Non–linear evolution & Gluon saturation in QCD at high energy

Part I

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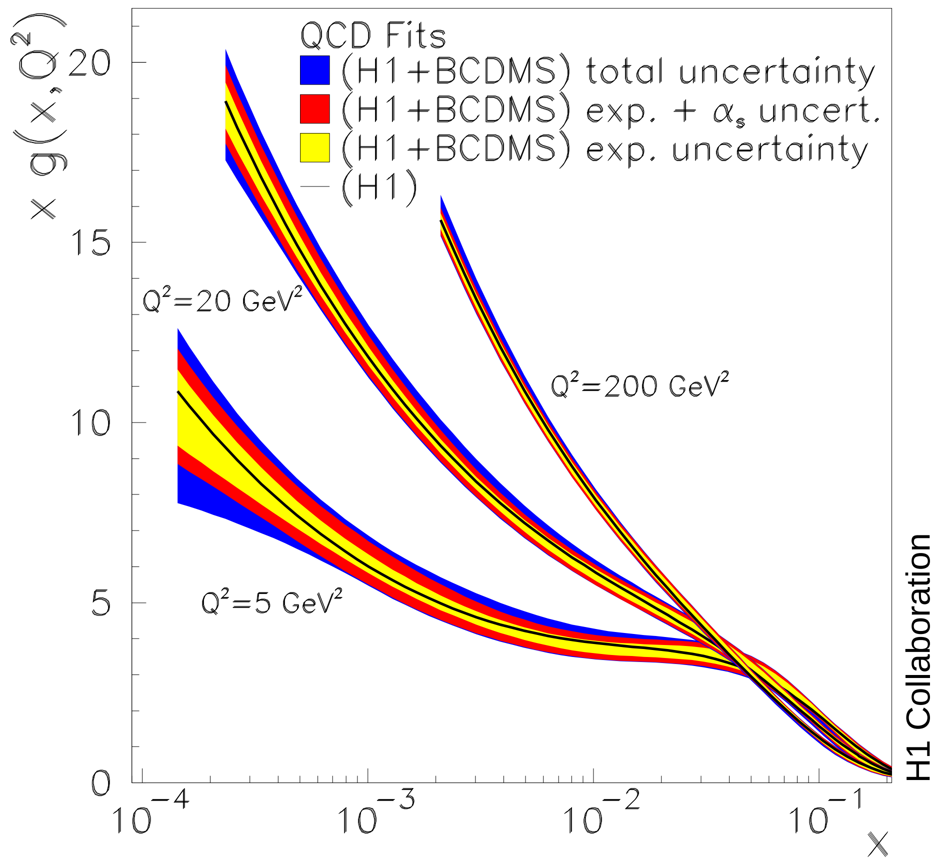
SPhT Saclay & CNRS

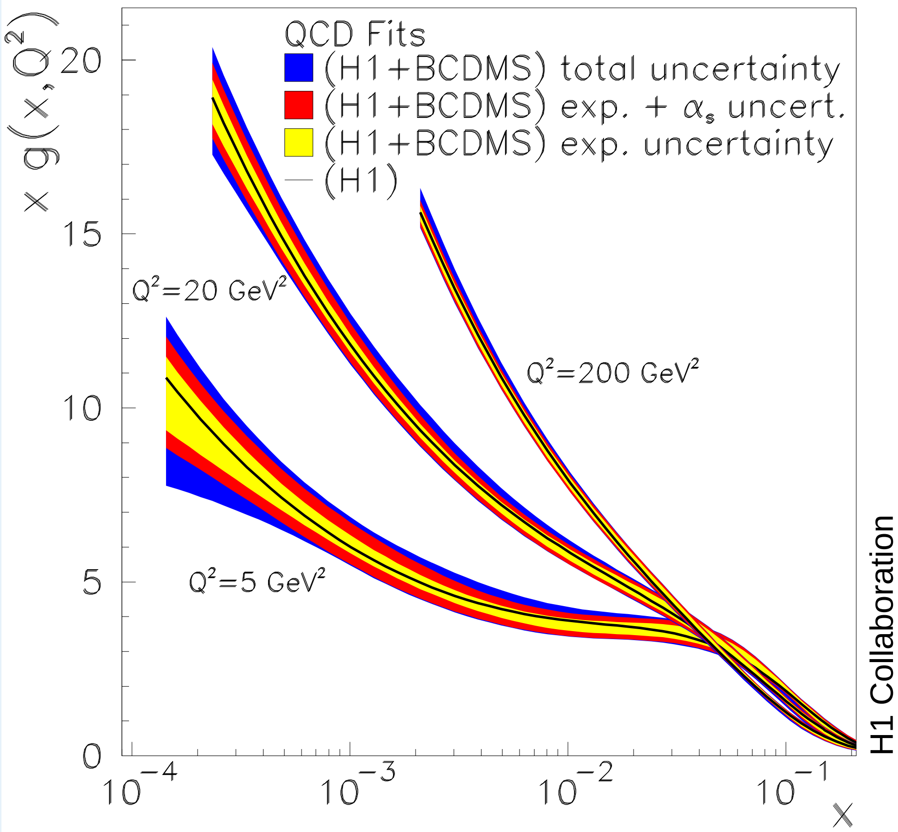
Motivation: Gluons at HERA

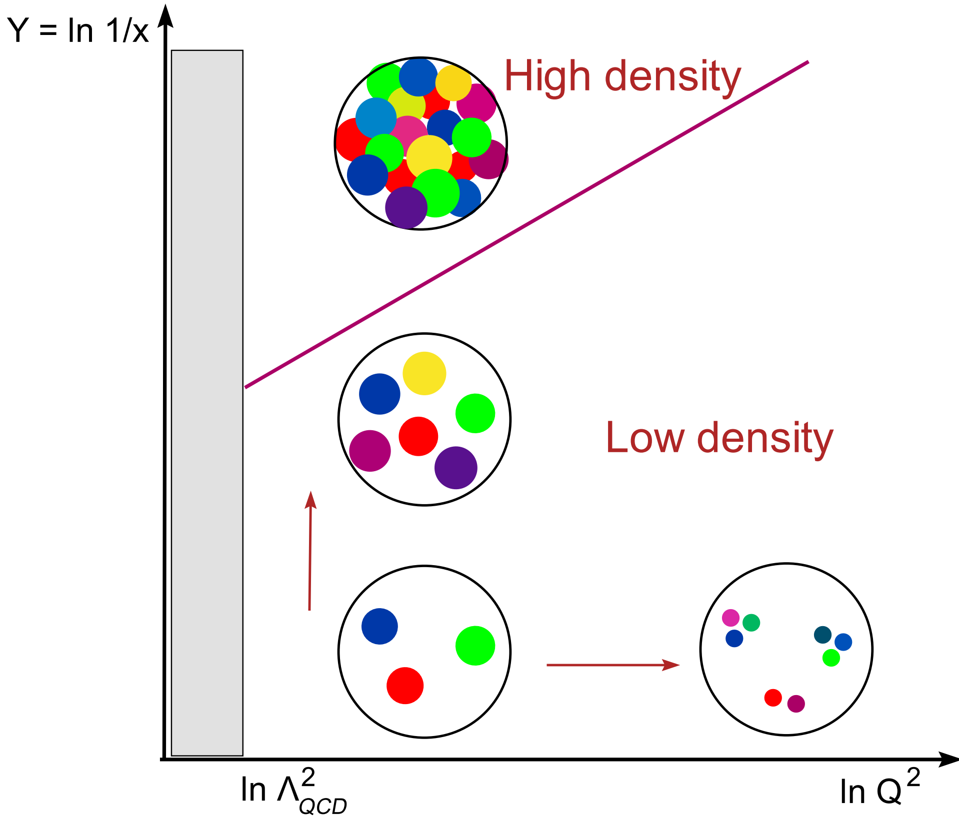
▷ The gluon distribution rises very fast at small x ! $(\sim 1/x^{\lambda})$



$xG(x, Q^2) \approx$ # of gluons with transverse size $\Delta x_{\perp} \sim 1/Q$ and $k_z = xP$

Motivation: Gluons at HERA

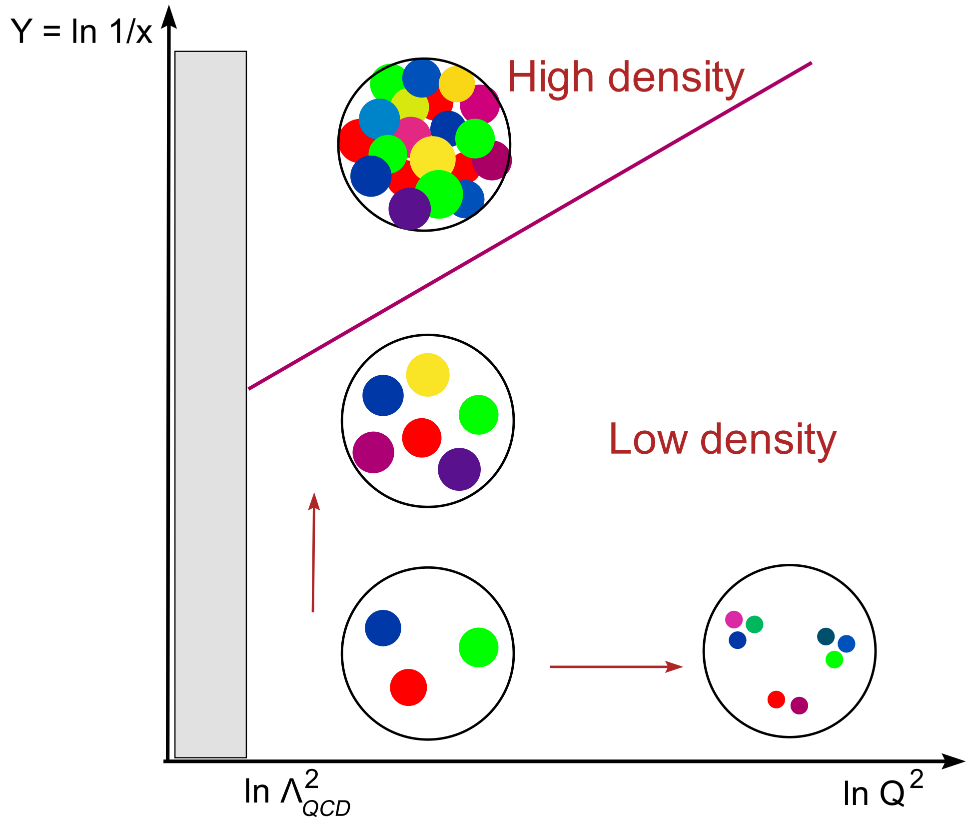




▷ High–Q^2 evolution : The parton density is decreasing

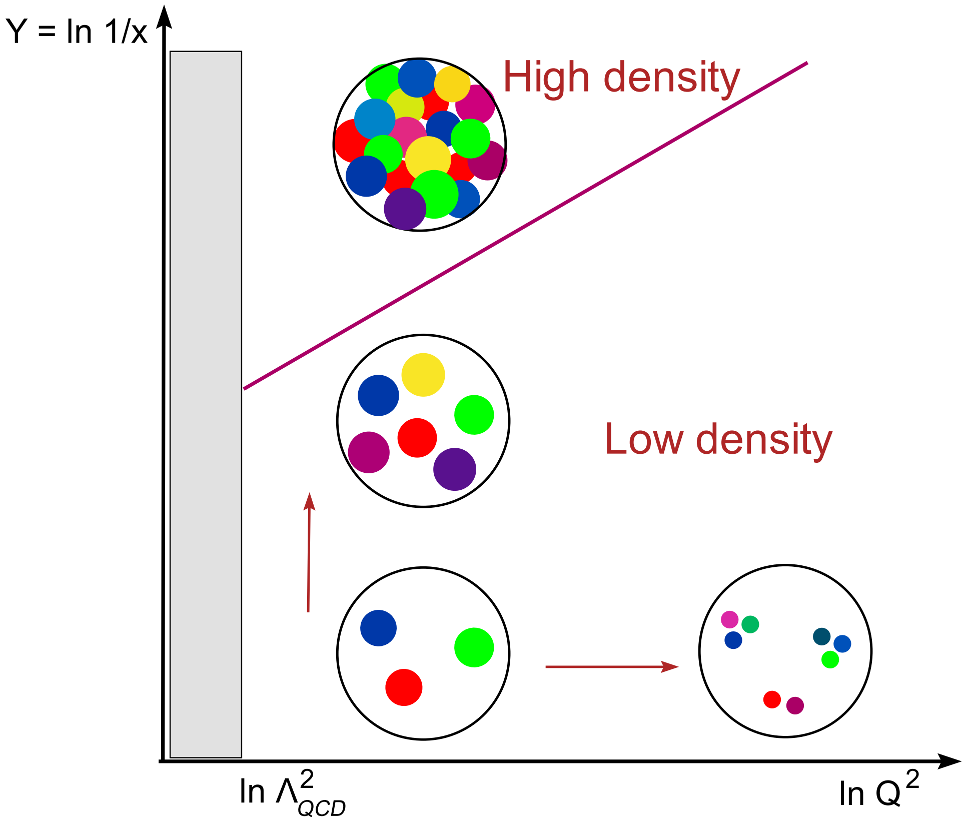
▷ High–energy evolution : An evolution towards increasing density !

Motivation: High density = Weak Coupling



- Asymptotic freedom of QCD
  - High density partonic matter should be weakly coupled !
  - First principles calculations in QCD at high energy

Motivation: High density = Non-linear

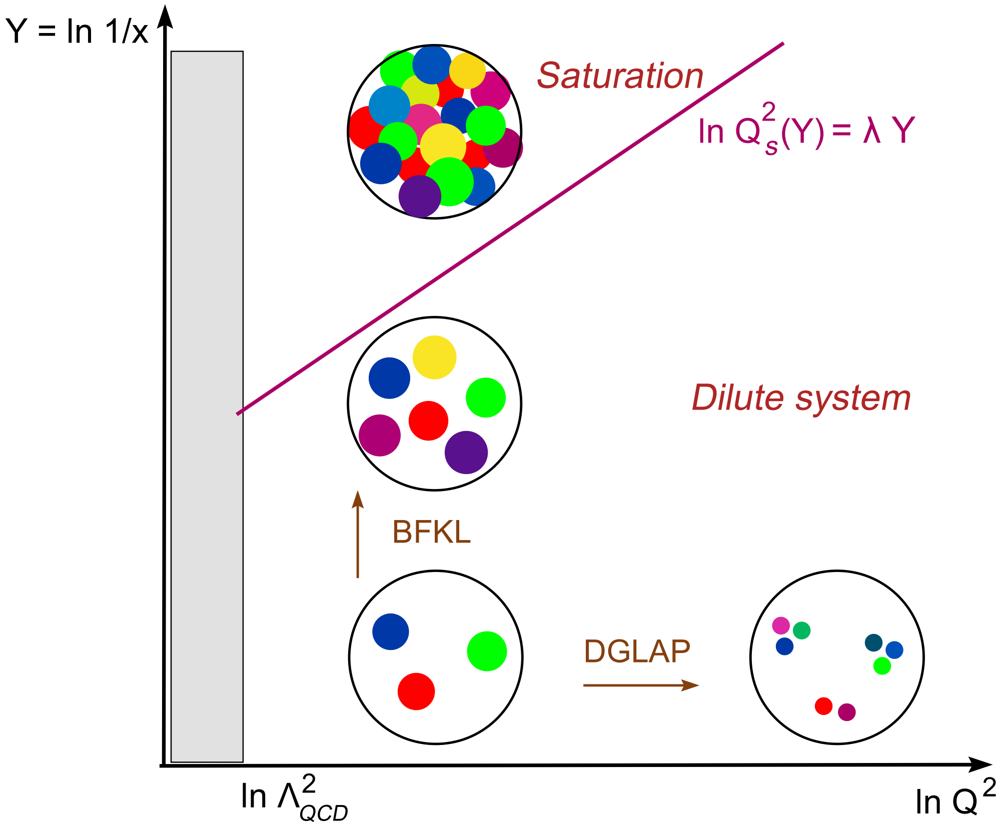


- A very challenging problem though !

  High density $\Longrightarrow$ strong non–linear effects

- The full non-linearity of QCD reveals itself at weak coupling !

Motivation: High density = Non-linear

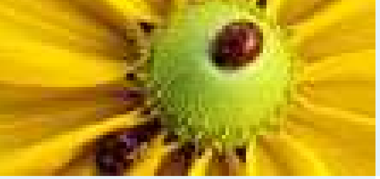


- DGLAP equation: naturally linear !
- BFKL equation: linear approximation (intermediate energies)
- In general, the high–energy evolution is non–linear

Introduction & Outline

▷ These lectures:

- A risky attempt to provide a pedagogical introduction to relatively recent developments in our understanding of non–linear evolution with increasing energy ...
  - BK equation *(Balitsky, 96; Kovchegov, 99)*
  - the Color Glass Condensate (CGC) effective theory *(McLerran, Venugopalan, 1994; E.I., Leonidov, McLerran, 2000)*
  - the JIMWLK equation *(Jalilian-Marian, E.I., McLerran, Weigert, Leonidov, Kovner, 97-00)*
  - the Pomeron loop equations *(E.I., Kovner, Lublinsky, Mueller, Munier, Shoshi, Triantafyllopoulos, Wong ... 04–05)*



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- ... themselves built on top of previous, fundamental, developments

  - the BFKL equation

    (Balitsky, Fadin, Kuraev, Lipatov, 75–78)

  - the Color Dipole Picture *(Mueller, 1994)*

  - ‘Pomeron vertices’ *(Bartels, Braun, Ewerz, Vacca..., since 1991)*

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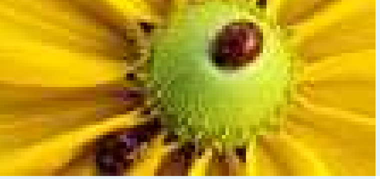

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- Lot of overlapping with previous lectures !

  BFKL, BK, CGC, JIMWLK, Wilson lines, Saturation momentum ...

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  (e.g., cartoons to be shamelessly presented as "Feynman graphs")

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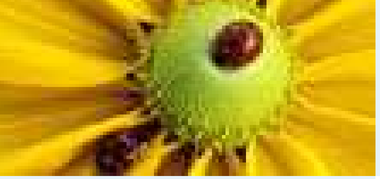
- My supreme authority argument: the "Eyes wide shut" slide

Eyes wide shut

- Theorists, please shut your eyes !

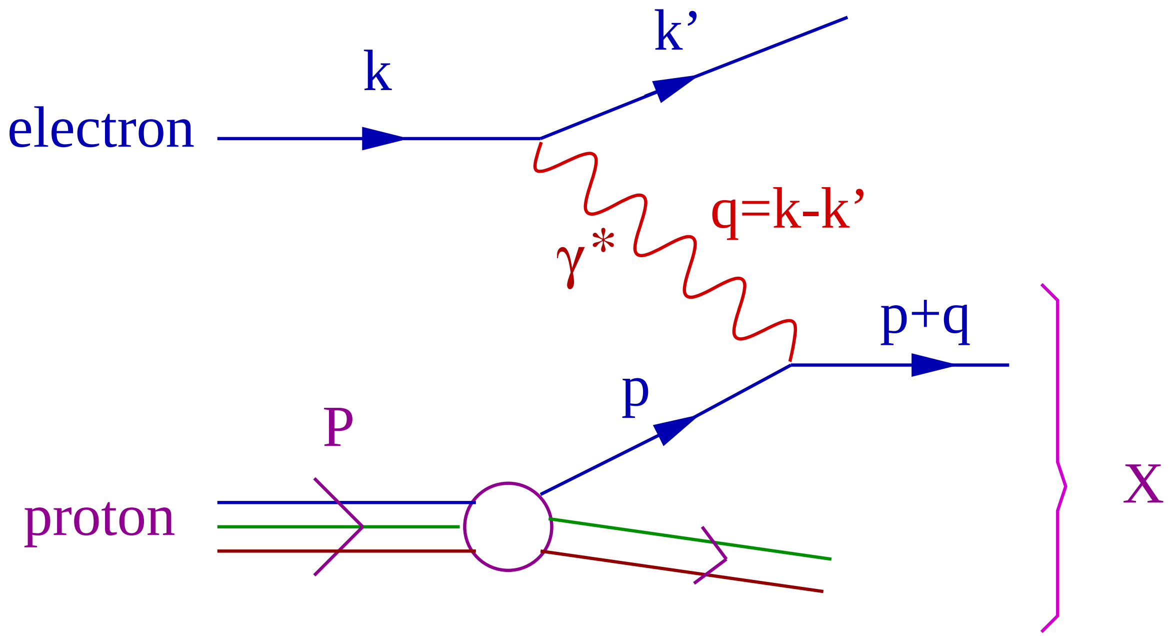
Lecture I

- The fundamental objectives
    - BK equation : the physical argument, a sketchy derivation, qualitative solution
    - BFKL equation (the linearized version of BK): an explicit solution (energy dependence of saturation momentum)
- I shall mostly use DIS as the experimental paradigm which motivates the physical problem



Deep Inelastic Scattering at Small–x

electron (k) + proton (P) $\longrightarrow$ electron (k') + X (P_X)



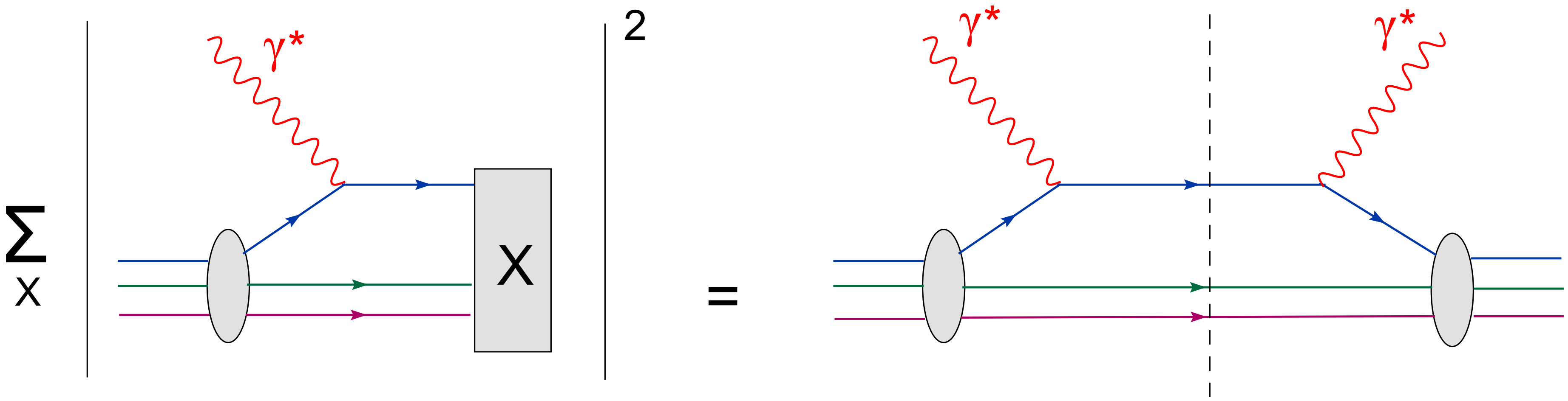
- Useful kinematical invariants (2 independent ones):

$$Q^2 \equiv -q^2 \geq 0\,, \qquad s \equiv (q+P)^2\,, \qquad x \equiv \frac{Q^2}{2P\cdot q} \simeq \frac{Q^2}{s+Q^2}$$

High–energy : $s \gg Q^2 \gg M_p^2 \iff$ Small–x : $x \simeq Q^2/s \ll 1$

The proton structure function

- Differential cross section for virtual photon absorbtion :



$$\sigma_{\gamma^* p}(x, Q^2) \;=\; \frac{4\pi^2 \alpha_{\rm em}}{Q^2}\, F_2(x, Q^2)$$

- $F_2(x, Q^2) =$ the ‘proton structure function’

Parton distributions

- The quark distribution function :

$$F_2(x,Q^2) \; = \; \sum_f e_f^2 \left[x q_f(x,Q^2) + x\bar{q}_f(x,Q^2) \right]$$

- $q_f(x,Q^2)\mathrm{d}x$: number of quarks of flavor f
  - with longitudinal momentum fraction between x and $x + \mathrm{d}x$ $\;(p_z = xP)$
  - localized in the transverse space within an area $1/Q^2$ $(\Delta x_\perp \sim 1/Q)$

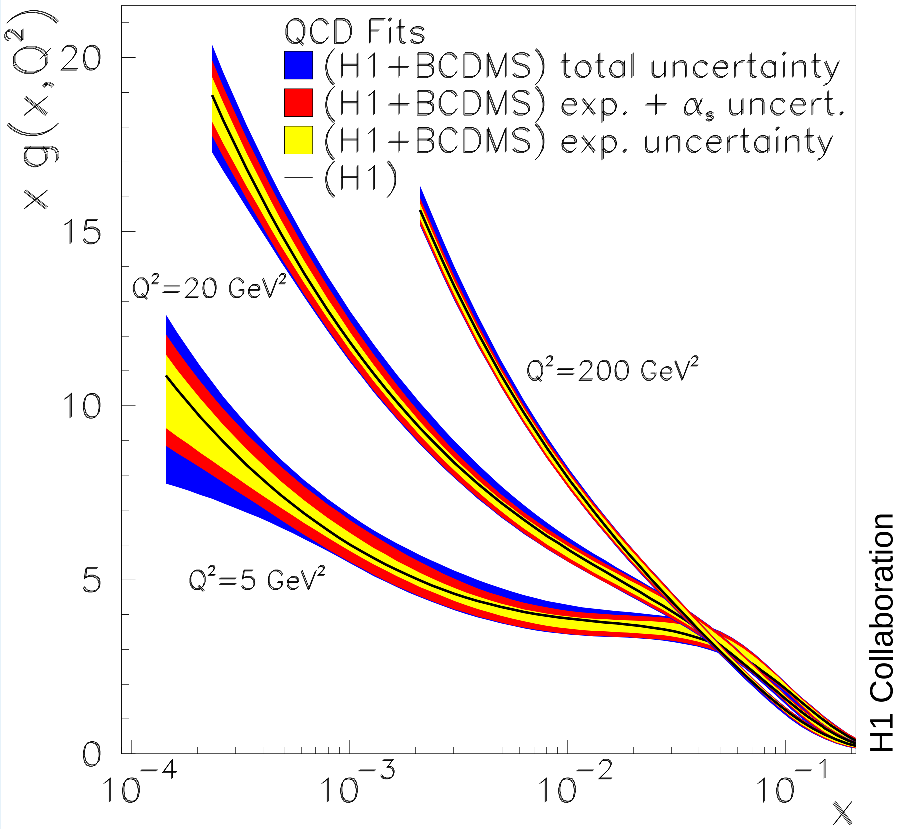
- Parton evolution $\Longrightarrow$ Gluon distribution :

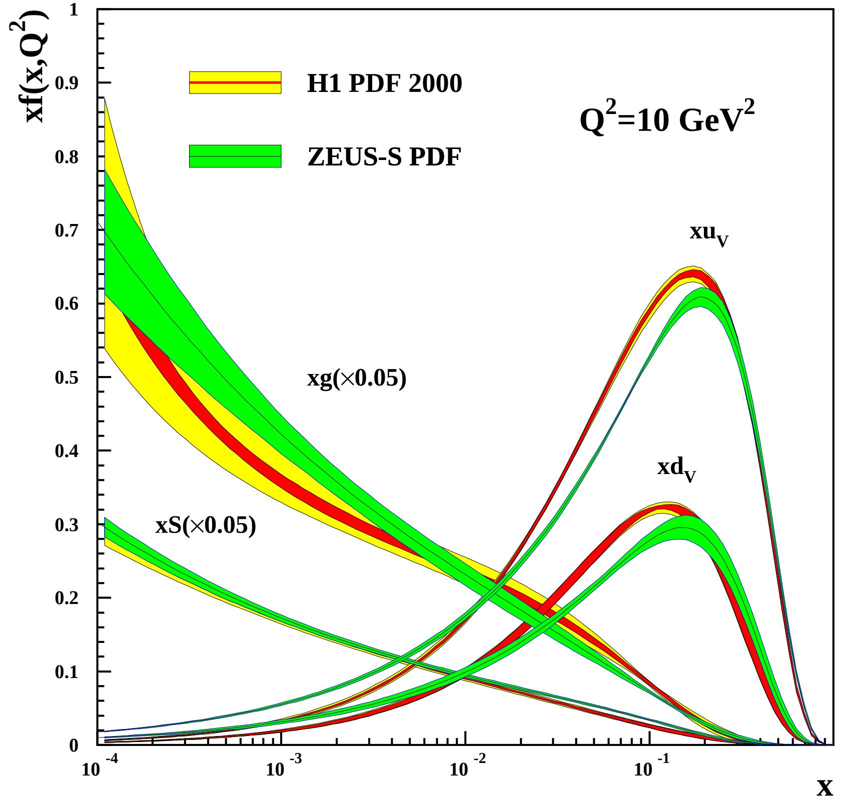
$$\frac{\partial F_2(x,Q^2)}{\partial \ln Q^2} \; \simeq \; \frac{\alpha_s}{3\pi} \Big(\sum_f e_f^2 \Big) \, xG(x,Q^2)$$

- DIS : a measure of parton distributions with longitudinal momentum resolution x and transverse space resolution Q^2

Partons at HERA

▷ The gluon distribution rises very fast at small x ! $(\sim 1/x^{\lambda})$

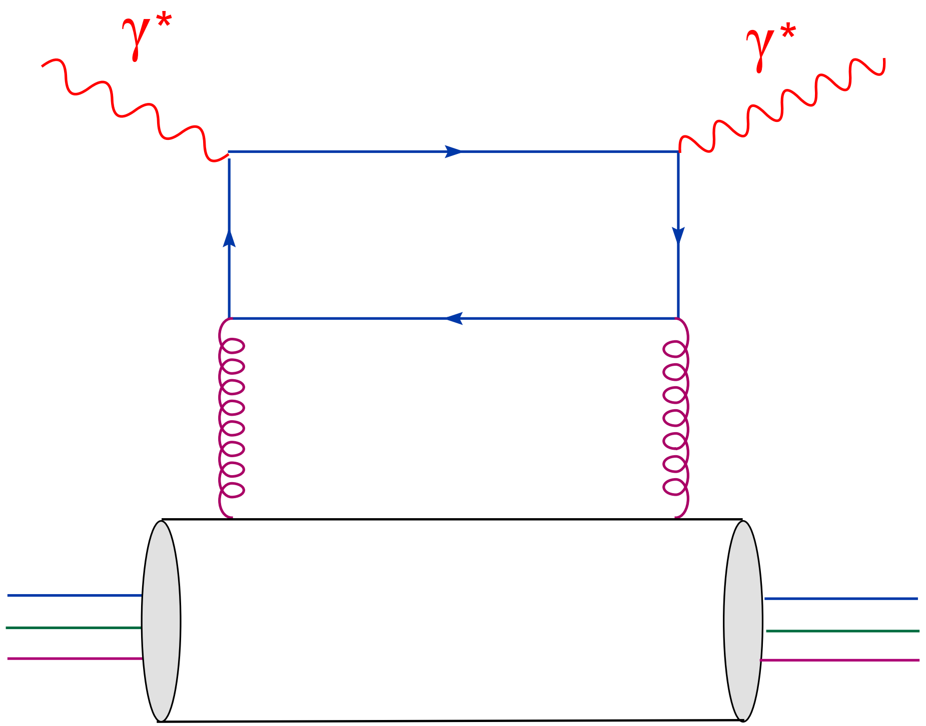




$xG(x,Q^2) \approx$ # of gluons with transverse size $\Delta x_\perp \sim 1/Q$ and $k_z = xP$

DIS at small x

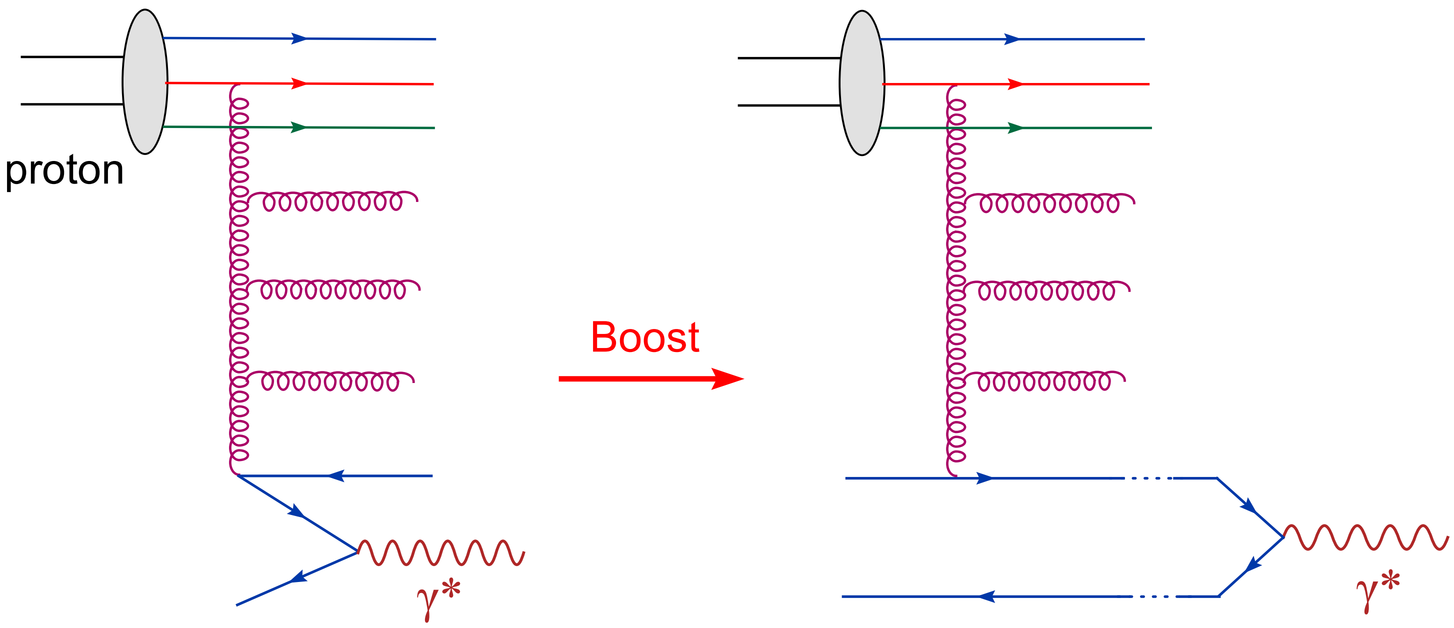
- At small x, the struck quark is typically a sea quark, emitted off the gluon distribution :



- The quark loop can be alternatively associated with the virtual photon wavefunction

Dipole frame

- Make a Lorentz boost to the ‘dipole frame’ :



- γ^* has a relatively high energy:

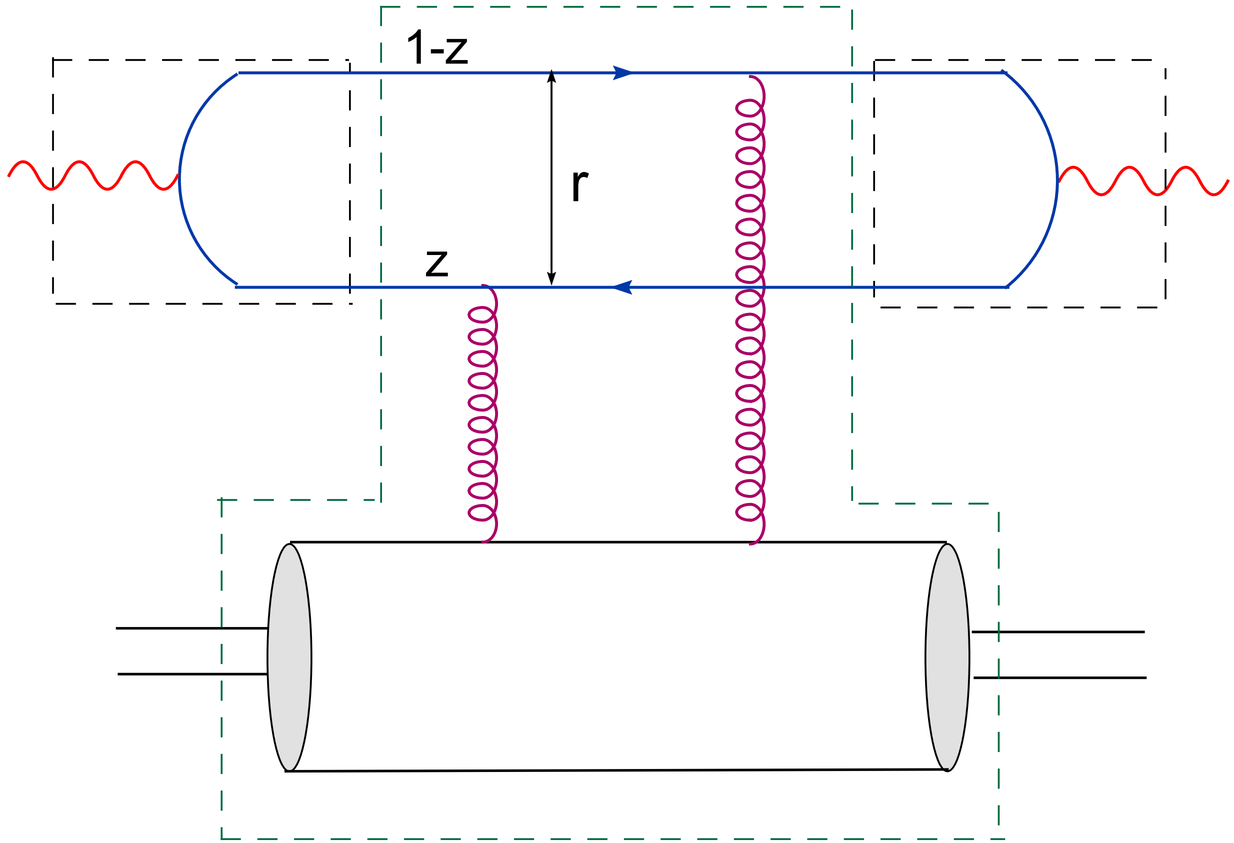
$$q^\mu \equiv (q_0, \boldsymbol{q}_\perp, q_z) = (\sqrt{q^2 - Q^2}, 0_\perp, -q) \quad \text{with} \quad q \gg Q$$

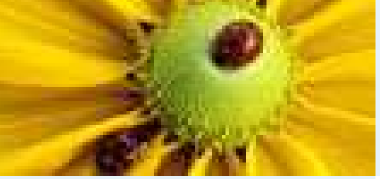
The proton still carries most of the total energy !

- γ^* fluctuates into a $q\bar{q}$ pair which then scatters off the proton

Dipole factorization for DIS

$$\sigma_{\gamma^* p}(x,Q^2) = \int_0^1 \mathrm{d}z \int \mathrm{d}^2\boldsymbol{r}\, |\Psi_\gamma(z,\boldsymbol{r};Q^2)|^2\, \sigma_{\mathrm{dipole}}(x,\boldsymbol{r})$$





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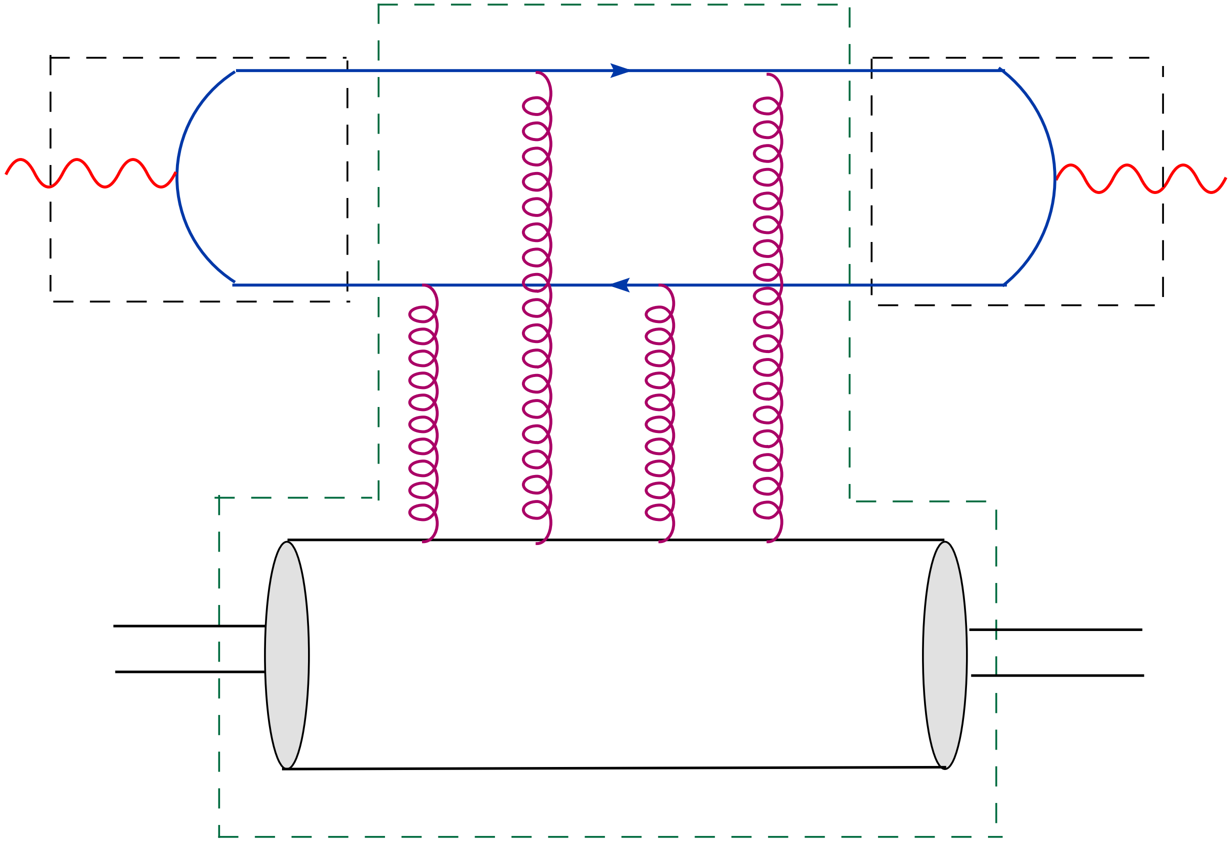
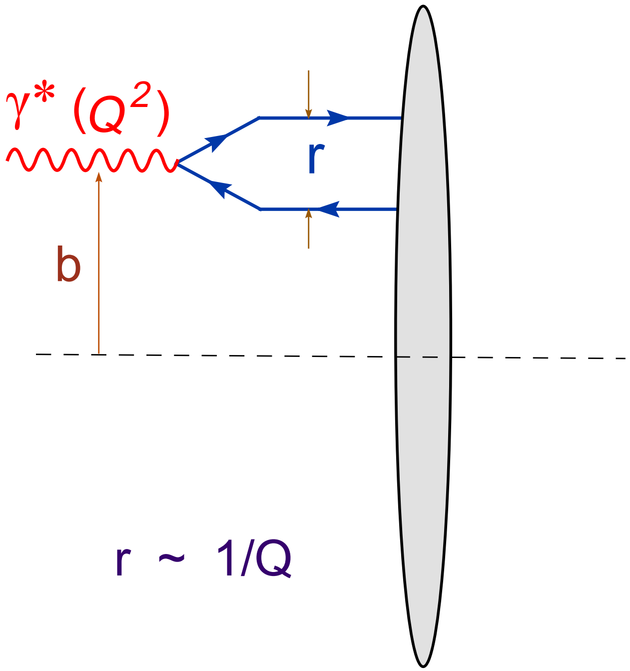


▷ Not restricted to single scattering !

The dipole cross–section

- Optical theorem: $S = \mathbb{1} + \mathrm{i}\mathcal{A}$

$$\sigma_{\rm dipole}(x, r) \;=\; 2\,\Im\mathrm{m}\,\mathcal{A}(s, t=0) \;=\; 2\int \mathrm{d}^2\boldsymbol{b}\,\Im\mathrm{m}\,\mathcal{A}(x, \boldsymbol{r}, \boldsymbol{b})$$



High energy: $\mathcal{A} \approx \mathrm{i}T$ with real T

$T = 1 - S$: ‘scattering amplitude’

$$\sigma_{\rm dipole}(x, r) \;=\; 2\int \mathrm{d}^2\boldsymbol{b}\;T(x, \boldsymbol{r}, \boldsymbol{b})$$

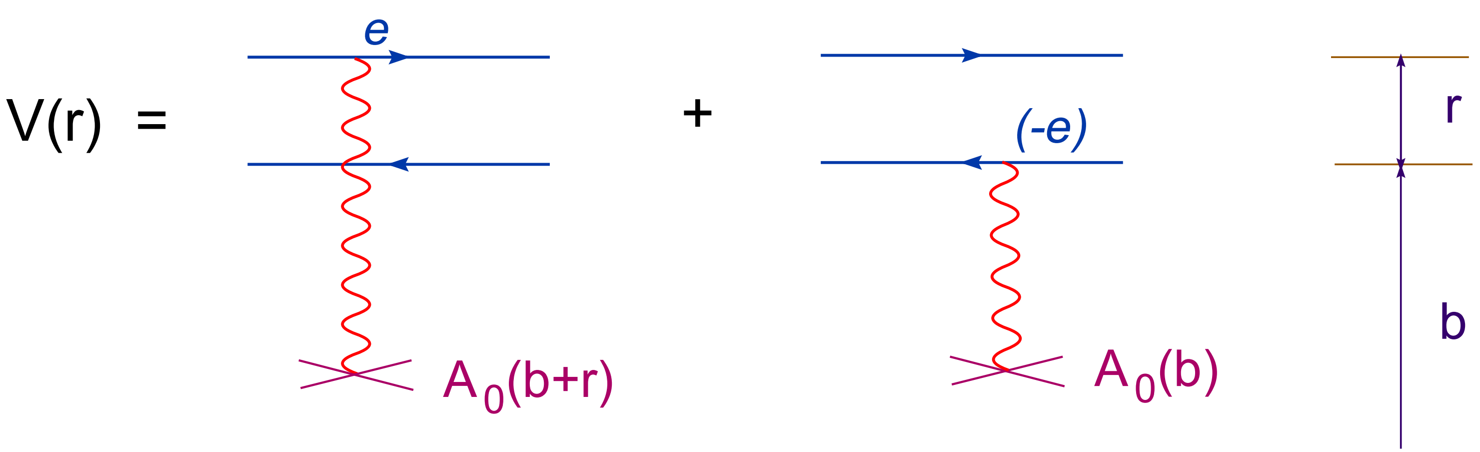
- Unitarity bound : $SS^\dagger = 1 \Longrightarrow T(x, \boldsymbol{r}, \boldsymbol{b}) \le 1$
  - $T \ll 1$: weak scattering
  - $T = 1$: ‘black disk limit’ (total absorption)

Dipole in a background field

- Reminder (classical electrodynamics) :

  A small dipole 'feels' the electric surrounding field:

$$V(\boldsymbol{r}) = e\big[A_0(\boldsymbol{b}+\boldsymbol{r}) - A_0(\boldsymbol{b})\big] \simeq e\, r^i \partial_i A_0(\boldsymbol{b}) = -e\, \boldsymbol{r}\cdot\boldsymbol{E}(\boldsymbol{b})$$



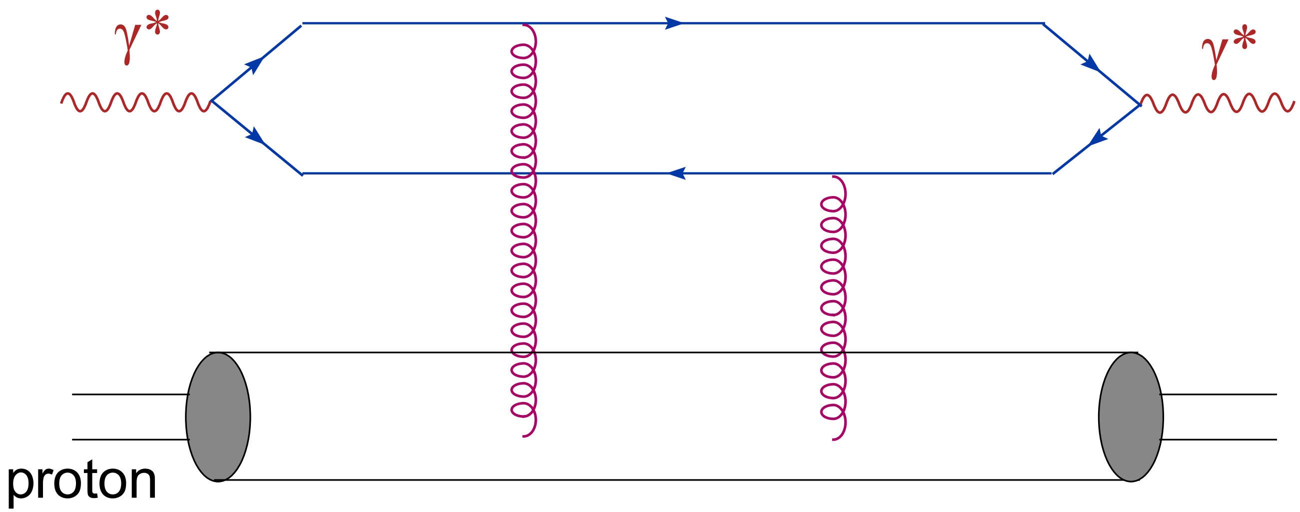
- QCD : 'Color dipole' = $q\bar{q}$ pair in a color singlet state

$$e\,\boldsymbol{r}\cdot\boldsymbol{E} \to g t^a \boldsymbol{r}\cdot\boldsymbol{E}_a \quad + \quad \text{average over color:} \quad \frac{1}{N_c}\mathrm{tr}\{\ldots\}$$

Color dipole: single scattering

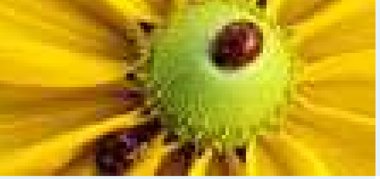
- A small color dipole scatters off the gluon fields in the target

$$V(\boldsymbol{r}) \simeq g t^a \boldsymbol{r} \cdot \boldsymbol{E}_a \implies T(x, \boldsymbol{r}, \boldsymbol{b}) \propto \frac{g^2}{N_c} r^2 \langle \boldsymbol{E}_a(\boldsymbol{b}) \cdot \boldsymbol{E}_a(\boldsymbol{b}) \rangle_x$$



$$T(x, r, b) \simeq \frac{\alpha_s}{N_c}\, r^2\, \frac{xG(x, 1/r^2)}{\pi R^2}$$

▷ A direct measure of the gluon distribution $xG(x, Q^2)$ at $Q^2 \sim 1/r^2$



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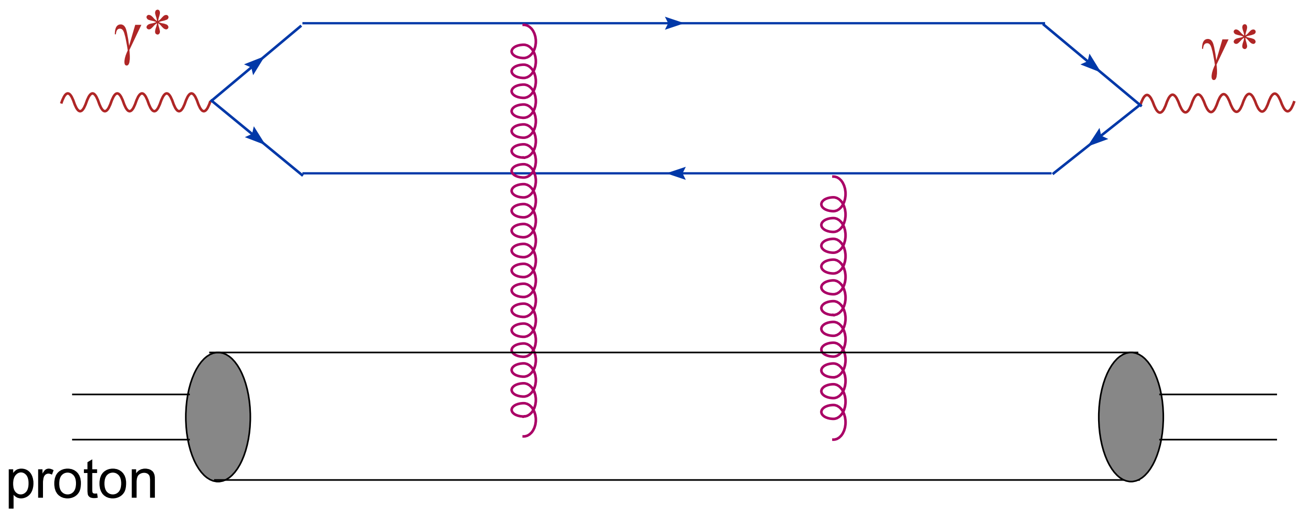
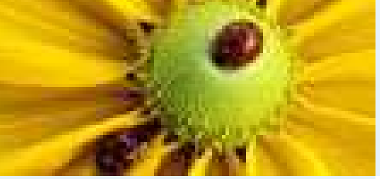


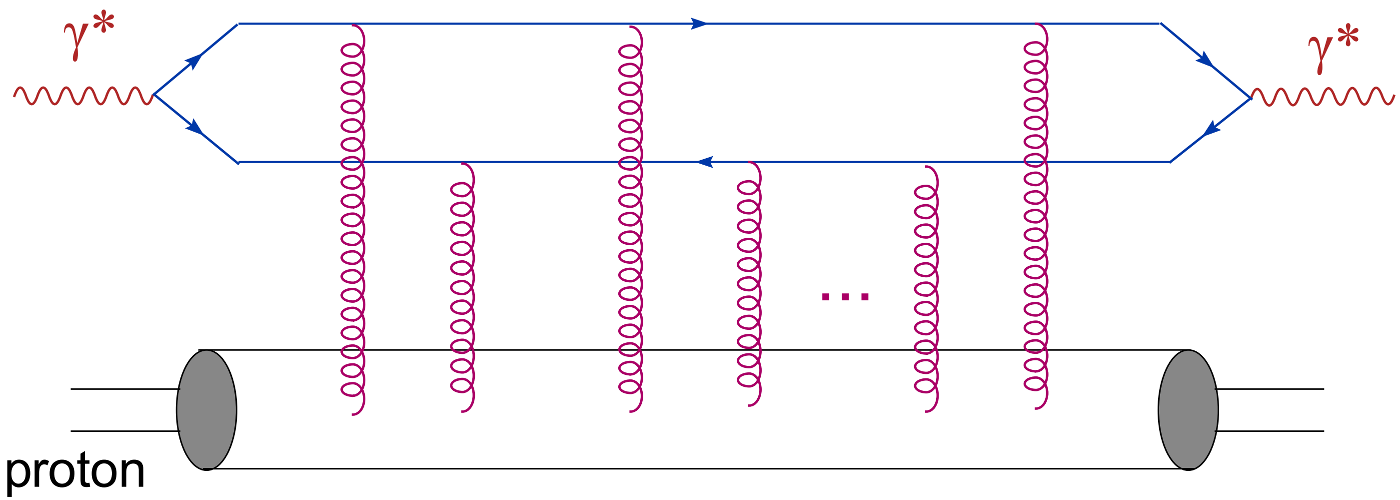
$$T(x, r, b) \propto \alpha_s^2 \, \frac{r^2}{R^2} \, \frac{1}{x^\lambda}$$

- $T(x, r, b) \to 0$ as $r \to 0$: ‘color transparency’



Multiple scattering: Unitarization

- When decreasing x and/or increasing r : $T(x,r) \sim \mathcal{O}(1)$
- Eventually, the single–scattering approximation violates unitarity

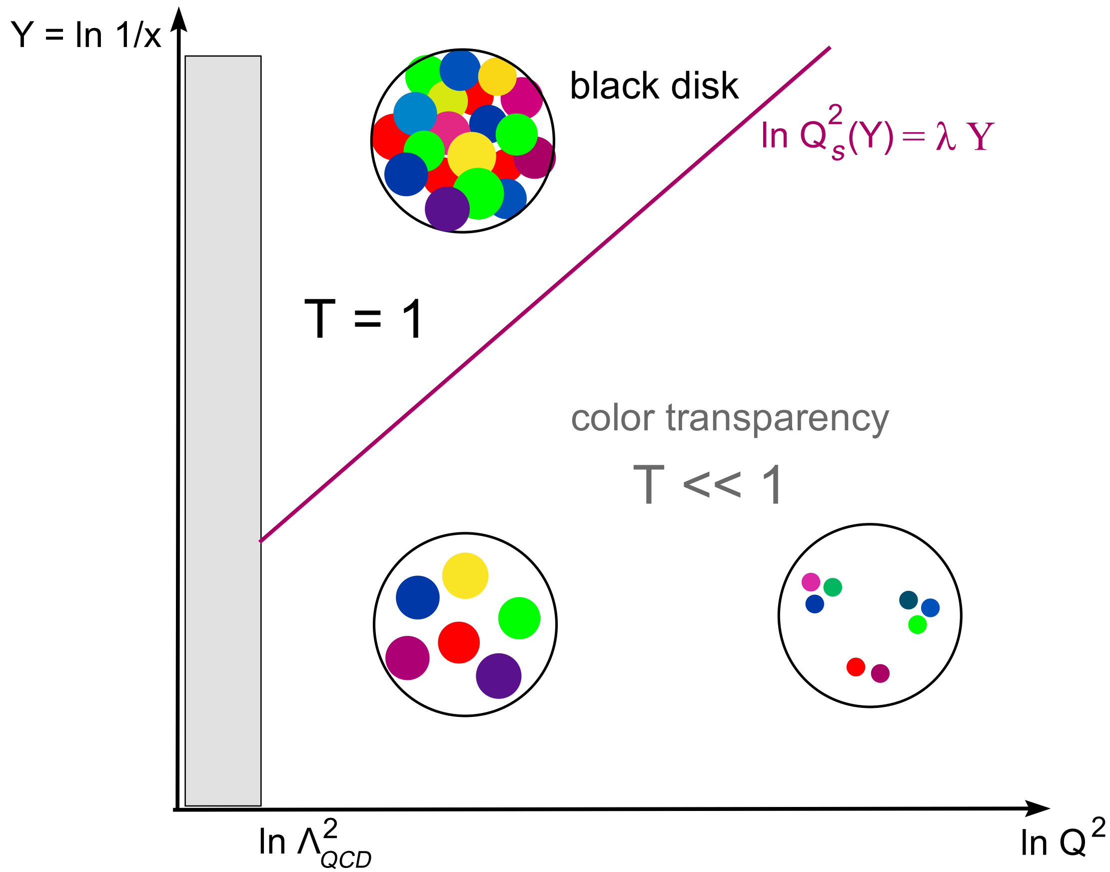


- Multiple scattering becomes important and restores unitarity

The Saturation Momentum

- Onset of unitarity corrections: $T(x,r) \sim 1 \Longleftrightarrow r \sim 1/Q_s(x)$

$$Q_s^2(x) \simeq \frac{\alpha_s}{N_c} \frac{xG(x,Q_s^2)}{\pi R^2} \sim x^{-\lambda}$$

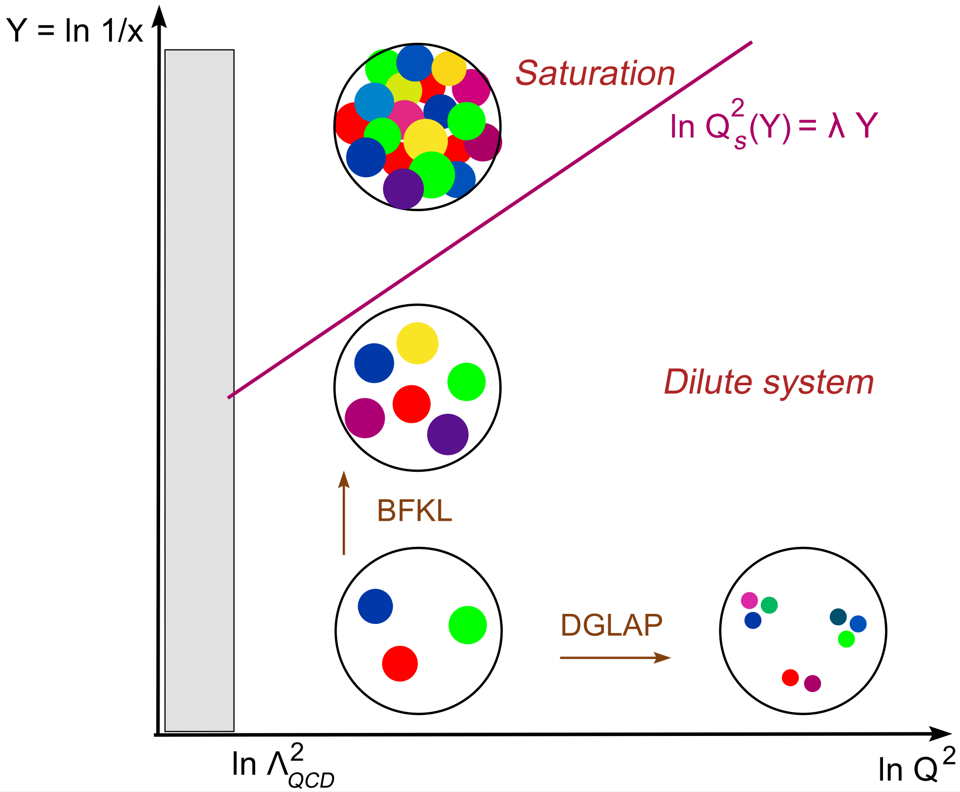


The Saturation Momentum

- The gluon occupation number (or ‘packing factor’):

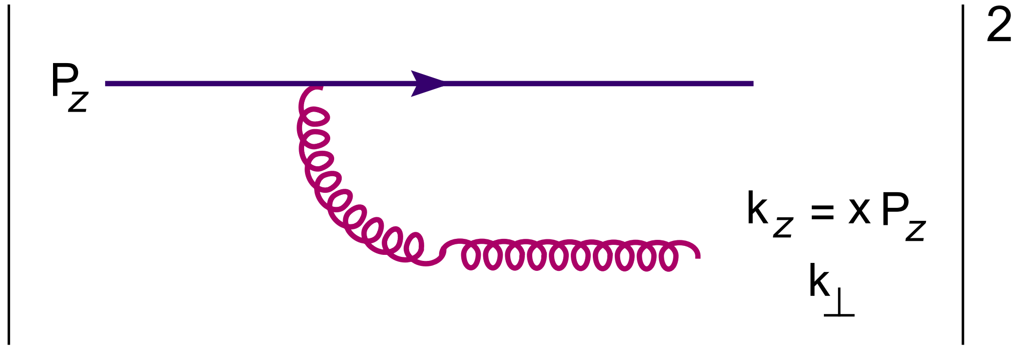
$$n(x, Q^2) \equiv \frac{\pi}{Q^2} \times \frac{xG(x, Q^2)}{\pi R^2}$$

- Onset of non–linear physics : $n \sim 1/\alpha_s \iff A_a^i \sim 1/g$



Bremsstrahlung: probability

- The ‘infrared sensitivity’ of bremsstrahlung favors the emission of ‘soft’ (= small–x) gluons



$$\mathrm{d}P_{\mathrm{Brem}} \simeq \frac{\alpha_s C_F}{\pi^2}\,\frac{\mathrm{d}^2k_\perp}{k_\perp^2}\,\frac{\mathrm{d}k_z}{k_z} \propto \alpha_s \frac{\mathrm{d}x}{x} \equiv \alpha_s\,\mathrm{d}Y$$

$$Y \equiv \ln\frac{1}{x} \sim \ln s \Longrightarrow \mathrm{d}Y = \frac{\mathrm{d}x}{x} : \quad \text{“rapidity”}$$

- A probability of $\mathcal{O}(\alpha_s)$ to emit one gluon per unit rapidity.

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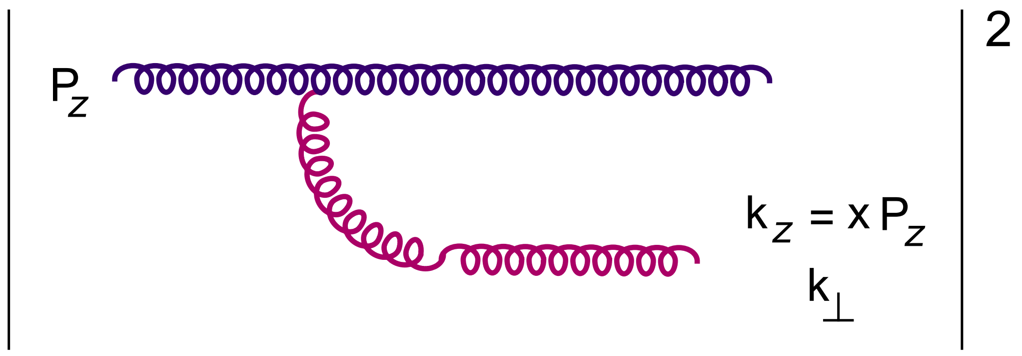

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Eyes wide shut

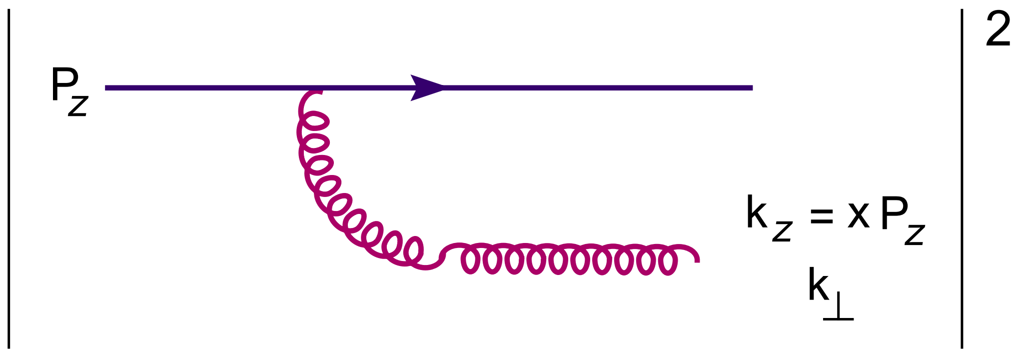
- Theorists, please shut your eyes !

The gluon distribution of a single quark

- $N(x, k_\perp) = \#$ of gluons with longitudinal momentum fraction x and transverse momentum $k_\perp$ radiated by a quark.

"unintegrated gluon distribution" : $$xG(x, Q^2) = \int^{Q^2} \mathrm{d}\, k_\perp^2\, N(x, k_\perp)$$

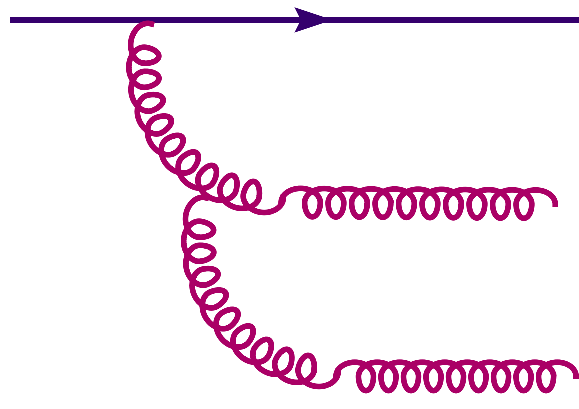
- Lowest order in α_s :



$$N(x, k_\perp) = x \frac{\mathrm{d}P_{\mathrm{Brem}}}{\mathrm{d}x\, \mathrm{d}k_\perp^2} \simeq \frac{\alpha_s C_F}{\pi} \frac{1}{k_\perp^2}$$

Gluon evolution with increasing energy

- $\mathcal{O}(\alpha_s \ln(1/x))$: First order correction ("leading logarithmic approximation")
- One intermediate gluon with x_1 in the range $x \ll x_1 \ll 1$



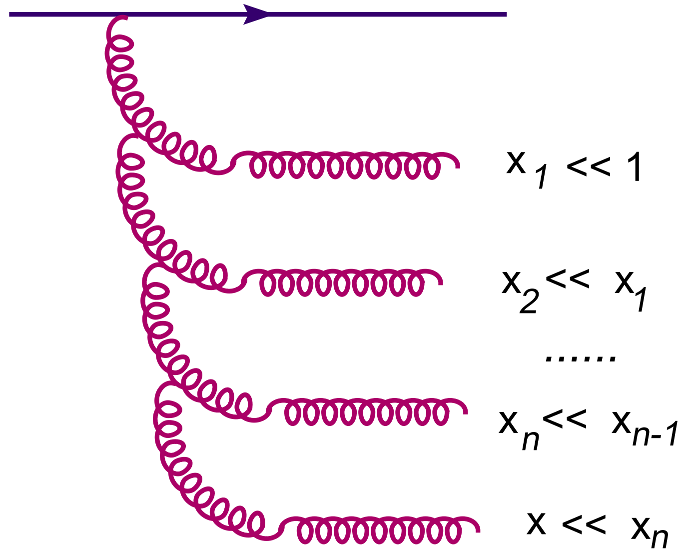
- A contribution of relative order

$$\mathcal{P}(1) \propto \alpha_s \int_x^1 \frac{\mathrm{d}x_1}{x_1} = \alpha_s \ln \frac{1}{x}$$

Gluon evolution with increasing energy

- $\mathcal{O}((\alpha_s Y)^n)$: n intermediate gluons strongly ordered in x

$$x \ll x_n \ll x_{n-1} \cdots \ll x_2 \ll x_1 \ll 1$$

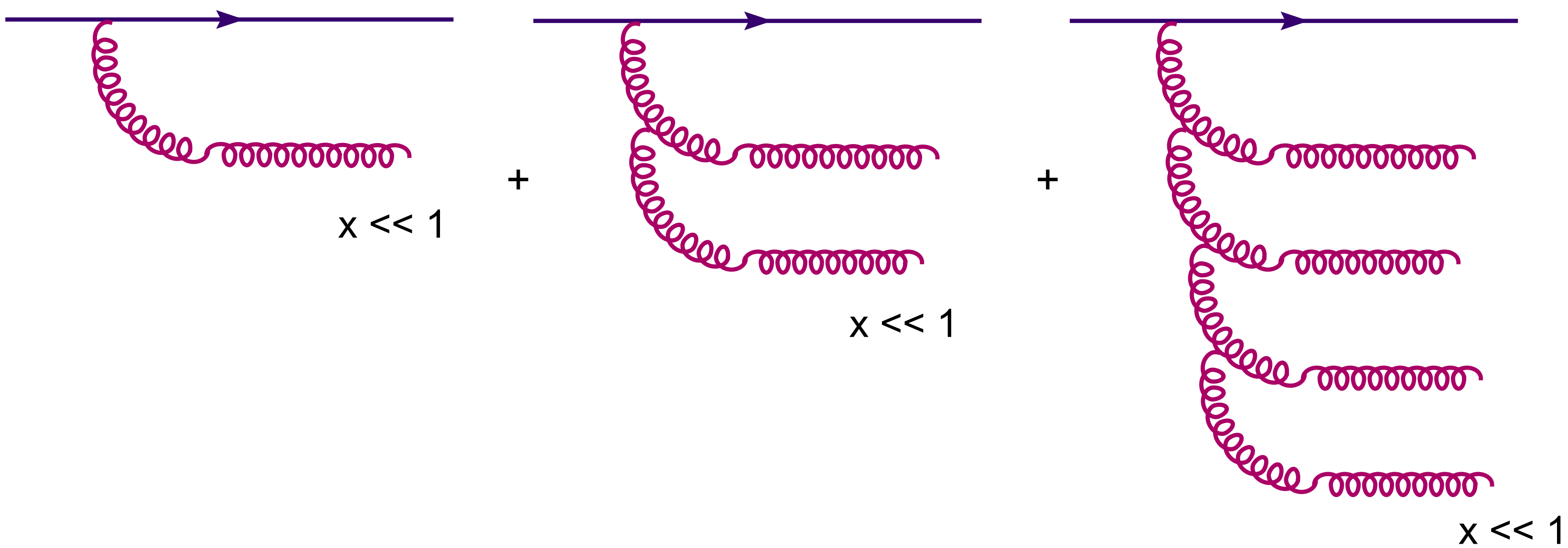


$$\mathcal{P}(n) \propto \frac{1}{n!}\left(\alpha_s \ln \frac{1}{x}\right)^n$$

Gluon evolution with increasing energy

- The sum of all contributions exponentiate:

$$\sum_n \frac{1}{n!}\left(\alpha_s \ln\frac{1}{x}\right)^n \sim \ \mathrm{e}^{\omega\alpha_s Y} \quad \text{with} \quad Y \equiv \ln\frac{1}{x}$$



$$N(Y, k_\perp) \approx \frac{\alpha_s C_F}{\pi}\frac{1}{k_\perp^2}\mathrm{e}^{\omega\alpha_s Y}$$

- "BFKL resummation" *(Balitsky, Fadin, Kuraev, Lipatov, 78)*

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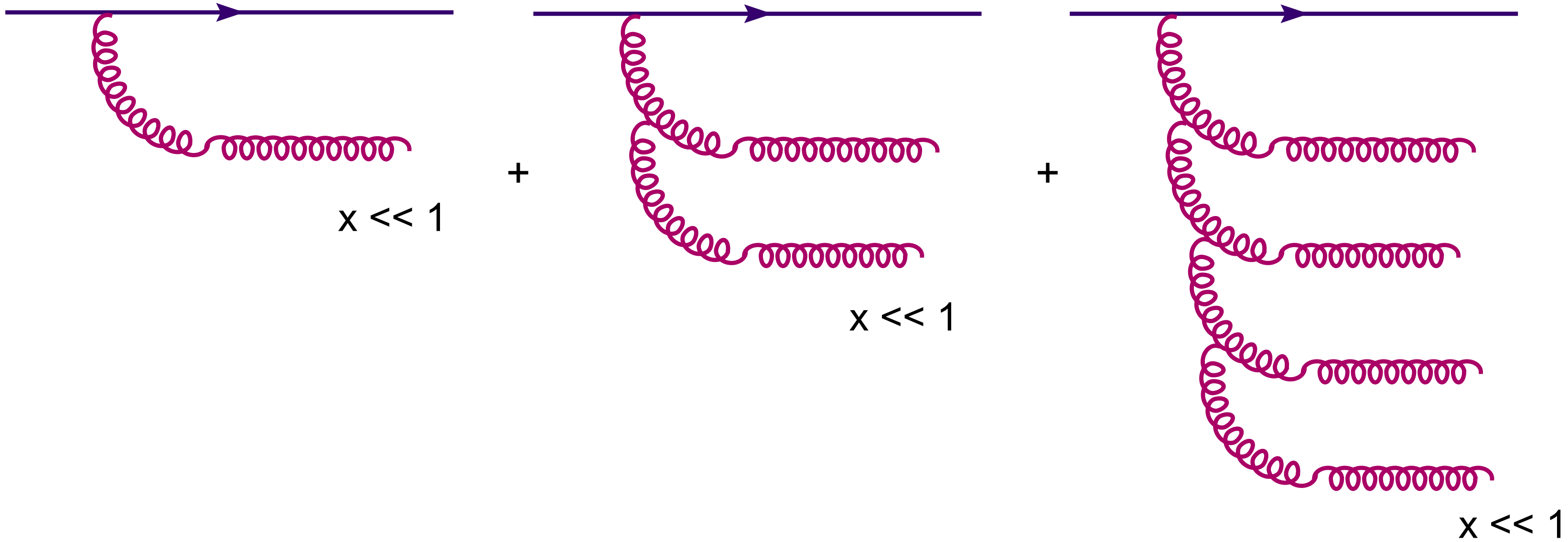

$$N(Y, k_\perp) \approx \frac{\alpha_s C_F}{\pi}\left(\frac{1}{k_\perp^2}\right)^\gamma \mathrm{e}^{\omega\alpha_s Y} \ldots$$

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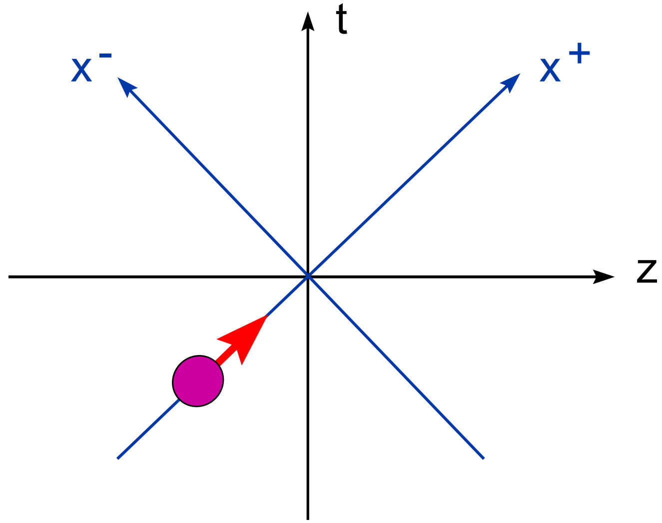
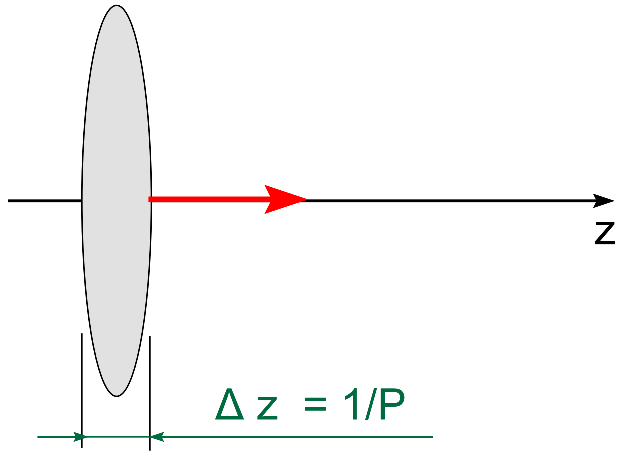
Light Cone notations & Kinematics

- The hadron moves in the positive z direction, with $v \simeq c = 1$



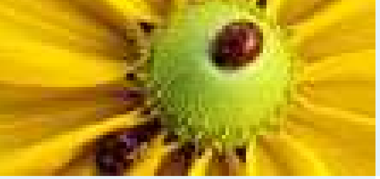


- Longitudinal momentum $P \gg M \Longrightarrow P^\mu = (E \approx P, 0, 0, P)$

$$P^+ \equiv \frac{1}{\sqrt{2}}(E+P) \simeq \sqrt{2}P\,, \quad P^- \equiv \frac{1}{\sqrt{2}}(E-P) \simeq 0$$

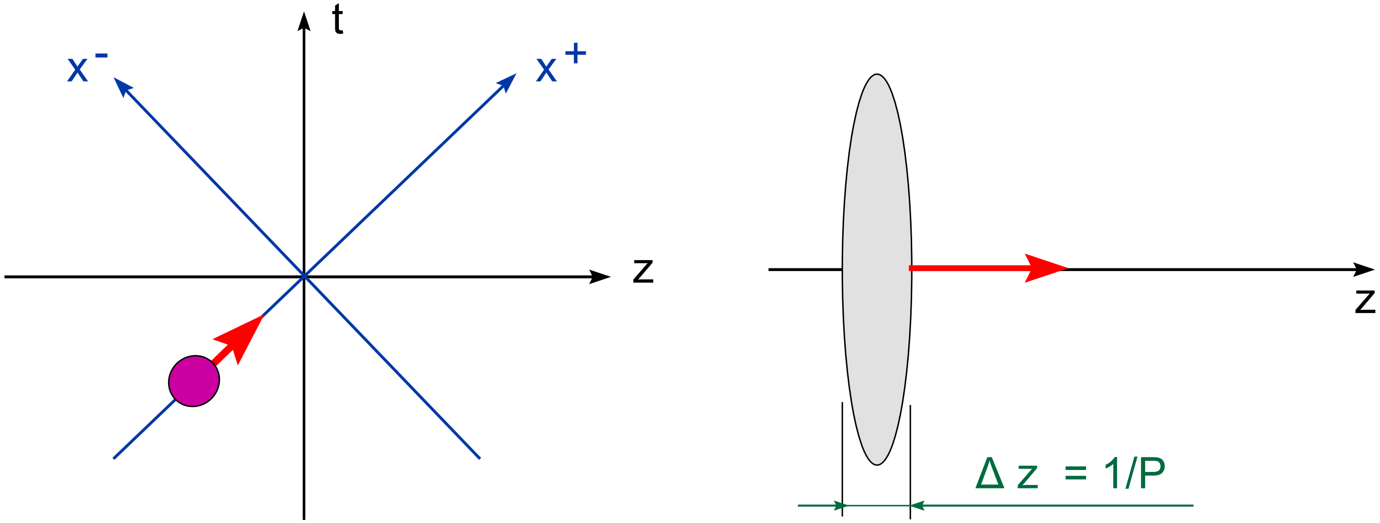
- A classical particle: $z \simeq t$ or $x^- \simeq 0$ with

$$x^+ \equiv \frac{1}{\sqrt{2}}(t+z) \simeq \sqrt{2}t\,, \quad x^- \equiv \frac{1}{\sqrt{2}}(t-z) \simeq 0$$



Light Cone notations & Kinematics

- The hadron moves in the positive z direction, with $v \simeq c = 1$



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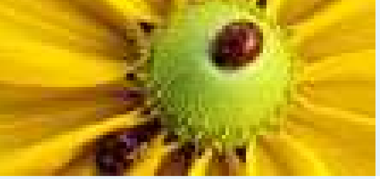
- Even for the quantum system, the wavefunction is strongly localized near $x^- = 0$ (“pancake”)

$$\Delta x^- \sim \frac{1}{P^+} \sim \frac{1}{\gamma M} \ll \frac{1}{M}$$



Light Cone notations & Kinematics

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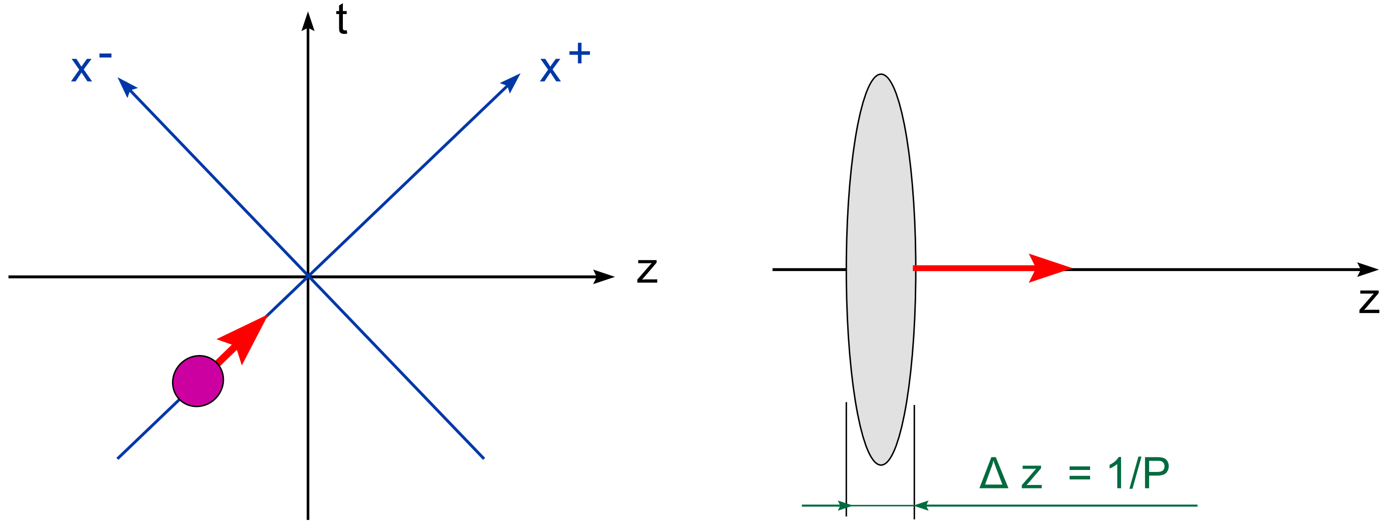
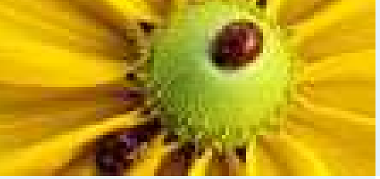


- Scalar product: $k_\mu x^\mu = k^- x^+ + k^+ x^- - \boldsymbol{k} \cdot \boldsymbol{y}$
  - x^+ = LC time $\Longrightarrow k^-$ = LC energy/Hamiltonian
  - x^- = LC longitudinal coordinate $\Longrightarrow k^+$ = LC momentum

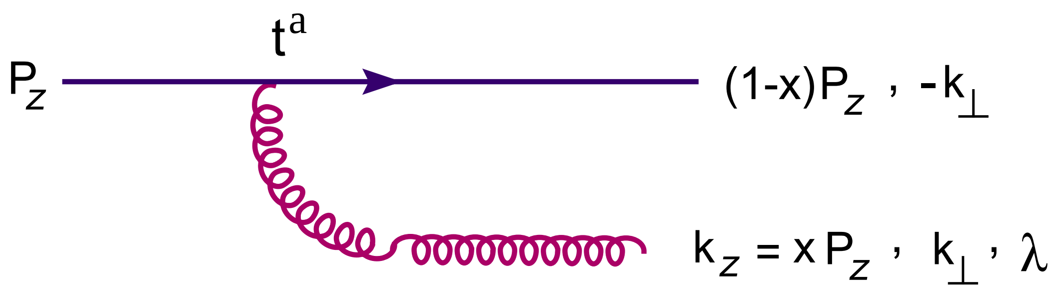
- Vectorial (gauge) interactions : $J_\mu A^\mu = J^+ A^-$

E.g.: $J^+(x) = g\bar{\psi}(x)\, t^a\, \gamma^+\, \psi(x)$ for a right–moving quark



Bremsstrahlung: amplitude

- The amplitude for emitting a ‘soft’ gluon off a fast source



- Use light–cone (LC) gauge $A_a^+ = 0$

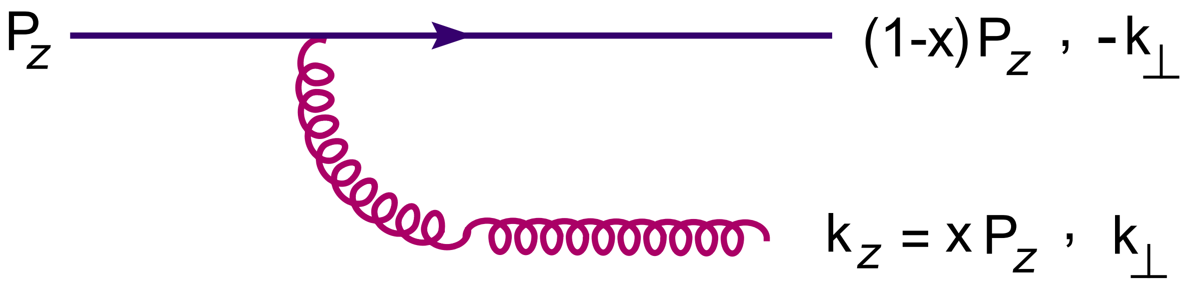
$$\mathcal{M}_\lambda^a(k^+, k_\perp) \simeq g t^a \epsilon_\lambda^- \frac{1}{\sqrt{(2\pi)^3 2k^+}} \frac{1}{\Delta E}$$

- The energy denominator = the inverse of the gluon lifetime

$$\Delta E = -p + \sqrt{x^2p^2 + k_\perp^2} + \sqrt{(1-x)^2p^2 + k_\perp^2} \approx \frac{k_\perp^2}{2xp} = \frac{k_\perp^2}{2k_z}$$

- Repeat the argument for the LC energy P^- : $\Delta E = k_\perp^2/2k^+$

Bremsstrahlung: amplitude

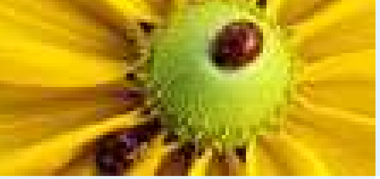


- NB: The lifetime of the virtual gluon is proportional to its longitudinal momentum (Lorentz time dilation)

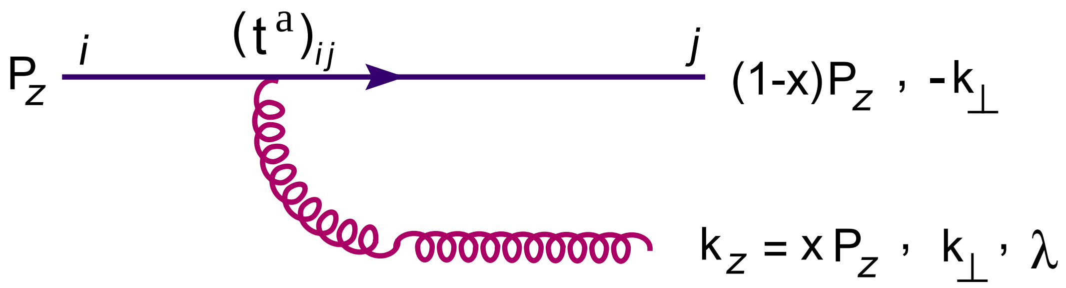
$$\Delta t \sim \frac{\hbar}{\Delta E} = \frac{2xp}{k_\perp^2}$$

- The smaller x, the shorter the lifetime !
- Even at small x, Δt is still large on natural time scales:

$$\Delta t \gg \frac{1}{xp} \qquad \text{so long as} \qquad k_\perp \ll k_z = xp$$



Bremsstrahlung: amplitude



- $k_\mu \epsilon^\mu_\lambda = 0$ & $\epsilon^\mu_\lambda = (0, \epsilon^-_\lambda, \boldsymbol{\epsilon}_\lambda) \implies \epsilon^-_\lambda = \boldsymbol{\epsilon}_\lambda \cdot k_\perp / k^+$

$$\mathcal{M}^a_\lambda(k^+, k_\perp) \simeq g t^a \epsilon^-_\lambda \frac{1}{\sqrt{(2\pi)^3 2k^+}} \frac{2k^+}{k^2_\perp} = \frac{2gt^a}{\sqrt{(2\pi)^3 2k^+}} \frac{\boldsymbol{\epsilon}_\lambda \cdot k_\perp}{k^2_\perp}$$

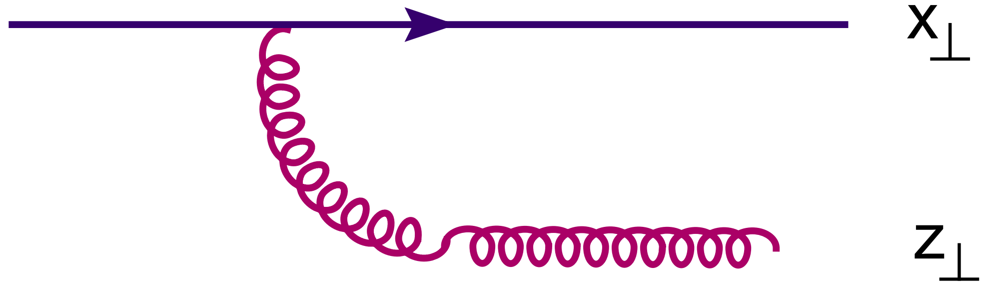
$$\mathrm{d}P_{\mathrm{Brem}} \equiv \sum_{a,\lambda} |\mathcal{M}^a_\lambda(k^+, k_\perp)|^2 \, \mathrm{d}^2 k_\perp \, \mathrm{d}k^+ \simeq \frac{\alpha_s C_F}{\pi^2} \frac{\mathrm{d}^2 k_\perp}{k^2_\perp} \frac{\mathrm{d}k^+}{k^+}$$

- NB: sum/average over color : $(1/N_c) t^a_{ij} (t^a_{ij})^* = C_F$

  sum over polarizations: $\sum_\lambda \epsilon^i_\lambda (\epsilon^j_\lambda)^* = \delta^{ij}$

Bremsstrahlung: amplitude

- The amplitude for soft gluon emission in coordinate space



- $\boldsymbol{r} = \boldsymbol{z} - \boldsymbol{x}$: Gluon position with respect to the parent quark

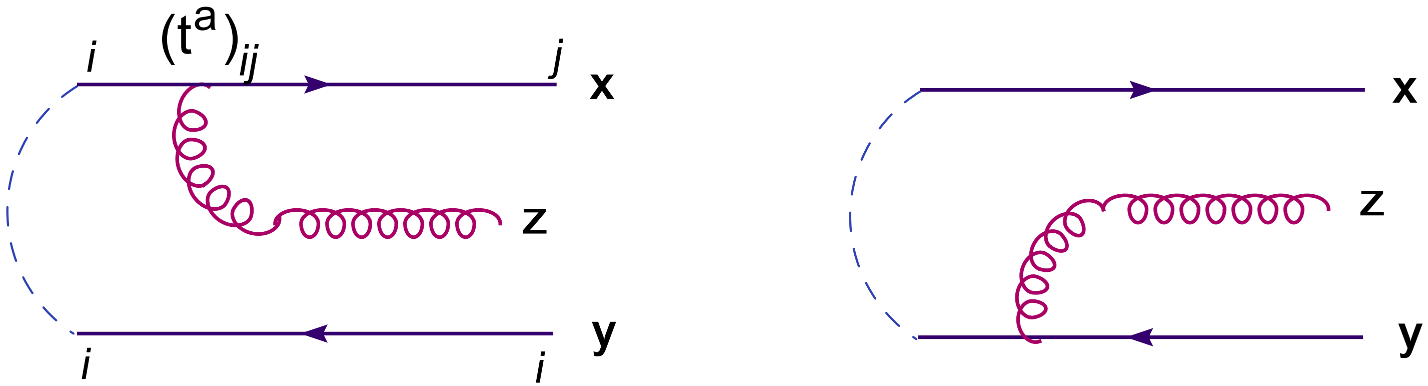
$$\mathcal{M}^a_\lambda(k^+, \boldsymbol{r}) = \int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\, \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}}\, \mathcal{M}^a_\lambda(k^+, \boldsymbol{k})$$

- The amplitude to emit one gluon at $\boldsymbol{z}$ from a quark at $\boldsymbol{x}$ (Exercise !)

$$\mathcal{M}^a_\lambda(k^+, \boldsymbol{x} - \boldsymbol{z}) = \frac{2igt^a}{\sqrt{(2\pi)^3 2k^+}} \frac{\boldsymbol{\epsilon}_\lambda \cdot (\boldsymbol{x} - \boldsymbol{z})}{(\boldsymbol{x} - \boldsymbol{z})^2}$$

Dipole splitting : amplitude

- Soft gluon emission from a color dipole $(\boldsymbol{x}, \boldsymbol{y})$

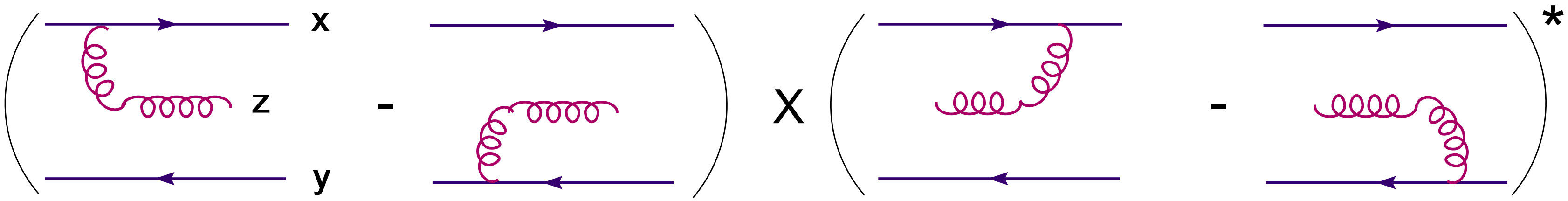


$$\mathcal{M}_\lambda^{a,ij}(k^+, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \frac{t^a_{ij}}{\sqrt{N_c}} \frac{2ig}{\sqrt{(2\pi)^3 2k^+}} \left\{ \frac{\boldsymbol{\epsilon}_\lambda \cdot (\boldsymbol{x} - \boldsymbol{z})}{(\boldsymbol{x} - \boldsymbol{z})^2} - \frac{\boldsymbol{\epsilon}_\lambda \cdot (\boldsymbol{y} - \boldsymbol{z})}{(\boldsymbol{y} - \boldsymbol{z})^2} \right\}$$

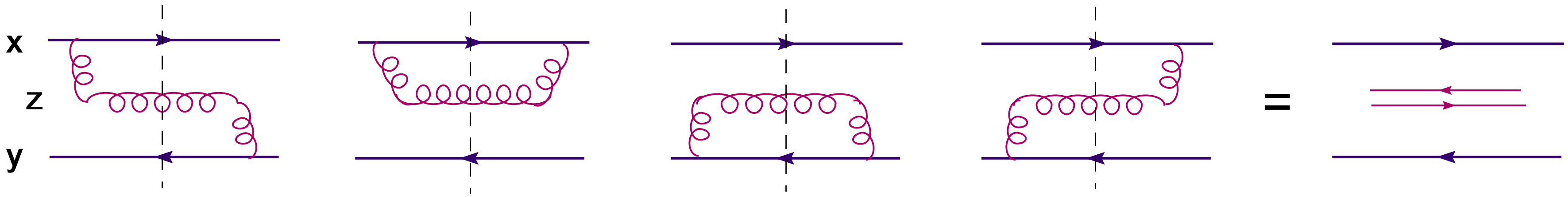
- NB: Before emitting the gluon, the quark and the antiquark composing the dipole have both the same color

Dipole splitting : probability

- Amplitude times complex conjugate amplitude yields this:



- ... which is the same as this :

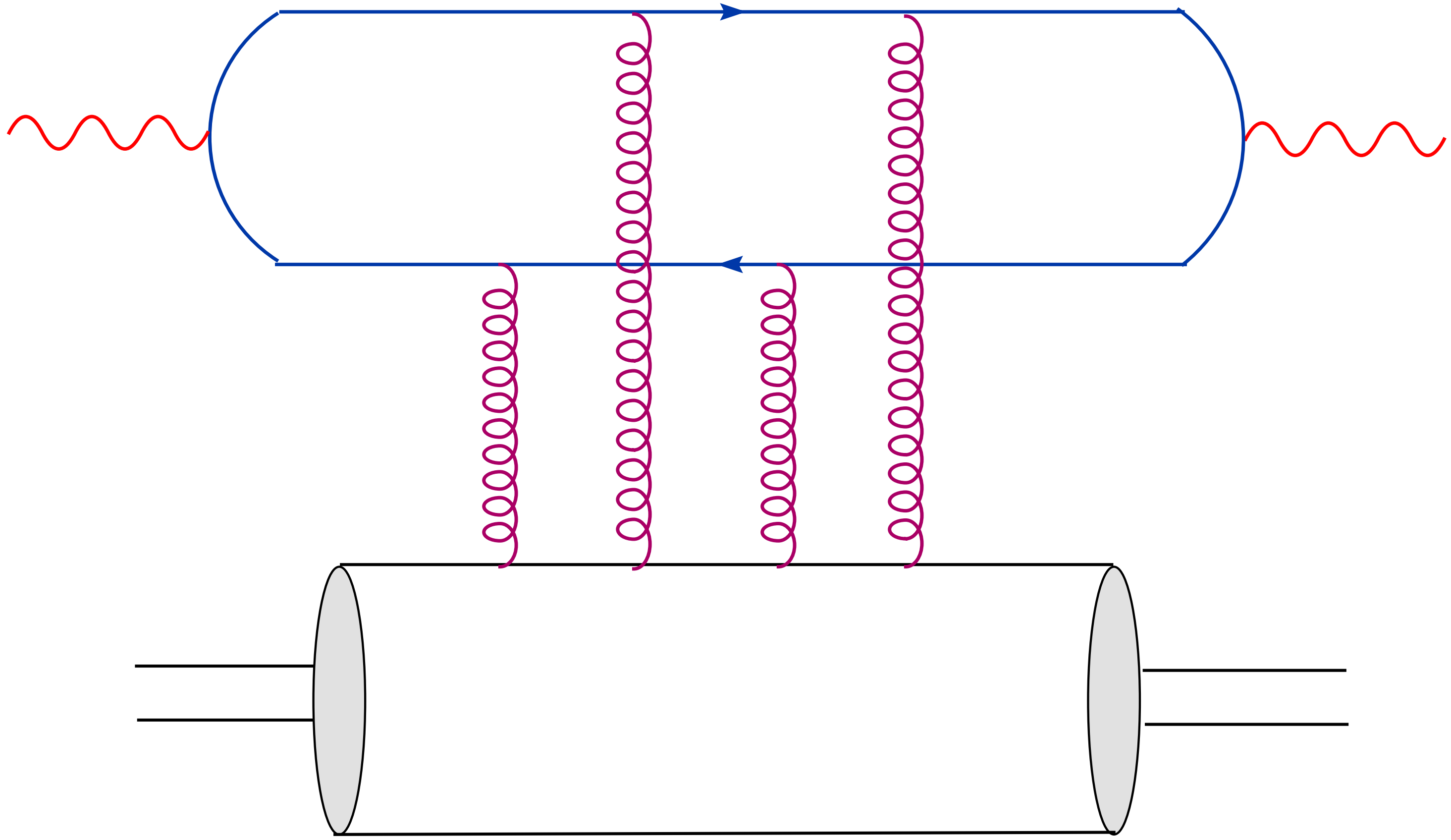


- Differential probability for 'dipole splitting' : (Exercise !)

$$\mathrm{d}P_{\mathrm{split}} = \frac{\alpha_s N_c}{2\pi^2} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \mathrm{d}^2\boldsymbol{z}\,\mathrm{d}Y$$

- Interpretation (large N_c) : one dipole $(\boldsymbol{x}, \boldsymbol{y})$ splits into two dipoles $(\boldsymbol{x}, \boldsymbol{z})$ and $(\boldsymbol{z}, \boldsymbol{y})$

Dipole evolution in DIS



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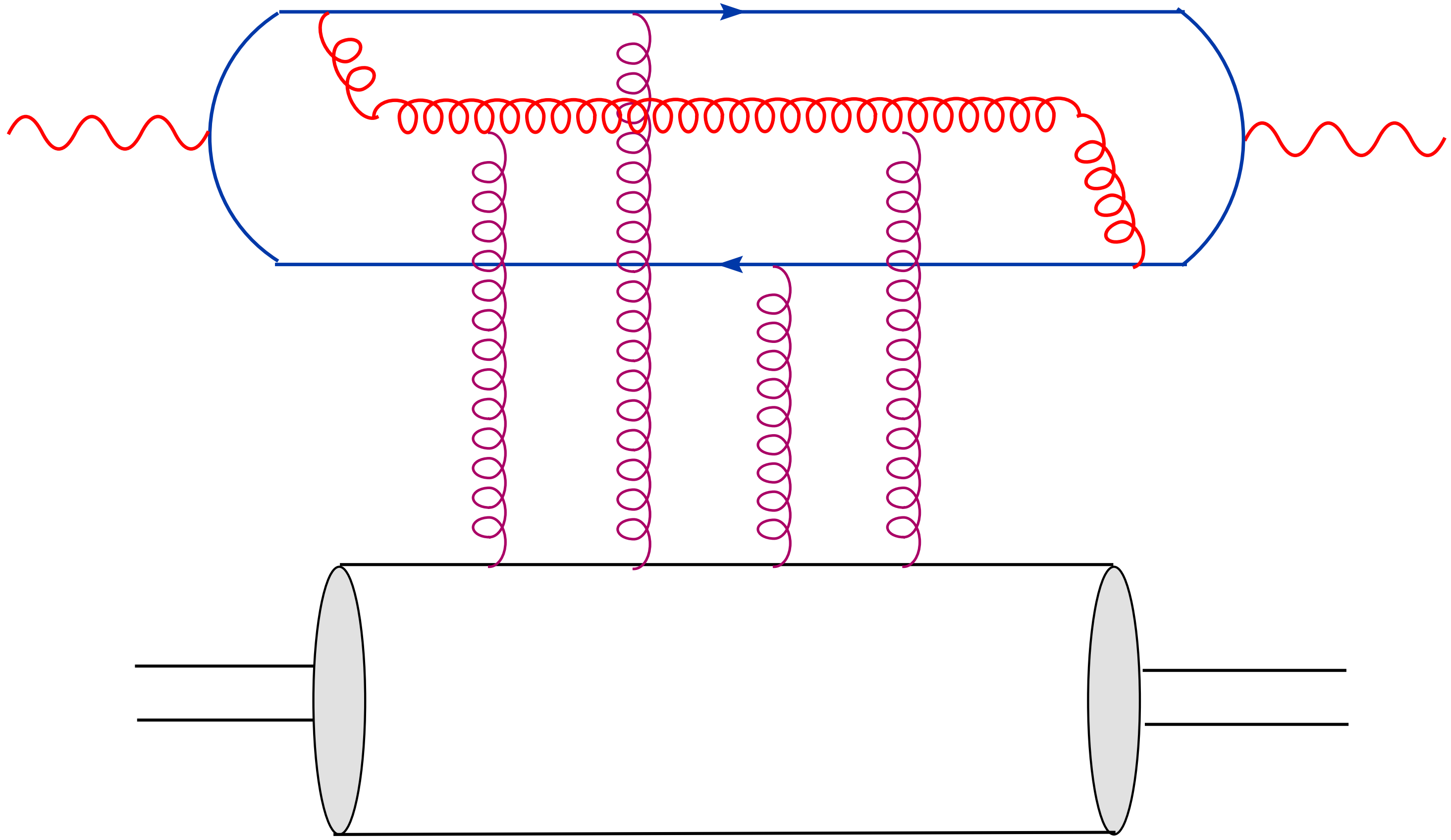
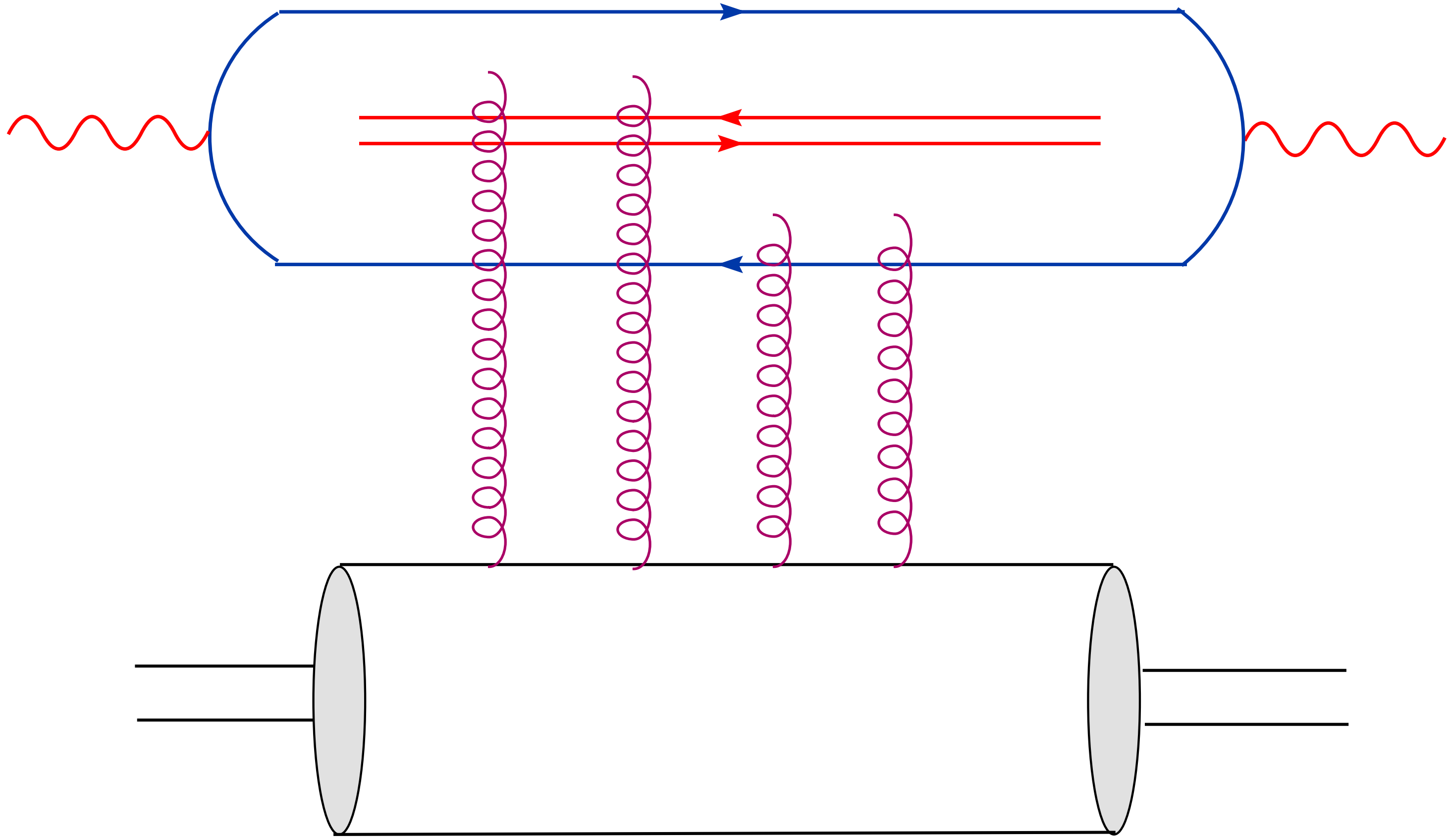
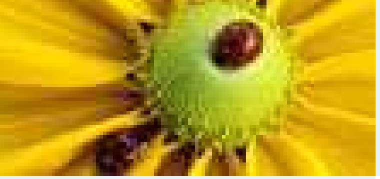


▷ Make one step in rapidity: $Y \to Y + \mathrm{d}Y$ and use the rapidity increment $\mathrm{d}Y$ to boost the color dipole.

Dipole evolution in DIS



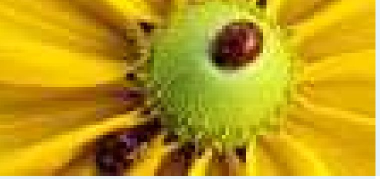
▷ The original dipole splits into two new dipoles which then can scatter off the target fields.



The first Balitsky equation

- $\langle S(\boldsymbol{x},\boldsymbol{y})\rangle_Y$: the S–matrix element for the scattering between the dipole $(\boldsymbol{x},\boldsymbol{y})$ and the target separated by a rapidity gap Y
  - $|\langle S(\boldsymbol{x},\boldsymbol{y})\rangle_Y|2 =$ the dipole survival probability
  - $\langle S(\boldsymbol{x}_1,\boldsymbol{y}_1)S(\boldsymbol{x}_2,\boldsymbol{y}_2)\rangle_Y$: the S–matrix for a projectile made with 2 dipoles: $(\boldsymbol{x}_1,\boldsymbol{y}_1)$ and $(\boldsymbol{x}_2,\boldsymbol{y}_2)$
- One evolution step : $Y \to Y+\mathrm{d}Y$ with $\alpha_s \mathrm{d}Y \ll 1$

$$
\begin{aligned}
\langle S(\boldsymbol{x},\boldsymbol{y})\rangle_{Y+\mathrm{d}Y} \quad &= \quad \left(1-\mathrm{d}Y\int \mathrm{d}^2\boldsymbol{z}\,\frac{\mathrm{d}P_{\rm split}}{\mathrm{d}Y\,\mathrm{d}^2\boldsymbol{z}}\right)\langle S(\boldsymbol{x},\boldsymbol{y})\rangle_Y \\
&+ \quad \mathrm{d}Y\int \mathrm{d}^2\boldsymbol{z}\,\frac{\mathrm{d}P_{\rm split}}{\mathrm{d}Y\,\mathrm{d}^2\boldsymbol{z}}\,\langle S(\boldsymbol{x},\boldsymbol{z})S(\boldsymbol{z},\boldsymbol{y})\rangle_Y
\end{aligned}
$$

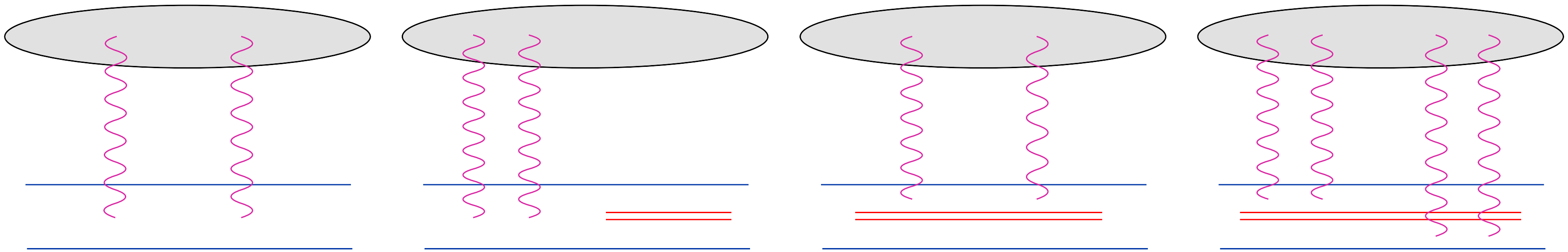


The first Balitsky equation

$$\frac{\partial}{\partial Y}\langle S(\boldsymbol{x},\boldsymbol{y})\rangle_Y = \frac{\bar{\alpha}_s}{2\pi}\int_{\boldsymbol{z}} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2}\Big\{-\langle S(\boldsymbol{x},\boldsymbol{y})\rangle_Y + \langle S(\boldsymbol{x},\boldsymbol{z})S(\boldsymbol{z},\boldsymbol{y})\rangle_Y\Big\}$$

- Not a closed equation ! (one dipole $\rightarrow$ two dipoles)
- Rewritten for the dipole scattering amplitude $T \equiv 1 - S$:

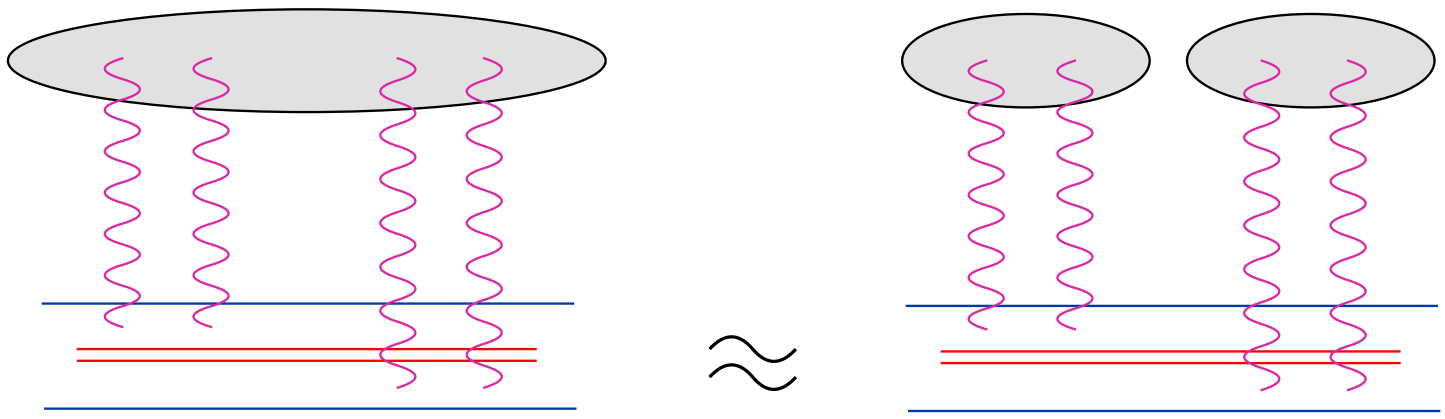
$$\frac{\partial}{\partial Y}\langle T(\boldsymbol{x},\boldsymbol{y})\rangle_Y = \frac{\bar{\alpha}_s}{2\pi}\int_{\boldsymbol{z}} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \Big\langle \underbrace{-T(\boldsymbol{x},\boldsymbol{y}) + T(\boldsymbol{x},\boldsymbol{z}) + T(\boldsymbol{z},\boldsymbol{y})}_{\text{BFKL (linear)}} - \underbrace{T(\boldsymbol{x},\boldsymbol{z})T(\boldsymbol{z},\boldsymbol{y})}_{\text{non–linear}}\Big\rangle_Y$$



The Balitsky–Kovchegov equation

- Mean field approximation assuming factorization:

$$\langle T(\boldsymbol{x},\boldsymbol{z})T(\boldsymbol{z},\boldsymbol{y})\rangle_Y \;\approx\; \langle T(\boldsymbol{x},\boldsymbol{z})\rangle_Y\,\langle T(\boldsymbol{z},\boldsymbol{y})\rangle_Y$$



- Neglects correlations in the target wavefunction.
- Some usual justifications in the literature:
  - large nucleus $A \gg 1$
  - large N_c
  - leads to a relatively simple (closed) equation ✓

The Balitsky–Kovchegov equation

$$\frac{\partial}{\partial Y} T(\boldsymbol{x},\boldsymbol{y}) = \frac{\bar{\alpha}_s}{2\pi} \int_{\boldsymbol{z}} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \Big\{ \underbrace{-T(\boldsymbol{x},\boldsymbol{y}) + T(\boldsymbol{x},\boldsymbol{z}) + T(\boldsymbol{z},\boldsymbol{y})}_{\text{BFKL (linear)}} - \underbrace{T(\boldsymbol{x},\boldsymbol{z})T(\boldsymbol{z},\boldsymbol{y})}_{\text{non–linear}} \Big\}$$

- Notations: $\bar{\alpha}_s \equiv \alpha_s N_c/\pi, \;\; T(\boldsymbol{x},\boldsymbol{y}) \equiv \langle T(\boldsymbol{x},\boldsymbol{y})\rangle_Y$
- Weak scattering $T \ll 1$ (low energy/small dipole) $\Longrightarrow$ BFKL
  - $T = 0$: unstable fixed point of BK equation
  - unitarity violations, infrared diffusion
- Strong scattering $T \sim 1$ $\Longrightarrow$ Non–linear effects
  - $T = 1$: stable fixed point of BK equation
- BK equation: a simple framework to study unitarization