Non–linear evolution & Gluon saturation in QCD at high energy

Part II

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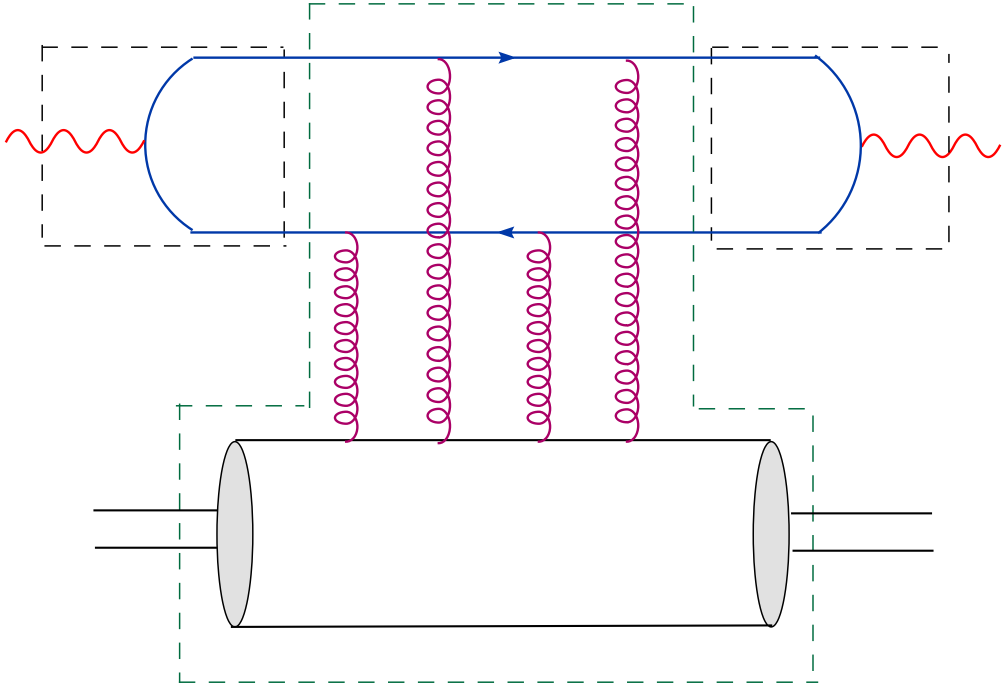
A brief reminder

- Previously on “Desperate Housewives”

Dipole factorization for DIS

$$\sigma_{\gamma^* p}(x, Q^2) = \int_0^1 \mathrm{d}z \int \mathrm{d}^2\boldsymbol{r}\; |\Psi_\gamma(z, \boldsymbol{r}; Q^2)|^2 \; \sigma_{\mathrm{dipole}}(x, \boldsymbol{r})$$

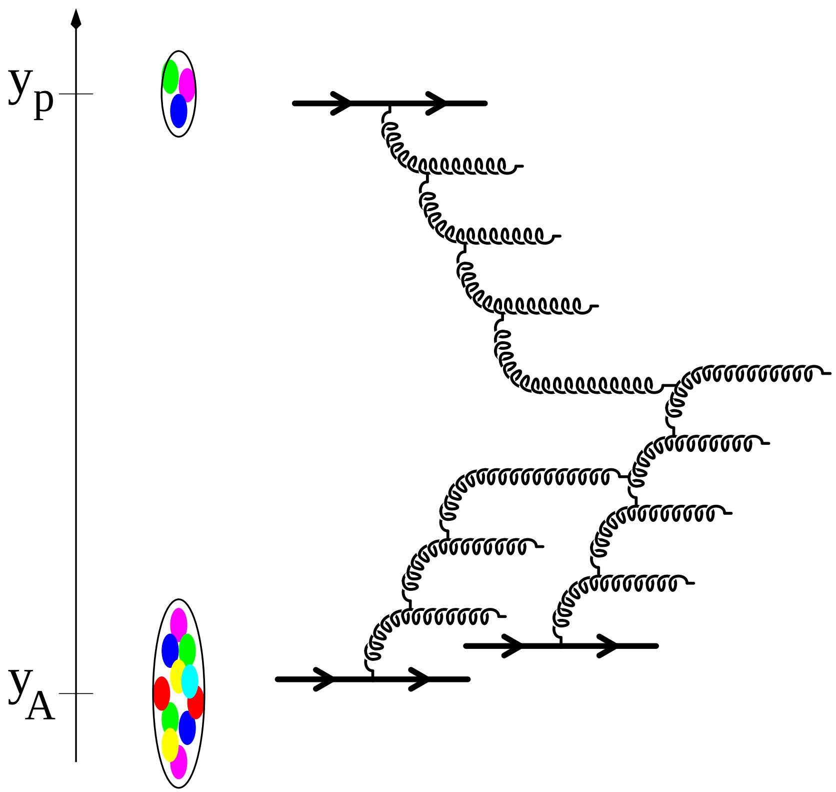


$$\sigma_{\mathrm{dipole}}(x, r) \;=\; 2 \int \mathrm{d}^2\boldsymbol{b}\; T(x, \boldsymbol{r}, \boldsymbol{b})$$

- Unitarity bound on the dipole amplitude: $T(x, \boldsymbol{r}, \boldsymbol{b}) \;\leq\; 1$

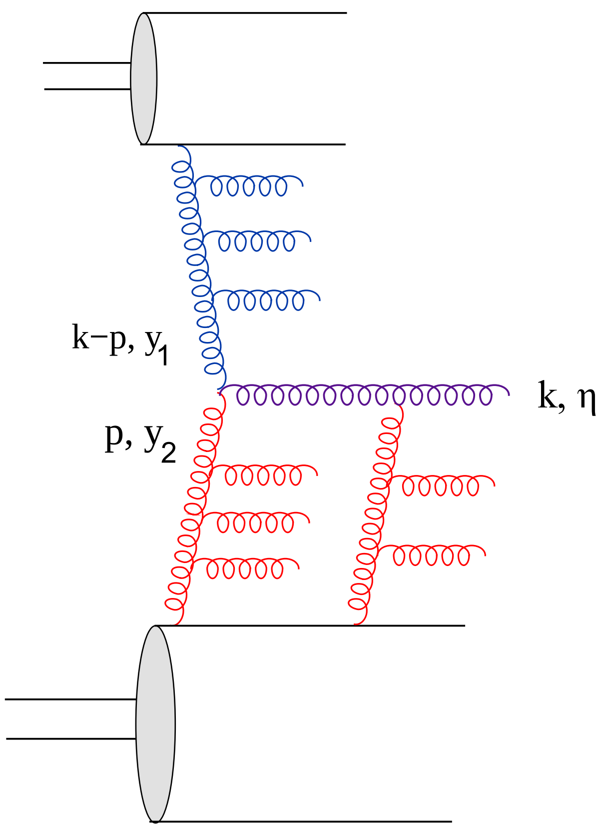
Forward gluon production in H–H collisions

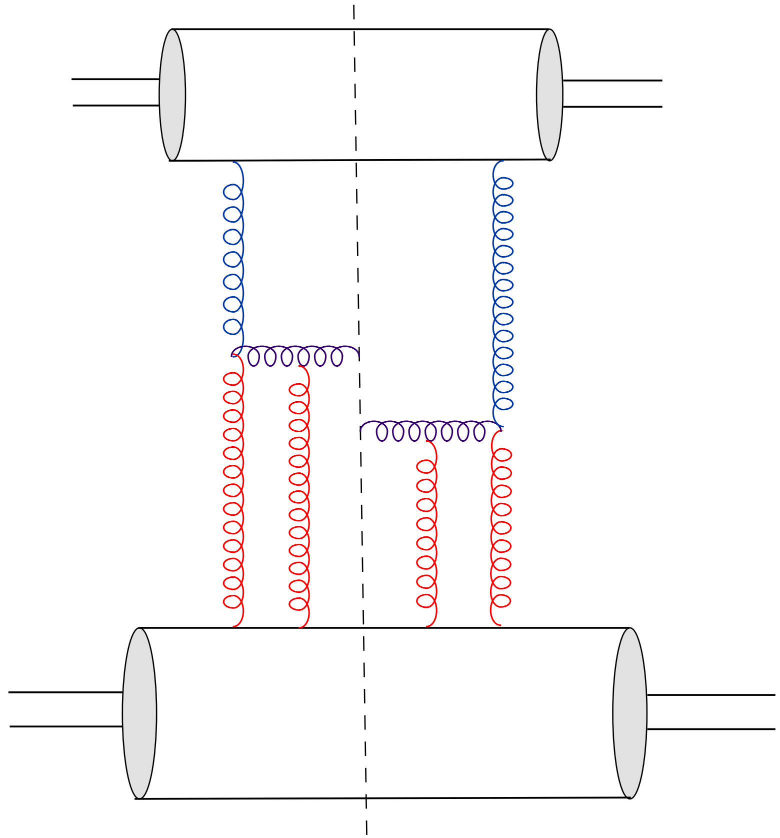
- ‘Dense–dilute’ scattering
  - pA collisions (RHIC, LHC)
  - pp collisions at forward rapidity (LHC)
- Only one parton from the dilute projectile gets involved



- The particles in the final state undergo multiple scattering

Forward gluon production in H–H collisions



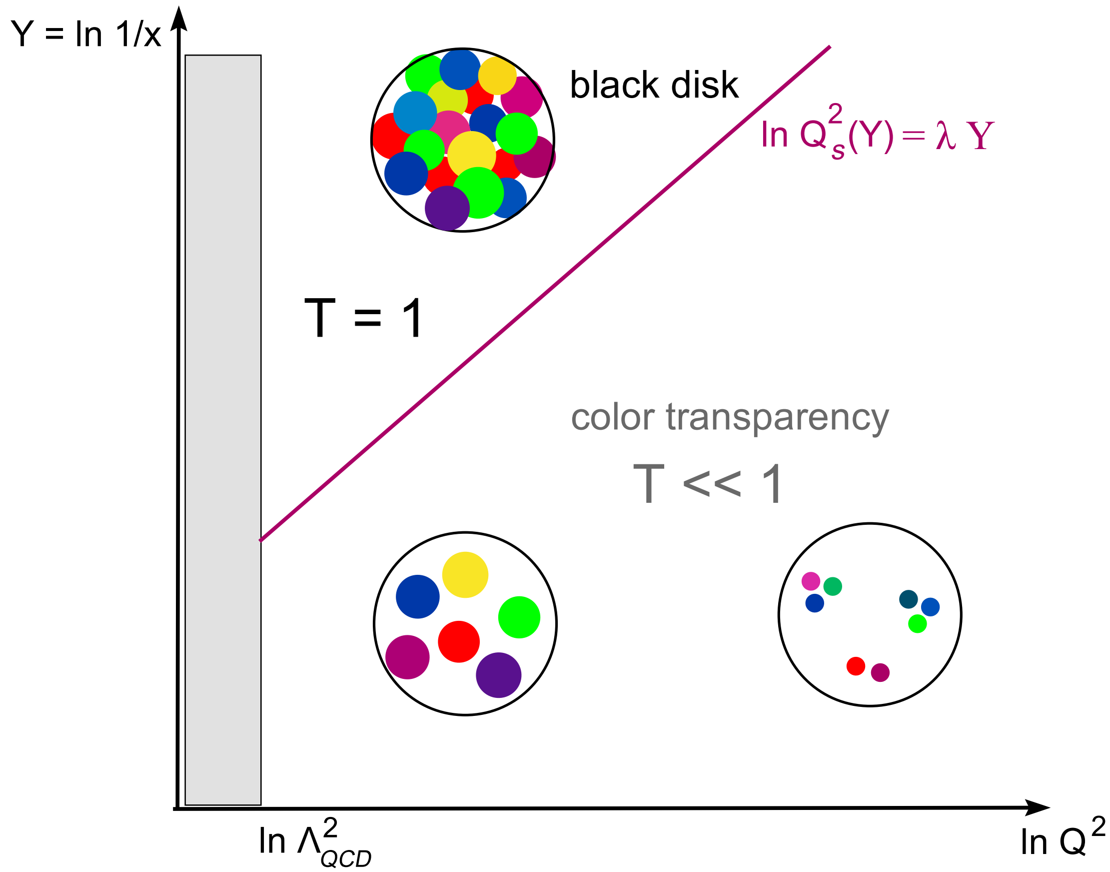


$$\Phi(\boldsymbol{p}, y_2) \equiv \int \mathrm{d}^2\boldsymbol{r}\ \mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{r}}\ \nabla^2_{\boldsymbol{r}} T_{gg}(\boldsymbol{r}, y_2)$$

- $y_i = \ln(1/x_i)$ where $x_{1,2} = (k_\perp/\sqrt{s})\mathrm{e}^{\pm\eta}$

The Saturation Line

- Dilute parton gas at low energy/large Q^2
- High gluon density at high energy/low Q^2
- Energy–dependent boundary: $Q_s^2(Y) = Q_0^2\, \mathrm{e}^{\lambda Y}$

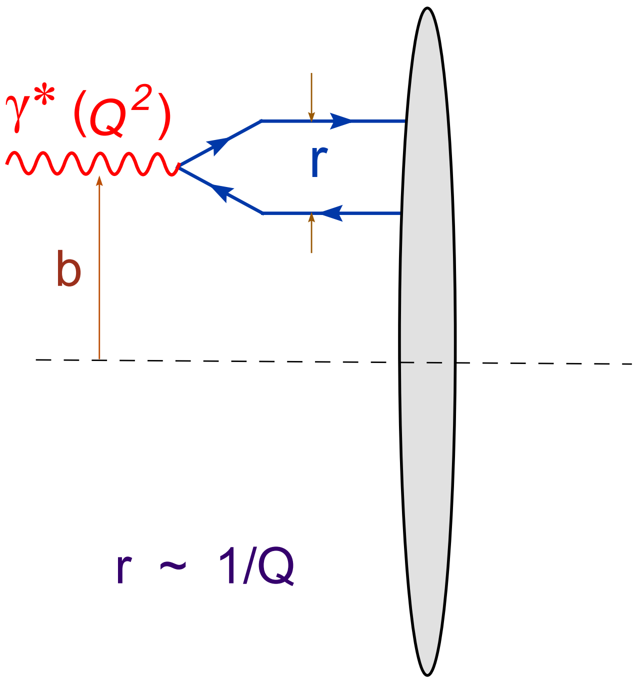


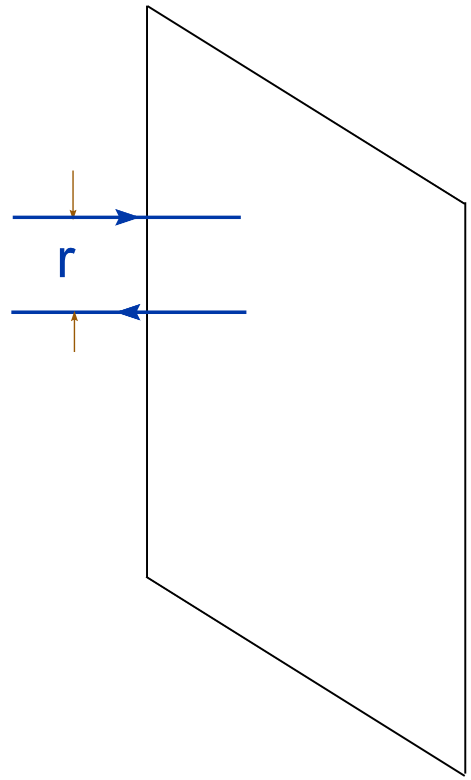
The Balitsky–Kovchegov equation

$$\frac{\partial}{\partial Y} T(\boldsymbol{x}, \boldsymbol{y}) = \frac{\bar{\alpha}_s}{2\pi} \int_{\boldsymbol{z}} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2 (\boldsymbol{y}-\boldsymbol{z})^2} \Big\{ \underbrace{-T(\boldsymbol{x}, \boldsymbol{y}) + T(\boldsymbol{x}, \boldsymbol{z}) + T(\boldsymbol{z}, \boldsymbol{y})}_{\text{BFKL (linear)}} - \underbrace{T(\boldsymbol{x}, \boldsymbol{z}) T(\boldsymbol{z}, \boldsymbol{y})}_{\text{non–linear}} \Big\}$$

- Weak scattering $T \ll 1$ (low energy/small dipole) $\Longrightarrow$ BFKL
  - $T = 0$: unstable fixed point of BK equation
  - unitarity violations, infrared diffusion
- Strong scattering $T \sim 1 \Longrightarrow$ Non–linear effects
  - $T = 1$: stable fixed point of BK equation
- BK equation: a simple framework to study unitarization

b -dependence, initial conditions ...



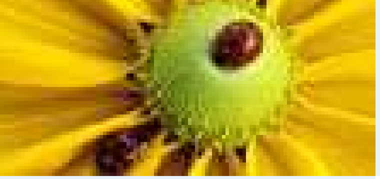


- $T(\boldsymbol{x}, \boldsymbol{y}) = T(\boldsymbol{r}, \boldsymbol{b})$

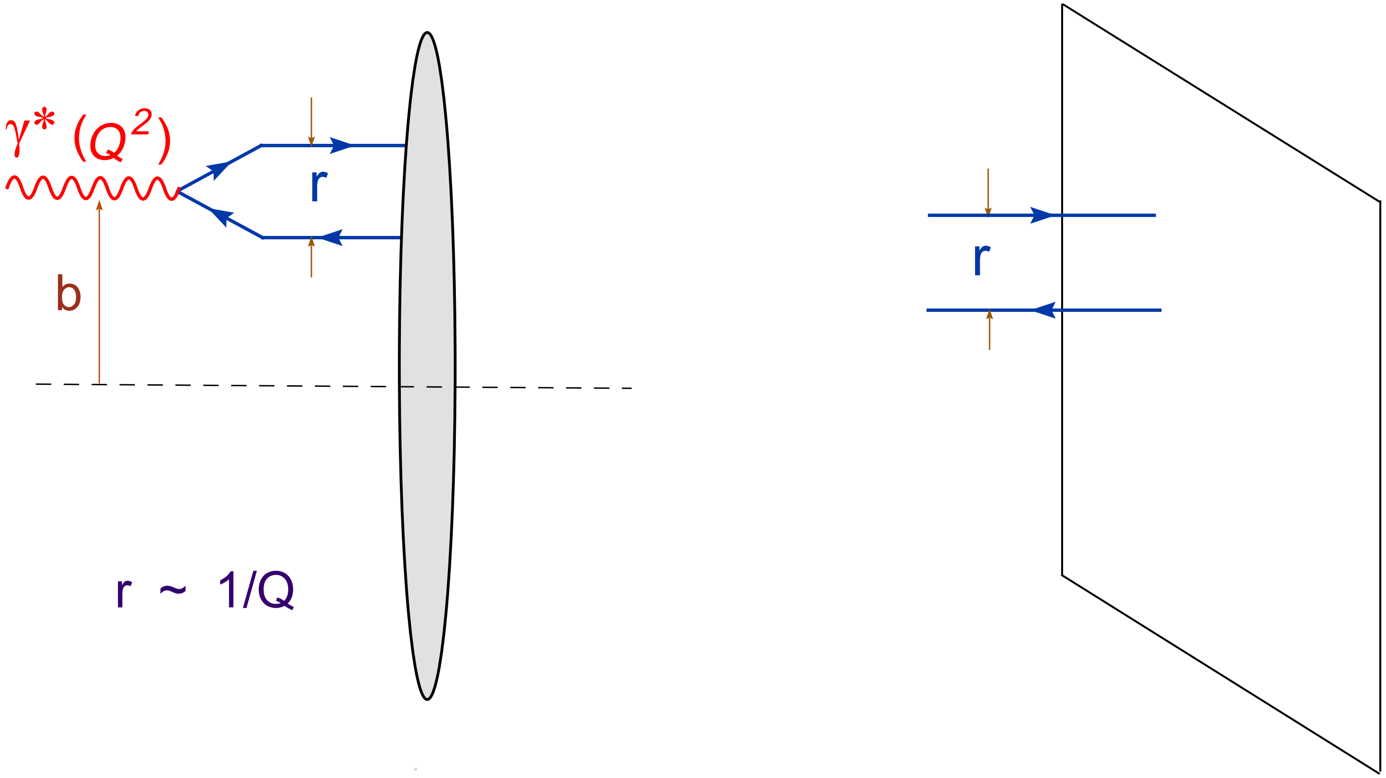
  $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$ (dipole size) and $\boldsymbol{b} = (\boldsymbol{x} + \boldsymbol{y})/2$ (impact parameter)

- Translational invariance of the ‘dipole kernel’ $\Longrightarrow$

  the equation admits $\boldsymbol{b}$–independent solutions: $T = T(r, Y)$



b -dependence, initial conditions ...



- Homogeneous initial conditions at $Y = 0$: ‘large nucleus’

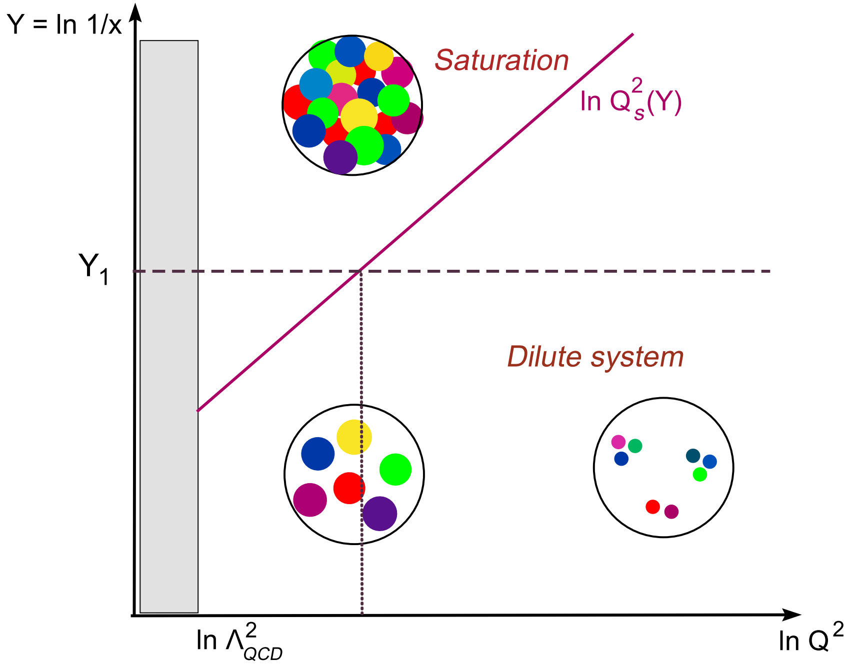
$$T(r, Y=0) \approx \begin{cases} r^2 Q_0^2 & \text{for } r \ll 1/Q_0 \\ 1 & \text{for } r \gtrsim 1/Q_0 \end{cases}$$

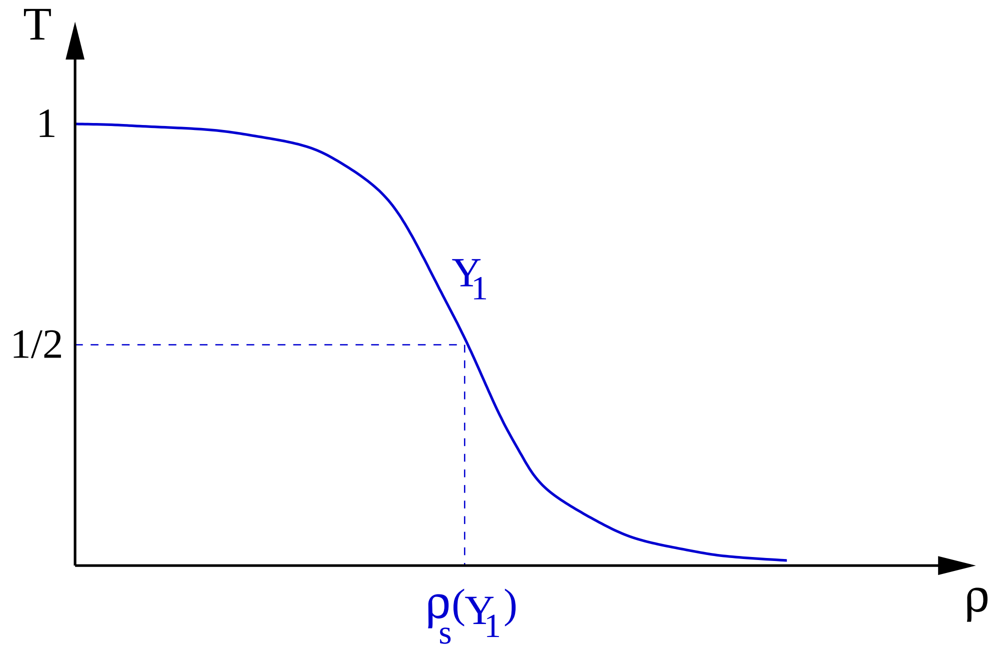
- Q_0 : saturation momentum in the initial conditions

The saturation front

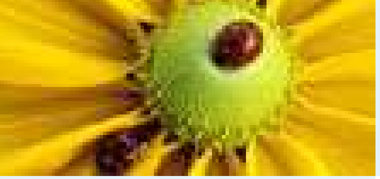
- The BK solution: $T(r, Y) \equiv T(\rho, Y)$ with $\rho \equiv \ln(1/Q_0^2 r^2)$

  a front interpolating between $T = 0$ and $T = 1$



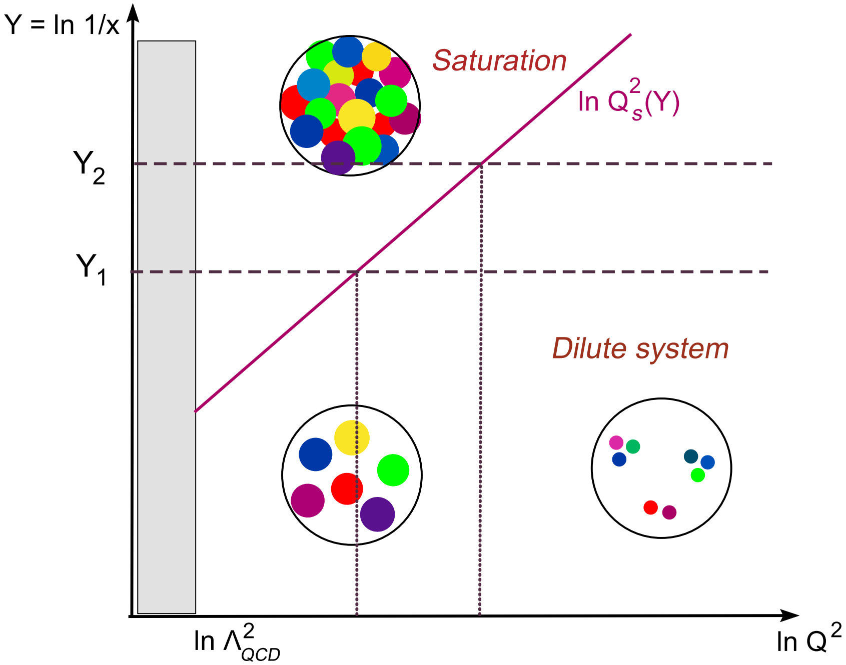


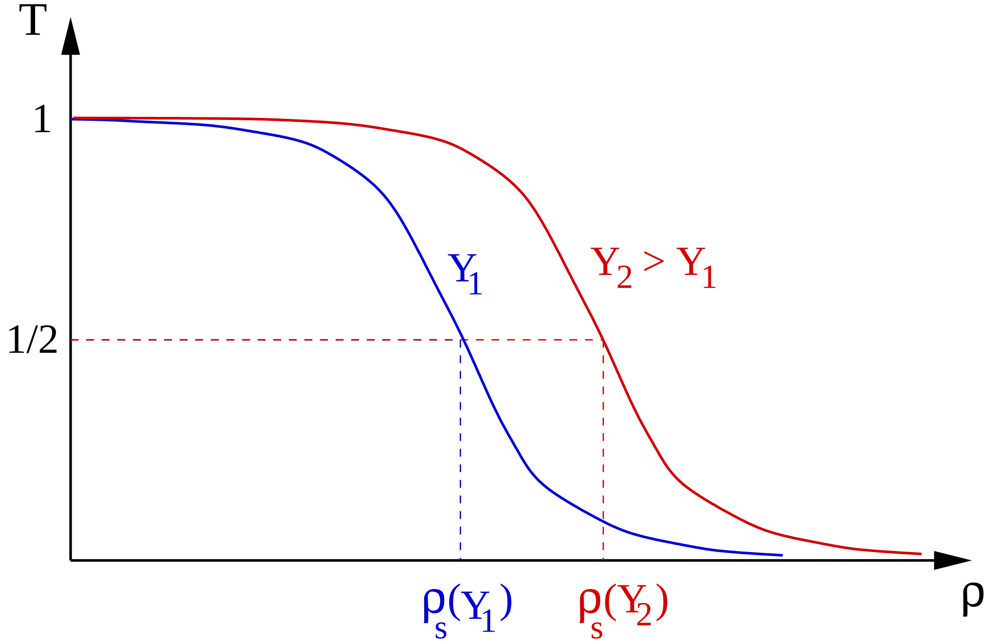
- The front position: the saturation scale $\rho_s \equiv \ln(Q_s^2/Q_0^2)$



The saturation front

- Increase Y : the front propagates towards larger values of ρ

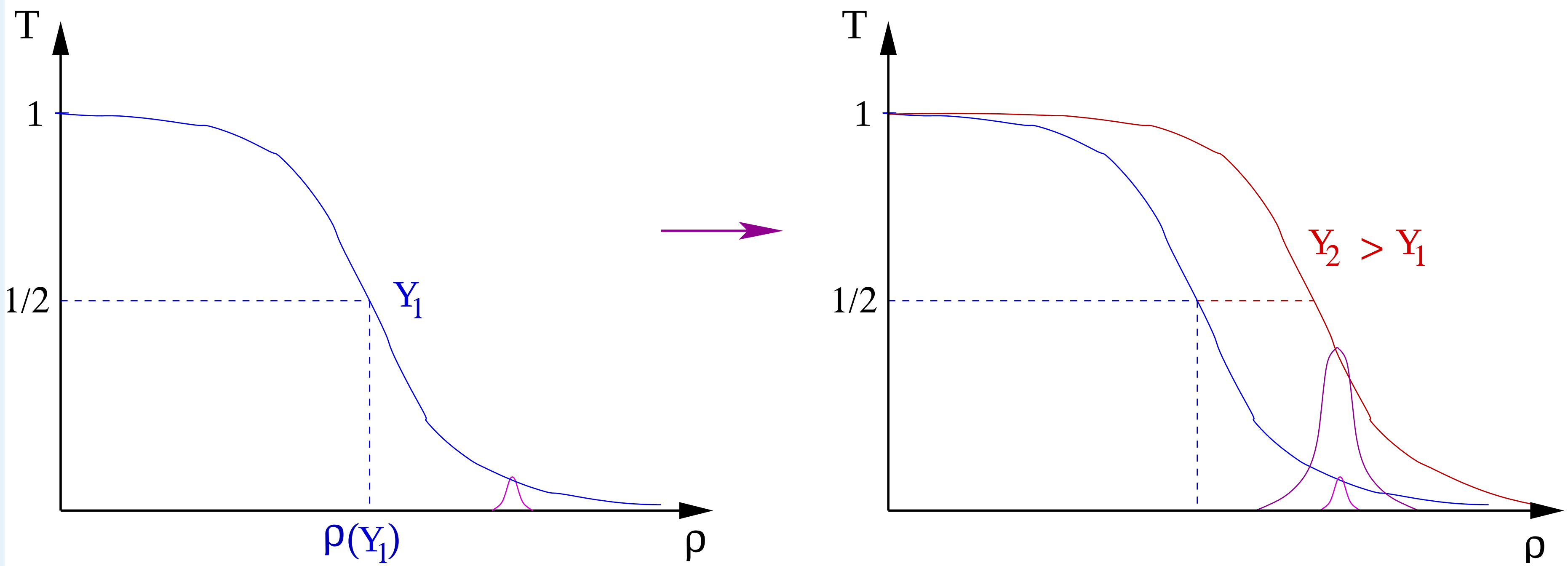




- $T = 1$ for $\rho \lesssim \rho_s(Y)$ and $T \propto \mathrm{e}^{-\rho} = r^2$ for $\rho \gg \rho_s(Y)$

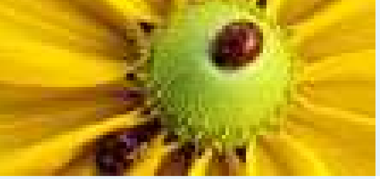
$$T(\rho, Y) \; = \; 1/2 \qquad \text{for} \qquad \rho \; = \; \rho_s(Y)$$

Pulled front



- The front is driven by the linear (BFKL) growth in the tail of the amplitude at large $\rho \gg \rho_s(Y)$, where $T \ll 1$
- The position of the front can be inferred from a study of the linearized (BFKL) equation

  "How fast does a small perturbation become of $\mathcal{O}(1)$?"



The BFKL equation

- BFKL equation : $T(\rho, Y) \ll 1 \iff \rho \gg \rho_s(Y)$

$$\frac{\partial}{\partial Y} T(\boldsymbol{x}, \boldsymbol{y}) = \frac{\bar{\alpha}_s}{2\pi} \int_{\boldsymbol{z}} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2 (\boldsymbol{y}-\boldsymbol{z})^2} \Big\{ - T(\boldsymbol{x}, \boldsymbol{y}) + T(\boldsymbol{x}, \boldsymbol{z}) + T(\boldsymbol{z}, \boldsymbol{y}) \Big\}$$

$$= \mathcal{K}_{\rm BFKL} \otimes T(r)$$

- The powers of r^2 are BFKL eigenfunctions:

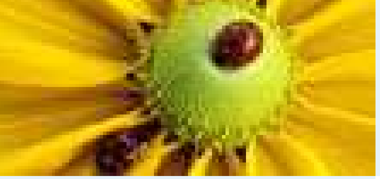
$$\mathcal{K}_{\rm BFKL} \otimes r^{2\gamma} = \bar{\alpha}_s \, \chi(\gamma) \, r^{2\gamma}$$

with the eigenvalue $\chi(\gamma)$ (‘BFKL characteristic function’) :

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma), \qquad \psi(\gamma) \equiv {\rm d} \ln \Gamma(\gamma)/{\rm d}\gamma$$

- Mellin representation of the solution $(r_0 \equiv 1/Q_0)$

$$T(r, Y) = \int_C \frac{{\rm d}\gamma}{2\pi i} \left(\frac{r^2}{r_0^2} \right)^{\gamma} \mathcal{T}(\gamma, Y)$$



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- BFKL equation becomes local in Mellin space:

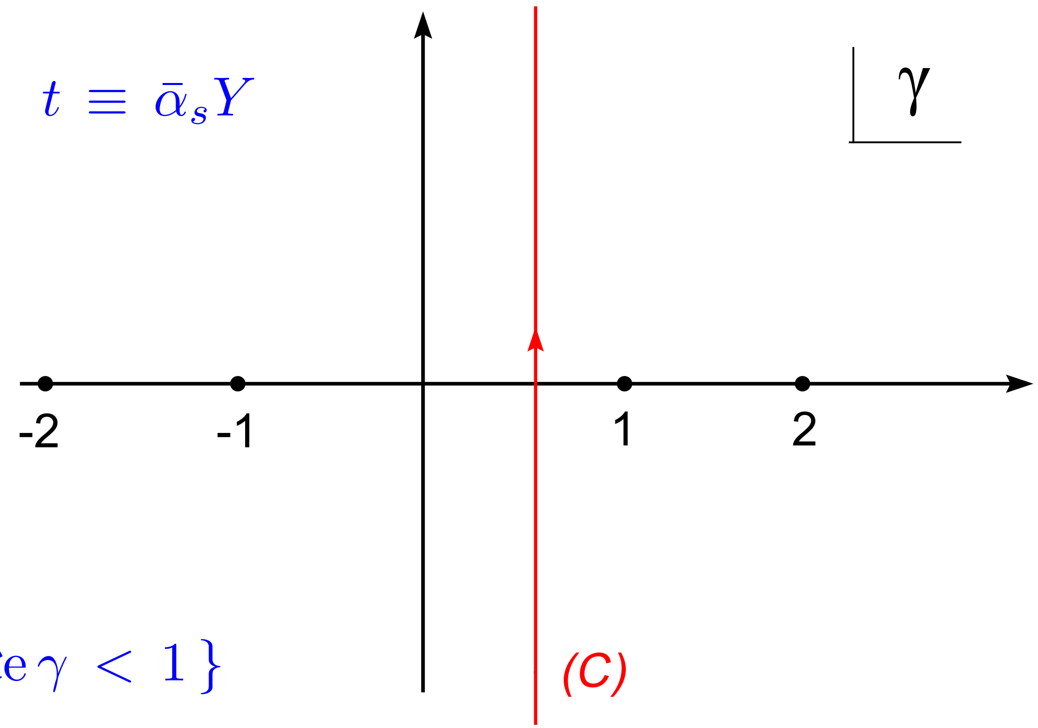
$$\frac{\partial}{\partial Y} \mathcal{T}(\gamma, Y) = \bar{\alpha}_s \, \chi(\gamma) \, \mathcal{T}(\gamma, Y) \implies \mathcal{T}(\gamma, Y) = \mathrm{e}^{\bar{\alpha}_s \chi(\gamma) Y} \, \mathcal{T}_0(\gamma)$$

The BFKL solution

$$T(r,Y) = \int_C \frac{\mathrm{d}\gamma}{2\pi i} \left(\frac{r^2}{r_0^2}\right)^{\gamma} \mathrm{e}^{\bar{\alpha}_s Y \chi(\gamma)}\, \mathcal{T}_0(\gamma) \equiv \int_C \frac{\mathrm{d}\gamma}{2\pi i}\, \mathrm{e}^{-\gamma\rho}\, \mathrm{e}^{t\chi(\gamma)}\, \mathcal{T}_0(\gamma)$$

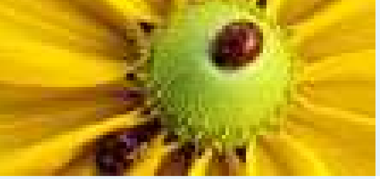
- Notations:

  $\rho \equiv \ln(r_0^2/r^2), \quad t \equiv \bar{\alpha}_s Y$



- $(C) = \{0 < \Re\mathrm{e}\, \gamma < 1\}$

$$\mathcal{T}_0(\gamma) = \int \mathrm{d}\rho\, \mathrm{e}^{\gamma\rho}\, T(\rho, Y=0) \quad \text{with} \quad T(\rho, Y=0) \approx \begin{cases} \mathrm{e}^{-\rho} & \text{for } \rho > 1 \\ 1 & \text{for } \rho \lesssim 0 \end{cases}$$

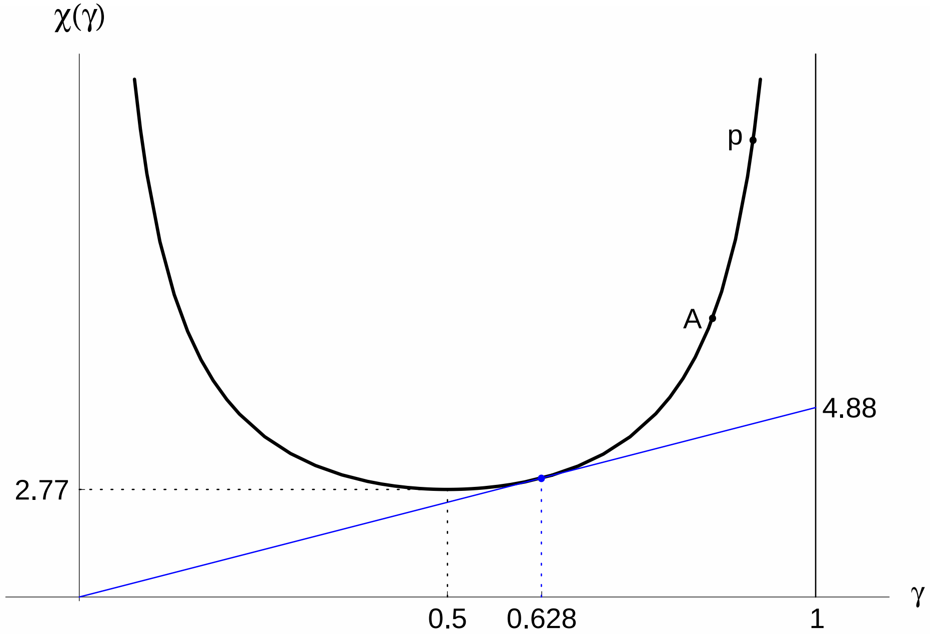


The saddle point approximation

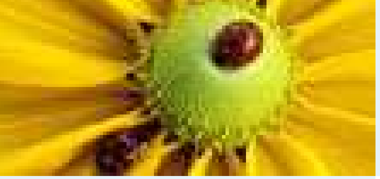
$$T(\rho, Y) = \int_C \frac{d\gamma}{2\pi i}\, e^{F(\gamma|\rho,t)}\, \mathcal{T}_0(\gamma) \quad \text{with} \quad F(\gamma|\rho, t) \equiv t\chi(\gamma) - \gamma\rho$$

- Large t and ρ : integral dominated by the saddle point γ_0

$$F'(\gamma_0) = 0 \quad \Longrightarrow \quad \chi'(\gamma_0) = \rho/t$$



- The formal high-energy limit: $t \to \infty$ at fixed $\rho \Longrightarrow \gamma_{\mathbb{P}} = 1/2$



The BFKL $\mathbb{P}$omeron: "High energy limit"

- Exercice: Using

$$\chi(\gamma) \approx \omega_{\mathbb{P}} + \frac{D}{2}\left(\gamma - \frac{1}{2}\right)^2, \quad \omega_{\mathbb{P}} = 4\ln 2 \approx 2.77\,, \quad D = 28\zeta(3) \approx 33.67$$

show that

$$T(\rho, Y) \simeq e^{\omega_{\mathbb{P}} t - \frac{1}{2}\rho}\, \frac{\exp\left\{-\frac{\rho^2}{2Dt}\right\}}{\sqrt{2\pi D t}}\, \mathcal{T}_0(1/2)$$

- More explicit notations: $\rho \equiv \ln(r_0^2/r^2)$ and $t \equiv \bar{\alpha}_s Y$

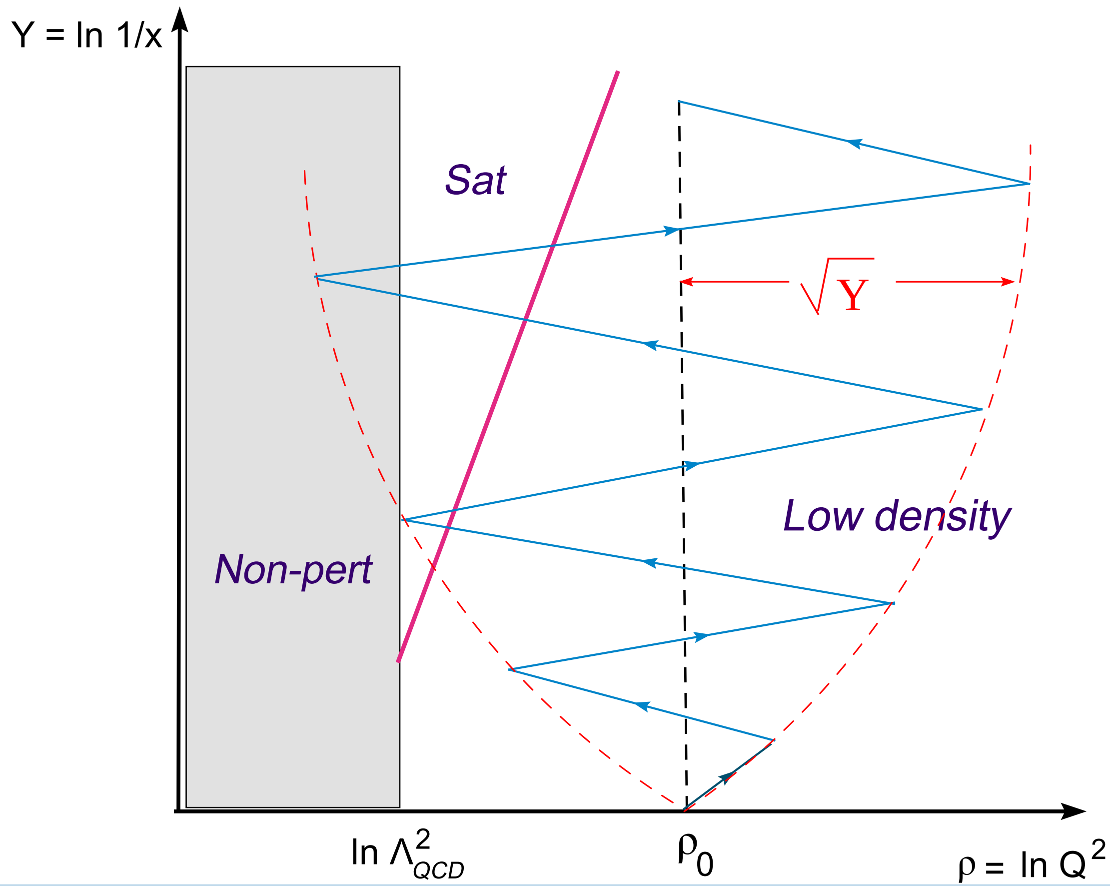
$$T(\rho, Y) \simeq \underbrace{\alpha_s^2}_{\mathcal{T}_0} \times \underbrace{e^{\omega_{\mathbb{P}}\bar{\alpha}_s Y}}_{\mathbb{P}\text{omeron}} \times \underbrace{\left(\frac{r^2}{r_0^2}\right)^{1/2}}_{\text{anomalous dim.}} \times \underbrace{\frac{\exp\left\{-\frac{\ln^2(r_0^2/r^2)}{2D\bar{\alpha}_s Y}\right\}}{\sqrt{2\pi D\bar{\alpha}_s Y}}}_{\text{diffusion}}$$

Conceptual problems

- **Unitarity violation :**

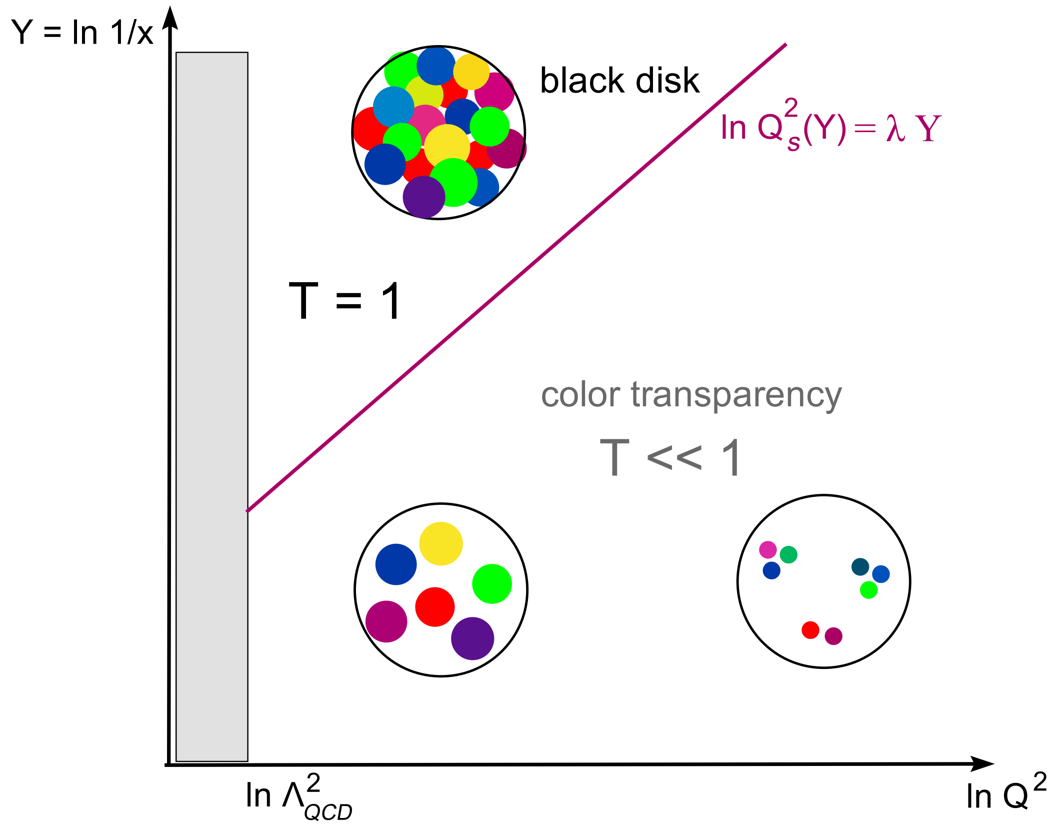
$$T(r \sim r_0, Y) \gtrsim 1 \quad \text{when} \quad Y \gtrsim Y_c \sim \frac{1}{\bar{\alpha}_s \omega_{\mathbb{P}}} \ln \frac{1}{\alpha_s^2}$$

- **Infrared diffusion :**



Saturation saddle point

- Non–linear terms in BK forbid the evolution in the region where $T \geq 1$

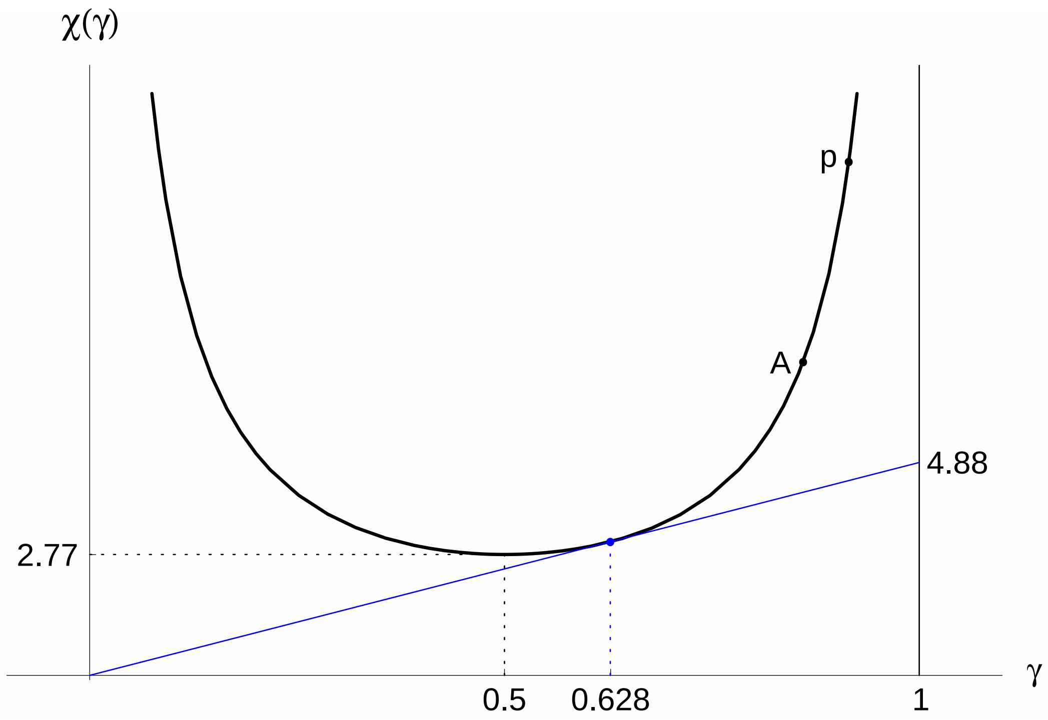


- Saturation line : a line of constant amplitude $T = 1$
- Y & ρ are simultaneously increased $\Rightarrow$ new saddle point γ_s

Saturation saddle point

- $F'(\gamma_s) = 0$ (saddle point) & $F(\gamma_s) = 0$ ($T =$ const)

$$\gamma_s = \frac{\chi(\gamma_s)}{\chi'(\gamma_s)} = 0.627..., \qquad \frac{\rho_s}{t} = \chi'(\gamma_s) = 4.883... \equiv \lambda_s$$

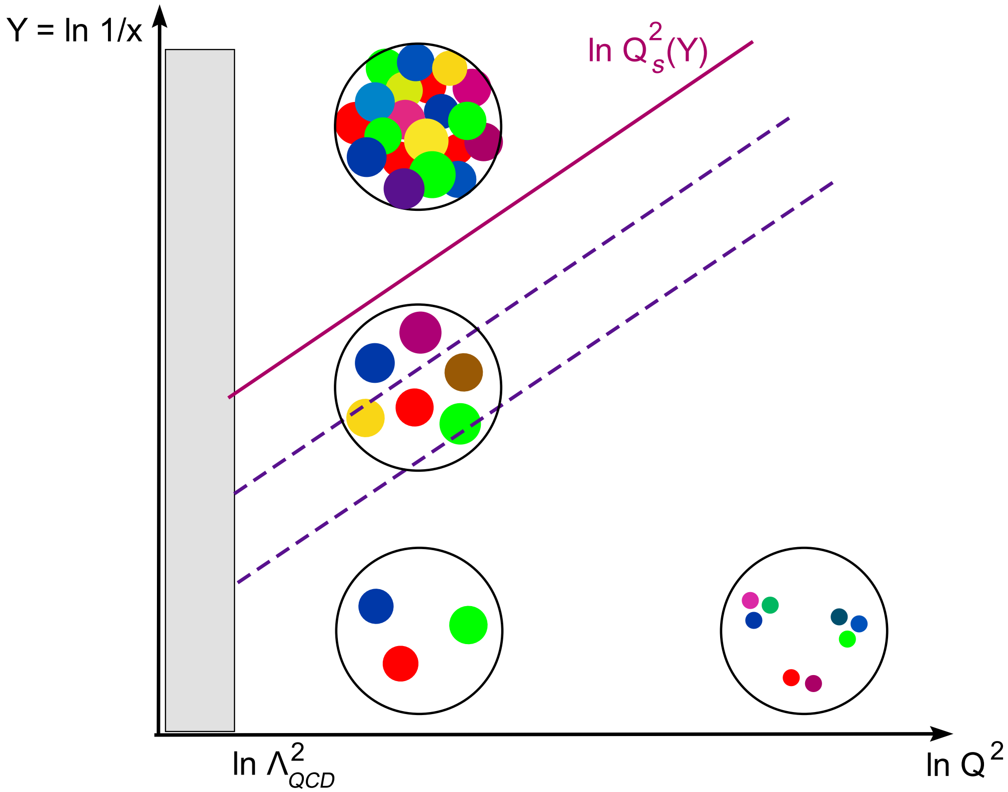


- γ_s : the point where a straight line drawn from the origin is tangent to the χ function

Saturation line

$$\boxed{\rho_s(Y) = \lambda_s\,\bar{\alpha}_s Y \quad\Longleftrightarrow\quad Q_s^2(Y) = Q_0^2\,\mathrm{e}^{\lambda_s \bar{\alpha}_s Y}}$$

- The saturation line is a straight line, as anticipated !



- What about the shape of the amplitude near saturation ?

The approach towards saturation

- $\rho \;>\; \rho_s$, but not **much** bigger

  $\Longrightarrow$ expansion around the saturation saddle point γ_s

- Exercice: Show that :

$$F(\gamma|\rho, Y) \;\simeq\; -(\rho-\rho_s)(\gamma-\gamma_s) \;+\; \frac{t}{2}\chi''(\gamma_s)(\gamma-\gamma_s)^2$$

and therefore (with $D \equiv \chi''(\gamma_s) = 48.518...$) :

$$T(\rho, Y) \;\simeq\; \mathrm{e}^{-\gamma_s(\rho-\rho_s)}\,\frac{\exp\left\{-\frac{(\rho-\rho_s)^2}{2D\bar{\alpha}_s Y}\right\}}{\sqrt{2\pi D\bar{\alpha}_s Y}}$$

valid for : $1 \;\ll\; \rho-\rho_s \;\ll\; 2\gamma_s D\bar{\alpha}_s Y$

$$T(\rho, Y) \;\simeq\; \underbrace{(r^2Q_s^2)^{\gamma_s}}_{\text{anomalous dim.}} \;\times\; \underbrace{\frac{\exp\left\{-\frac{\ln^2(r^2Q_s^2)}{2D\bar{\alpha}_s Y}\right\}}{\sqrt{2\pi D\bar{\alpha}_s Y}}}_{\text{diffusion}}$$

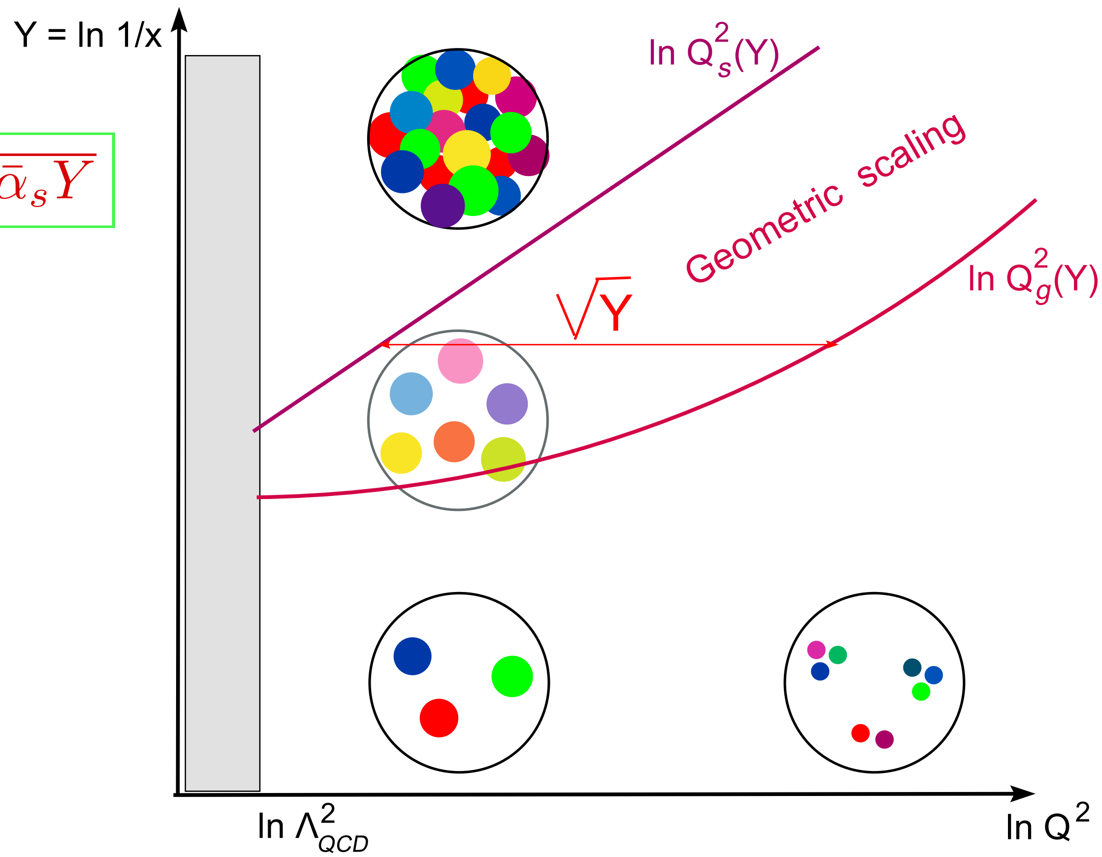
Geometric scaling

- The ‘diffusion’ can be ignored within the ‘scaling window’:

$$\boxed{1 \;<\; \rho - \rho_s \;\ll\; \sqrt{2D\bar{\alpha}_s Y}}$$

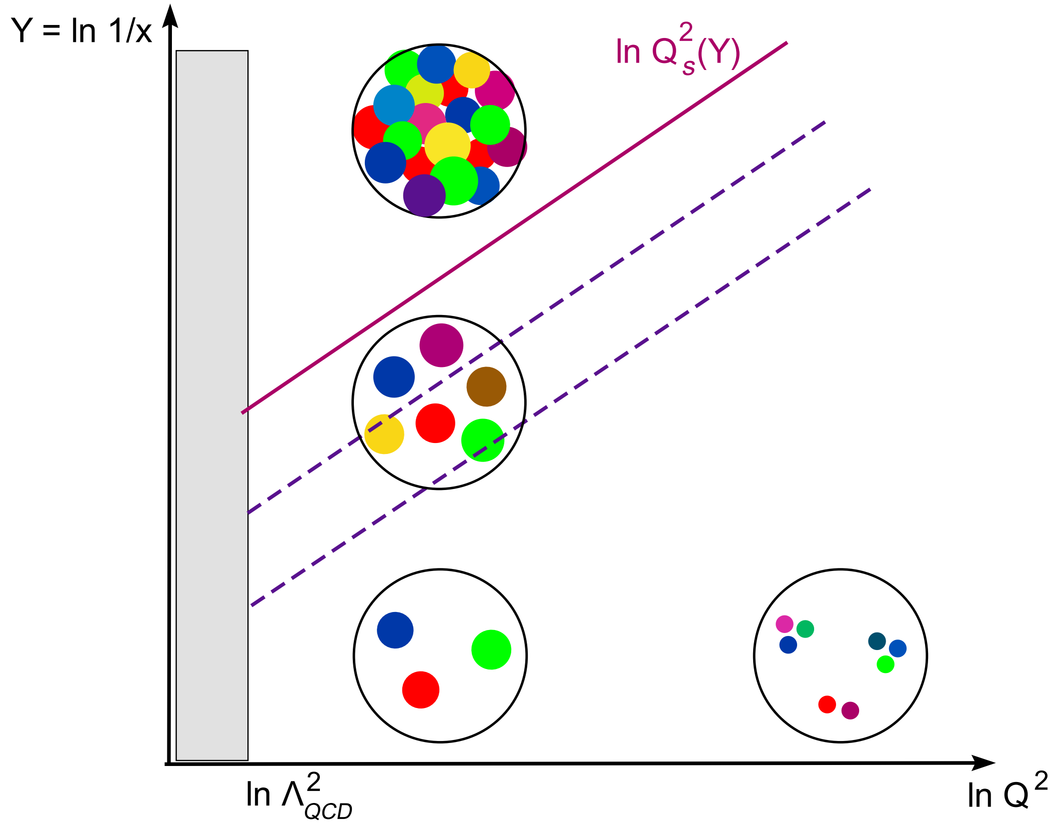
$$T(\rho, Y) \;\simeq\; \mathrm{e}^{-\gamma_s(\rho-\rho_s)}$$

$$T(r, Y) \;\simeq\; (r^2 Q_s^2)^{\gamma_s}$$



$$\boxed{T(\rho, Y) \;\simeq\; T(\rho - \rho_s(Y)) \;=\; T(r^2 Q_s^2(Y))}$$

Geometric scaling



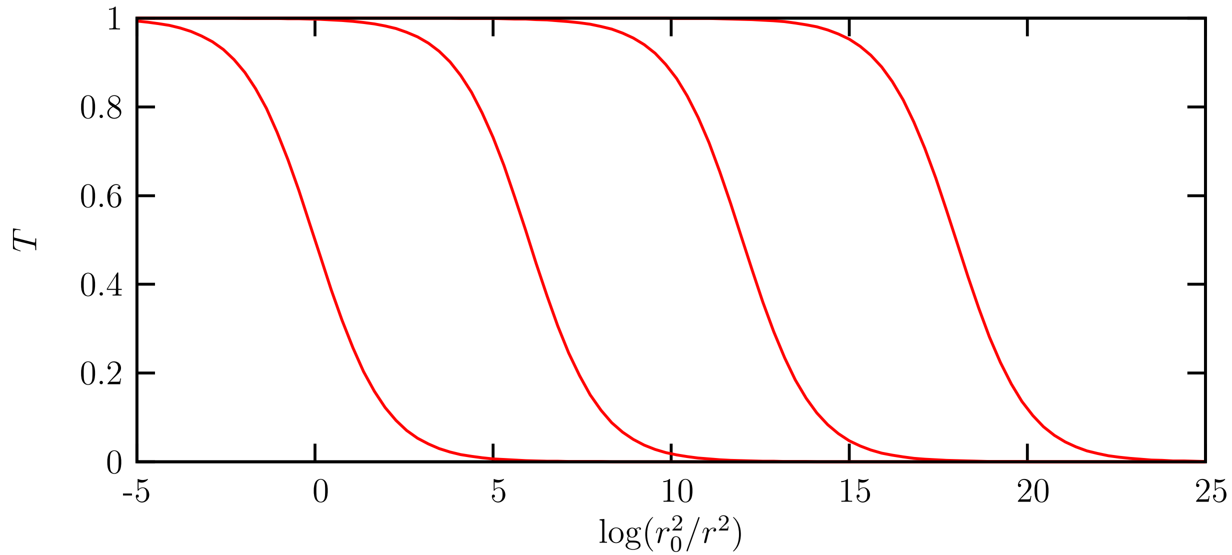
- $\rho - \rho_s(Y) = const.$: A line of constant gluon occupancy

  $n(Y, Q^2) \ = \ const \ \ll \ 1/\alpha_s$ (dilute !)

- Physics must be invariant along any such a line !

Traveling wave

- The shape of the front is not altered by the evolution



$$T(\rho, Y) \simeq \mathcal{T}(\rho - \rho_s(Y)) \equiv \mathcal{T}\big(r^2 Q_s^2(Y)\big)$$

- BFKL equation + saturation boundary
  E.I., Itakura, McLerran (02) ; Mueller, Triantafyllopoulos (02)
- Saturation exponent at NLO (NLO BFKL + saturation) :
  $\lambda \approx 0.3$ *(Triantafyllopoulos, 02)*
- Relation with the FKPP equation : *Munier, Peschanski (03)*

Geometric Scaling at HERA: Inclusive

(Staśto, Golec-Biernat and Kwieciński, 2000)

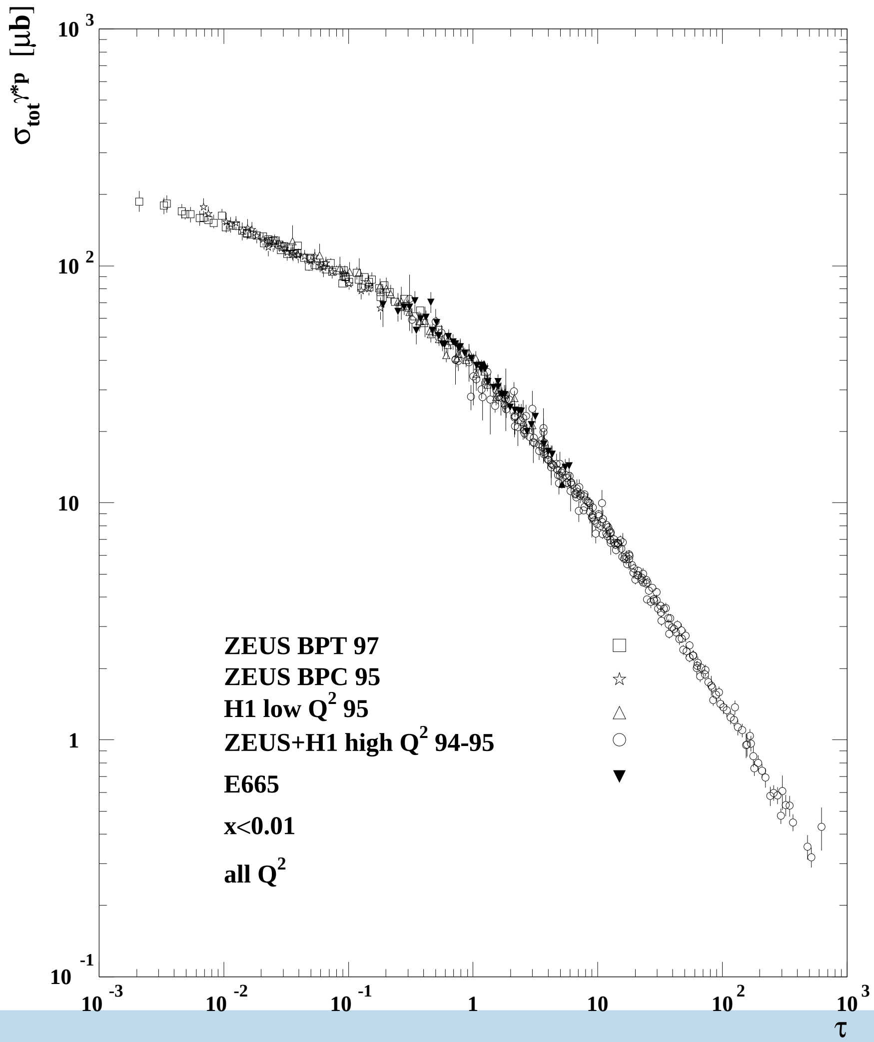
$$\sigma(x,Q^2) \approx \sigma(\tau) \quad \text{with} \quad \tau \equiv Q^2/Q_s^2(x), \quad Q_s^2(x) = (x_0/x)^\lambda\,\mathrm{GeV}^2\,, \quad \lambda \simeq 0.3$$

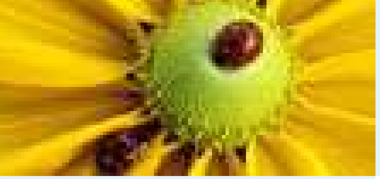
$x \leq 0.01$

$Q^2 \leq 450$ GeV^2

$Q_s^2 \sim 1$ GeV^2

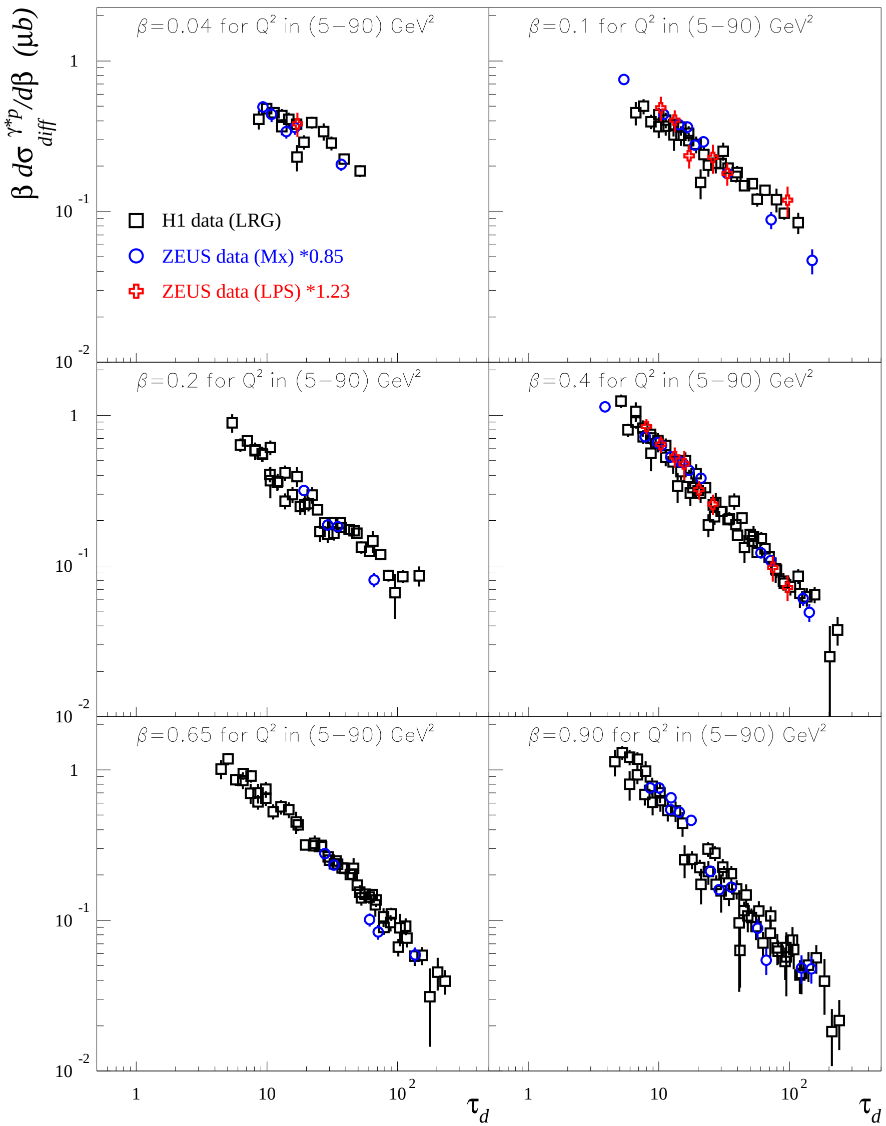
for $x \sim 10^{-4}$

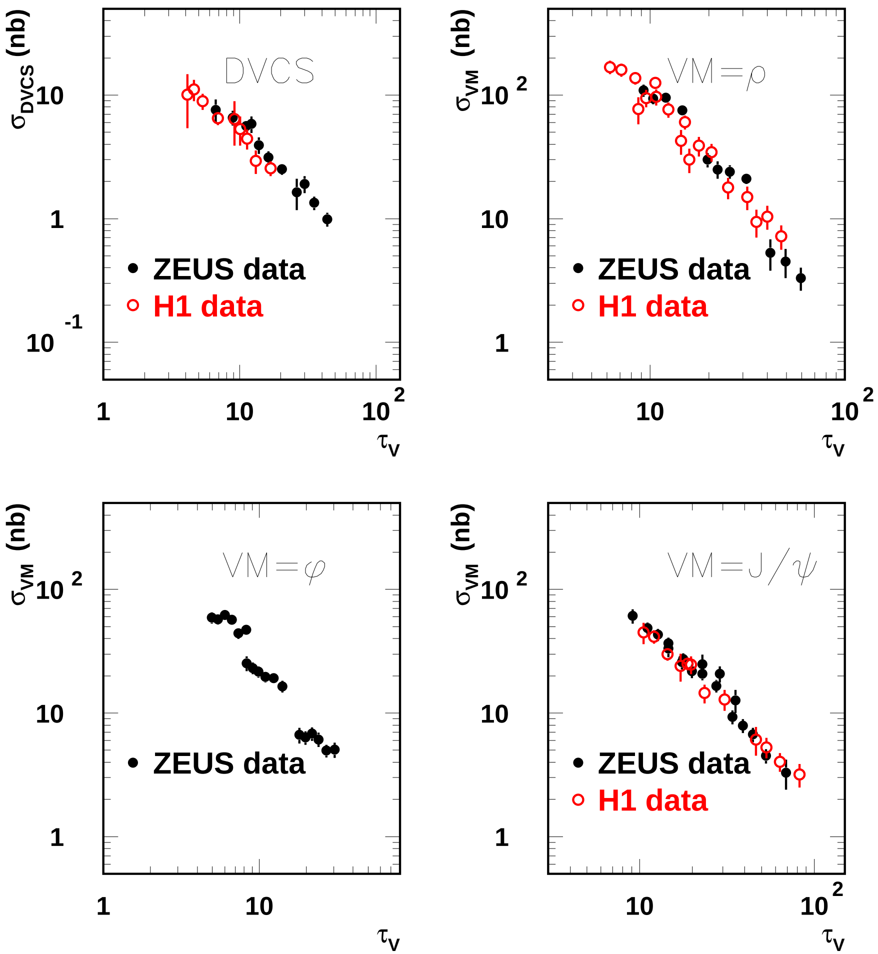


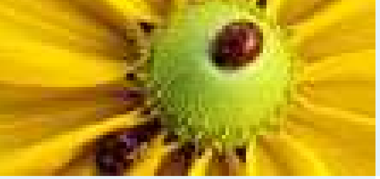


Geometric Scaling at HERA: Diffraction

(Marquet and Schoeffel 2006)

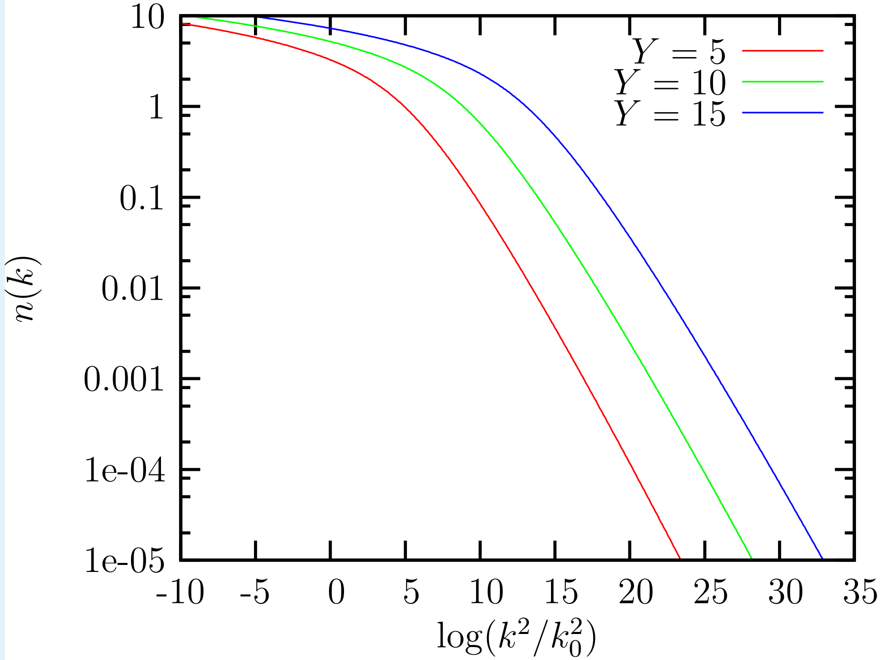


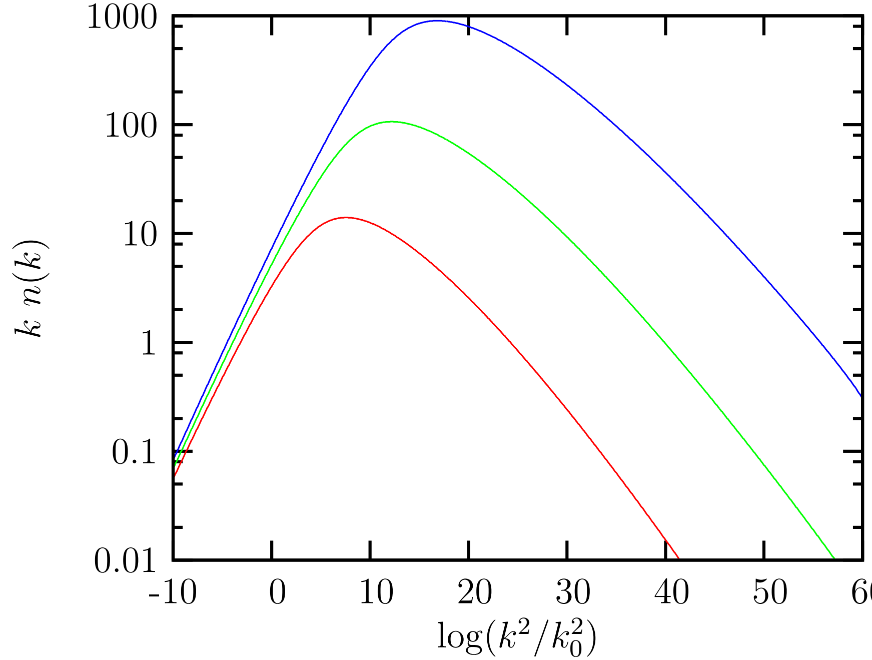




Gluon occupation number

$$xG(x,Q^2) = \int \mathrm{d}^2 b \int^{Q} \mathrm{d}k\, k\, n(x,k)$$

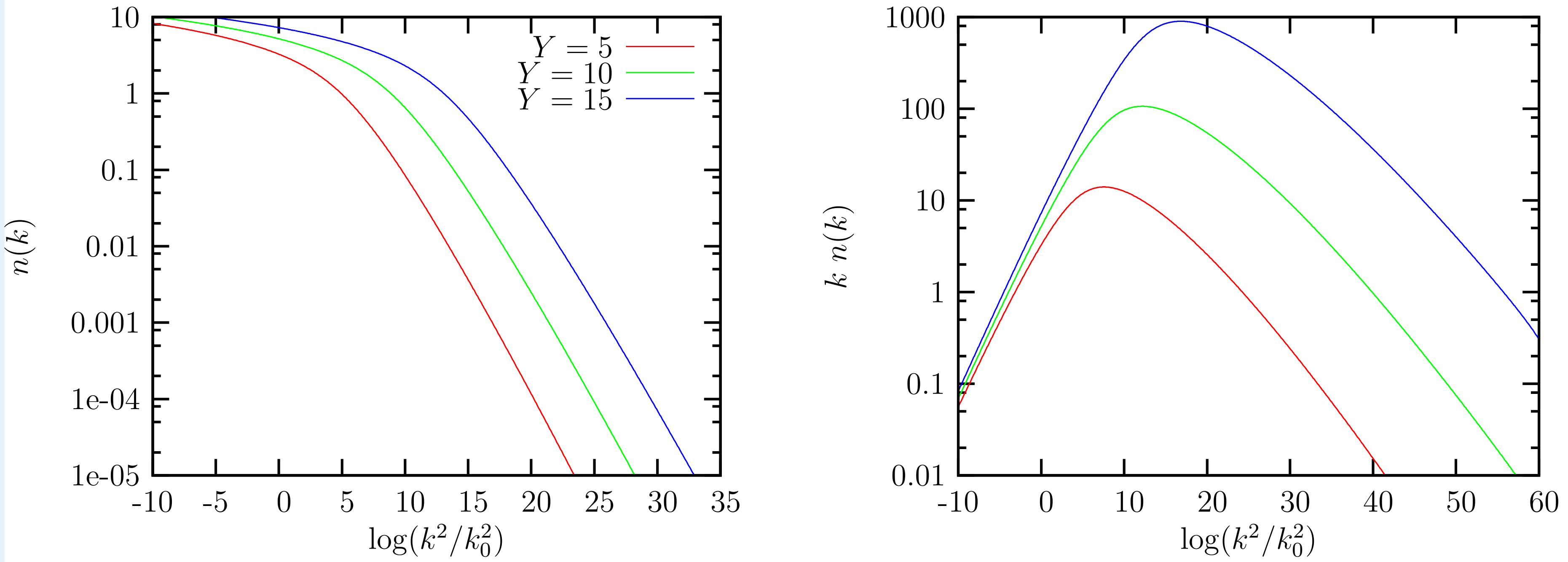




- $Q_s(Y)$: the typical transverse momentum of the gluons
- For $k_\perp \lesssim Q_s(Y)$: Gluon saturation

Gluon occupation number

$$xG(x,Q^2) = \int d^2b \int^Q dk\, k\, n(x,k)$$

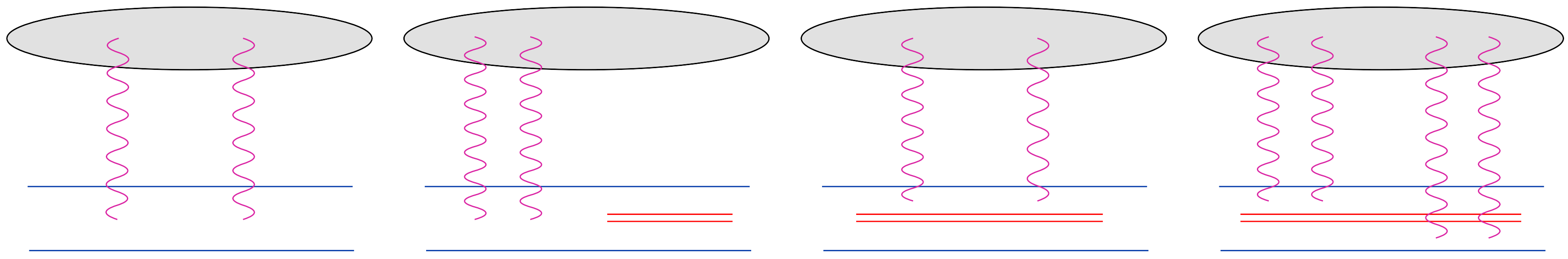


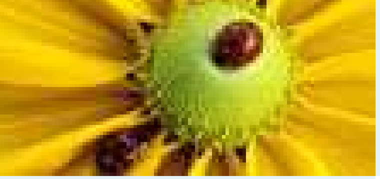
- $Q_s(Y)$: the typical transverse momentum of the gluons
- Unitarization in DIS teaches us about gluon saturation

Non–linear gluon evolution

- By changing frame, the evolution step can be associated with either the projectile, or the target
- Non–linear evolution of the dipoles amplitudes
  $\Longrightarrow$ Non–linear evolution of the gluons in the target

$$\frac{\partial}{\partial Y}\langle T(\boldsymbol{x},\boldsymbol{y})\rangle_Y = \frac{\bar{\alpha}_s}{2\pi}\int_{\boldsymbol{z}} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \Big\langle \underbrace{-T(\boldsymbol{x},\boldsymbol{y})+T(\boldsymbol{x},\boldsymbol{z})+T(\boldsymbol{z},\boldsymbol{y})}_{\text{BFKL (linear)}} - \underbrace{T(\boldsymbol{x},\boldsymbol{z})T(\boldsymbol{z},\boldsymbol{y})}_{\text{non–linear}}\Big\rangle_Y$$



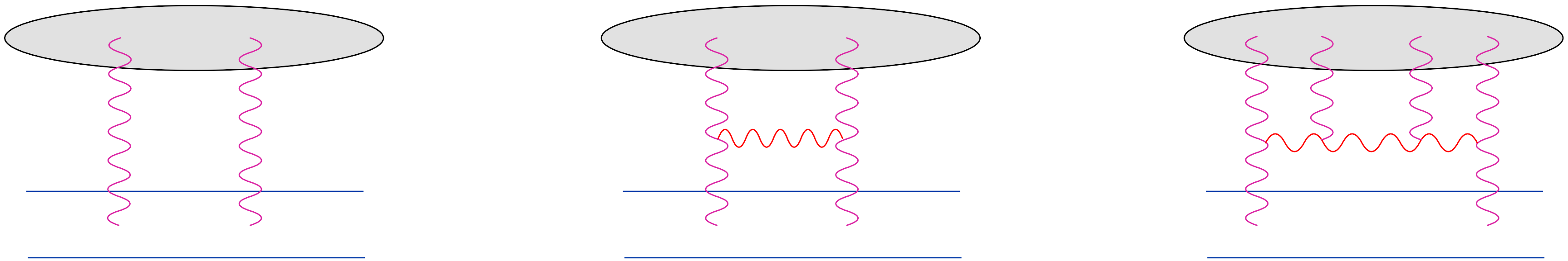


Non–linear gluon evolution

- By changing frame, the evolution step can be associated with either the projectile, or the target

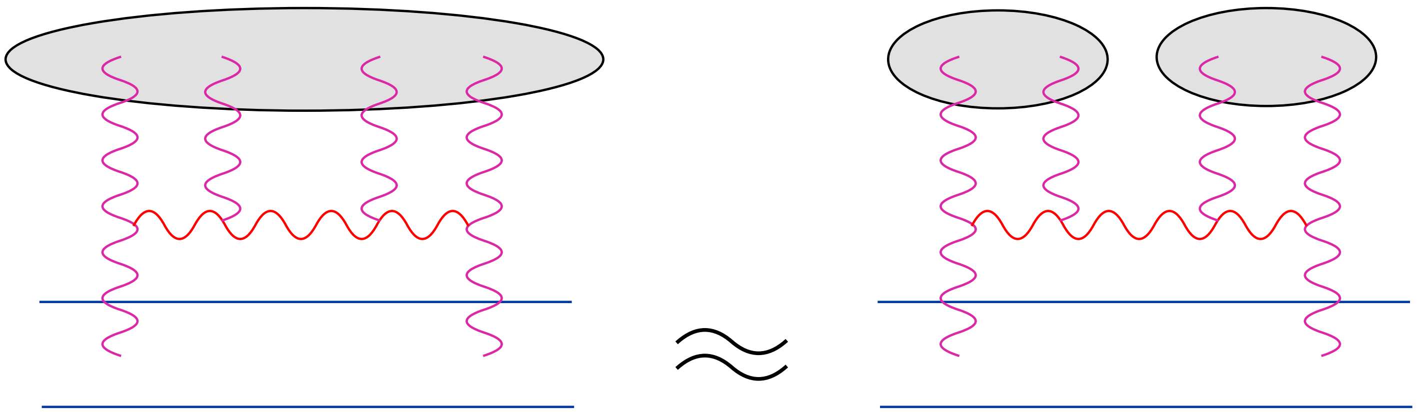
- Non–linear evolution of the dipoles amplitudes
  $\Longrightarrow$ Non–linear evolution of the gluons in the target

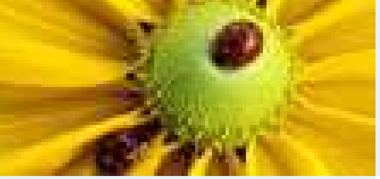
$$\frac{\partial}{\partial Y}\langle T(\boldsymbol{x},\boldsymbol{y})\rangle_Y = \frac{\bar{\alpha}_s}{2\pi}\int_{\boldsymbol{z}} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \Big\langle \underbrace{-T(\boldsymbol{x},\boldsymbol{y})+T(\boldsymbol{x},\boldsymbol{z})+T(\boldsymbol{z},\boldsymbol{y})}_{\text{BFKL (linear)}} - \underbrace{T(\boldsymbol{x},\boldsymbol{z})T(\boldsymbol{z},\boldsymbol{y})}_{\text{non–linear}} \Big\rangle_Y$$



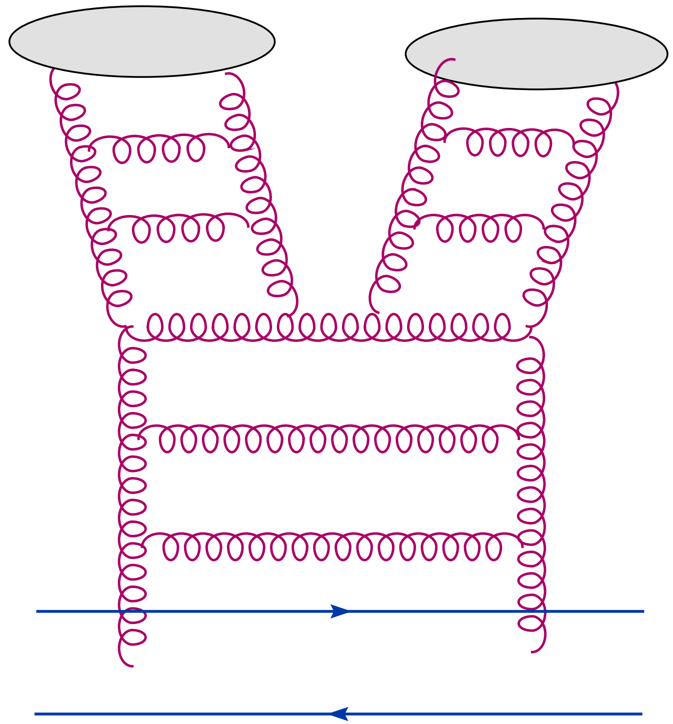
The mean field approximation (BK)

$$\langle T(\boldsymbol{x},\boldsymbol{z})T(\boldsymbol{z},\boldsymbol{y})\rangle_Y \;\approx\; \langle T(\boldsymbol{x},\boldsymbol{z})\rangle_Y\,\langle T(\boldsymbol{z},\boldsymbol{y})\rangle_Y$$



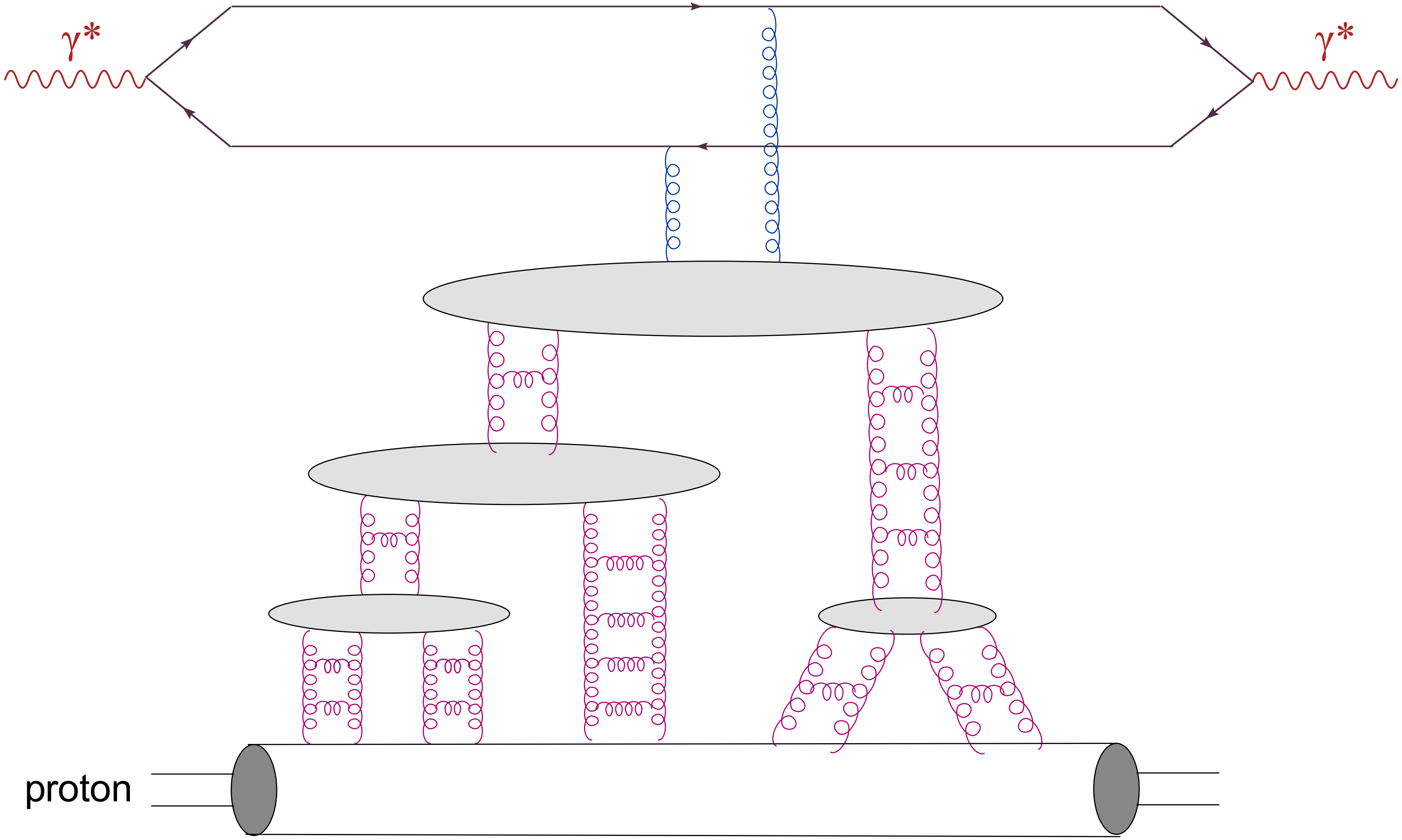


The mean field approximation (BK)



- Through iterations (several steps in the evolution)
  - Two '$\mathbb{P}$omerons' (color–singlets) evolve independently from each other, before melting into one $\mathbb{P}$omeron
  - $2 \;\rightarrow\; 1$ 'triple $\mathbb{P}$omeron vertex' = Dipole kernel

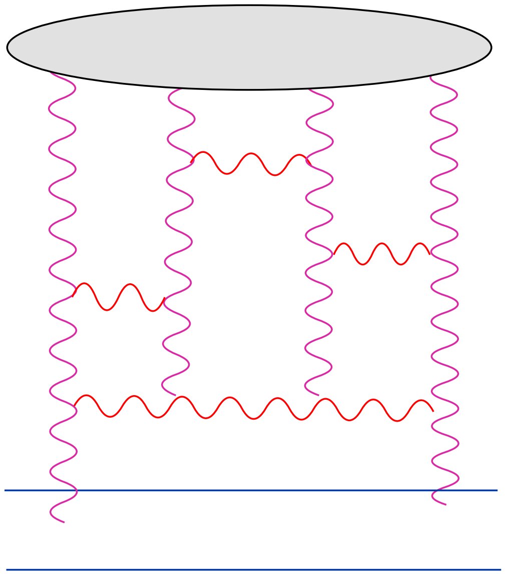
DIS in the MFA: ‘Fan diagrams’

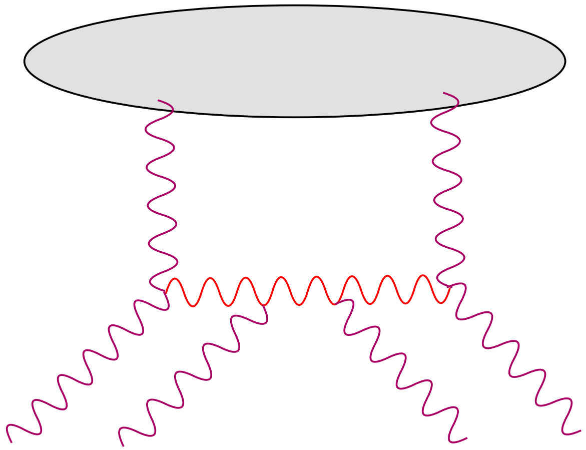


- BFKL evolution + gluon recombination at large N_c

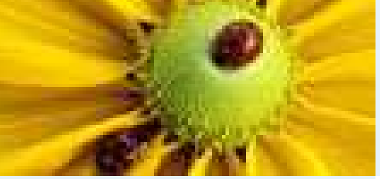
Gluon correlations

- There are several mechanisms for generating correlations



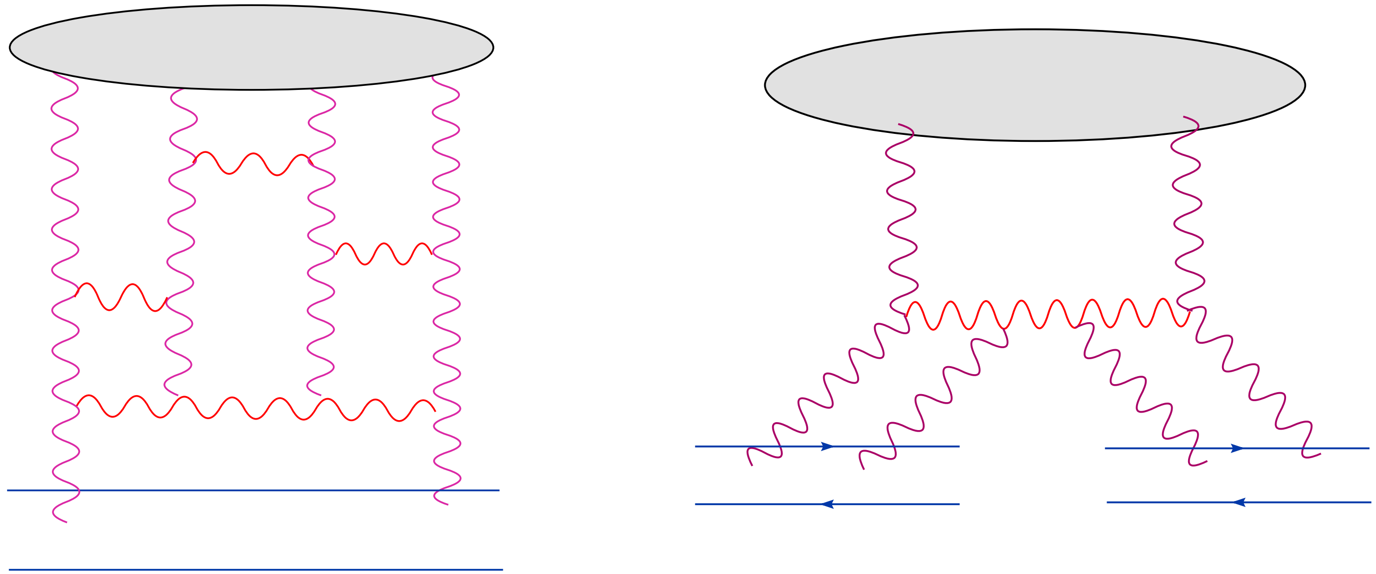


- Color exchanges :
  - Linear evolution ($3 \to 3$, $4 \to 4$, $n \to n$) : BKP equation
  - Non–linear evolution ($n \to 2$ with $n \geq 2$) : JIMWLK eq
- Gluon splitting $2 \to 4$ (generally, $2 \to n$)



Gluon correlations

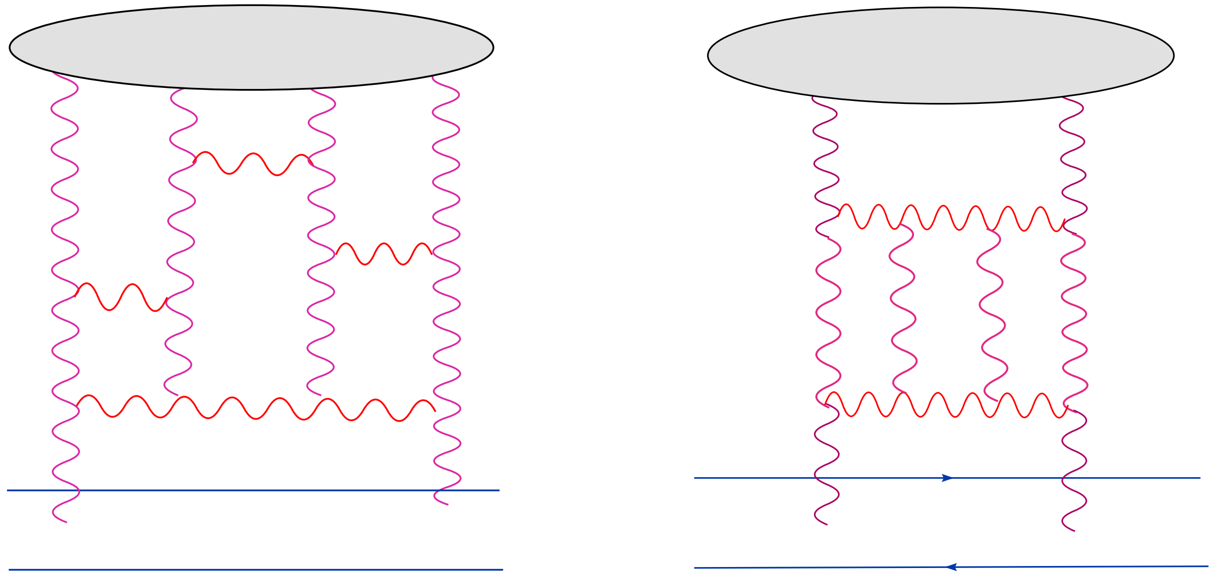
- There are several mechanisms for generating correlations



- Color exchanges
- Gluon splitting $2 \rightarrow 4$ (generally, $2 \rightarrow n$)
  - The 4 child gluons can be probed by 2 external dipoles

Gluon correlations

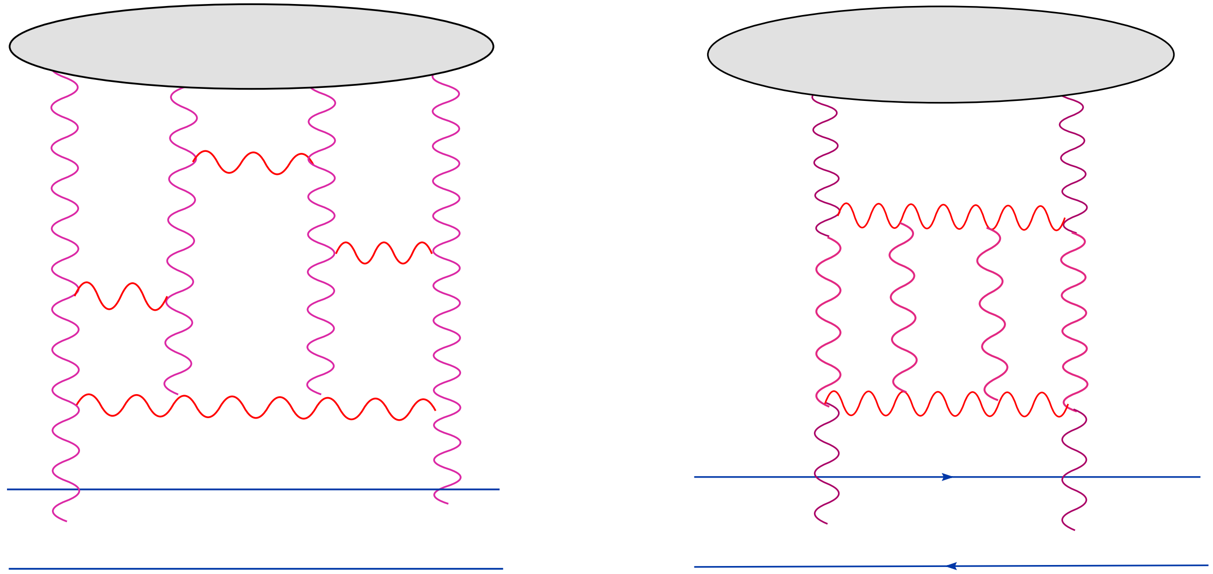
- There are several mechanisms for generating correlations



- Color exchanges
- Gluon splitting $2 \to 4$ (generally, $2 \to n$)
  - ... or by a single one, after one additional recombination

    splitting + merging = '$\mathbb{P}$omeron loop'

Are such correlations important ?



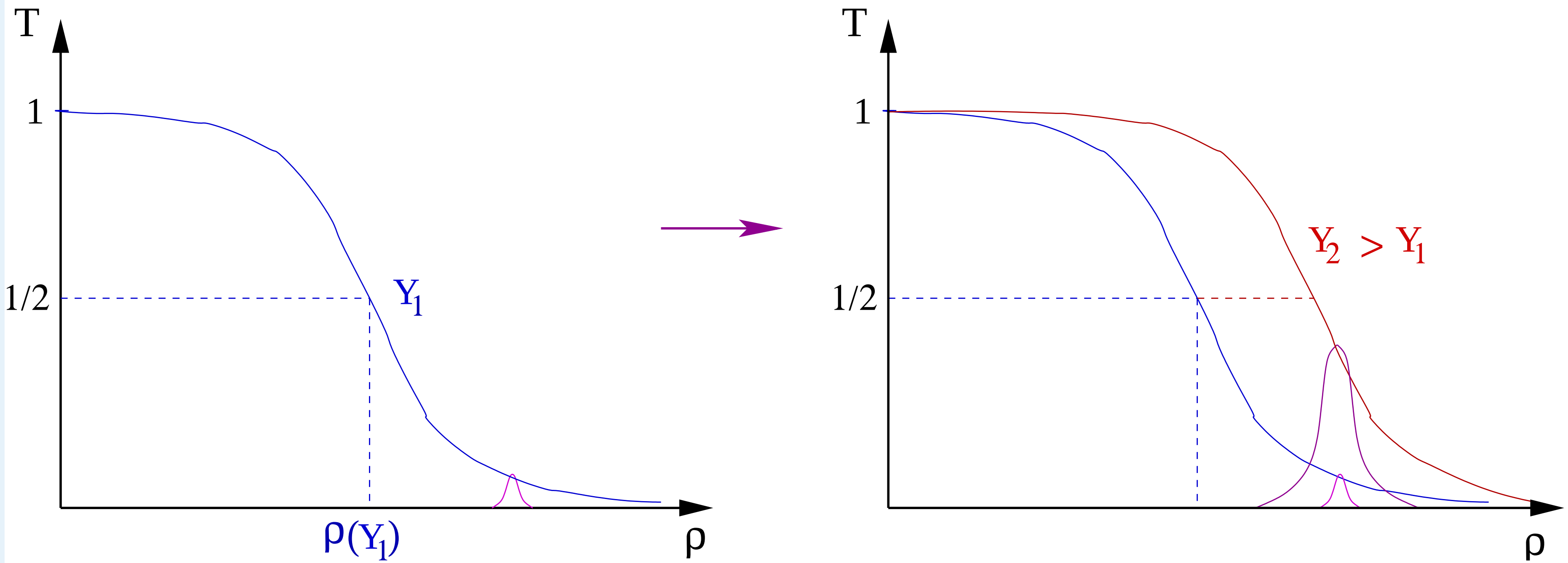
- Color exchanges are suppressed at large N_c !

  ... but they are inherent in the theory of gluon saturation (CGC), which is built in terms of color fields

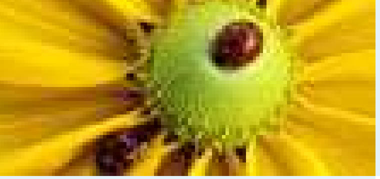
- Gluon splitting $2 \to n$ is negligible at high density !

  ... but what is dense at some rapidity has started by being dilute at some earlier one !

Reminder: Pulled front



- The front is driven by the linear (BFKL) growth in the tail of the amplitude at large $\rho \gg \rho_s(Y)$, where $T \ll 1$



The Color Glass Condensate

(McLerran, Venugopalan, 1994; E.I., Leonidov, McLerran, 2000)

- An effective theory for the small–x gluons in the high–density environment characteristic of saturation
- Large occupation numbers ($n \sim 1/\alpha_s$)
  - The gluons can be described as classical color fields ✓
  - No fluctuations in the gluon number ($2 \rightarrow n$ splitting) ~~✓~~
- Separation of scales (longitudinal momentum/time)
  - The smaller x, the shorter the lifetime of the gluon

$$\Delta t \sim \frac{\hbar}{\Delta E} = \frac{2xp}{k_\perp^2}$$

  - The gluons with $x' \gg x$ are 'frozen' over the typical time scale for the dynamics at x