation

- Propagation through region 3:



▷ One must solve the Dirac equation :

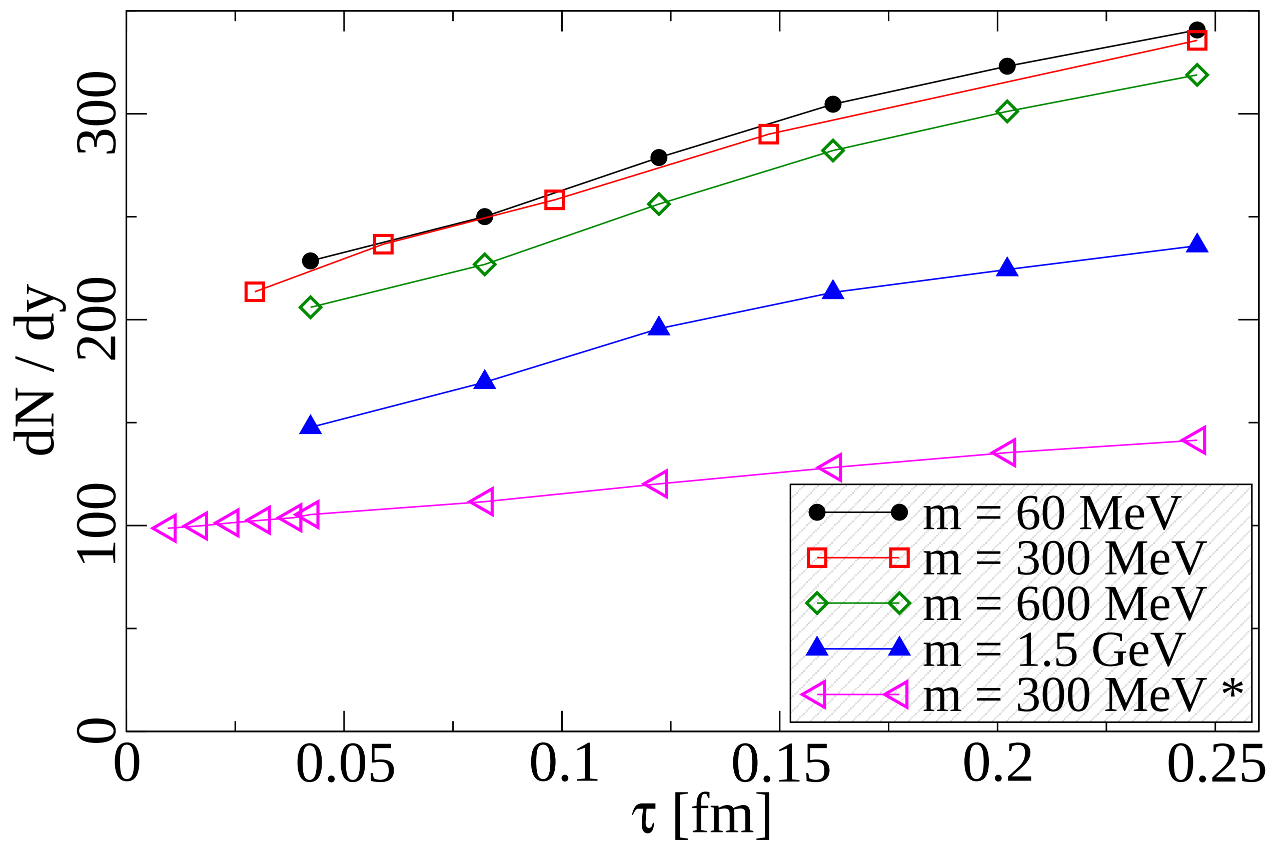
$$\left[i\partial\!\!\!/ - gA\!\!\!/ - m\right] \psi_{\boldsymbol{q}}(\tau, \eta, \vec{\boldsymbol{x}}_\perp) = 0$$

▷ initial condition: $\psi_{\boldsymbol{q}}(\tau_i) = \psi_{\boldsymbol{q},1}(\tau_i) + \psi_{\boldsymbol{q},2}(\tau_i)$

($\tau_i = 0^+$ in practice)

Time dependence

- $g^2\mu = 2$ GeV , (*) $g^2\mu = 1$ GeV :



Spectra for various quark masses

- $g^2\mu = 2$ GeV , $\tau = 0.25$ fm :