Non–linear evolution & Gluon saturation in QCD at high energy

Part III

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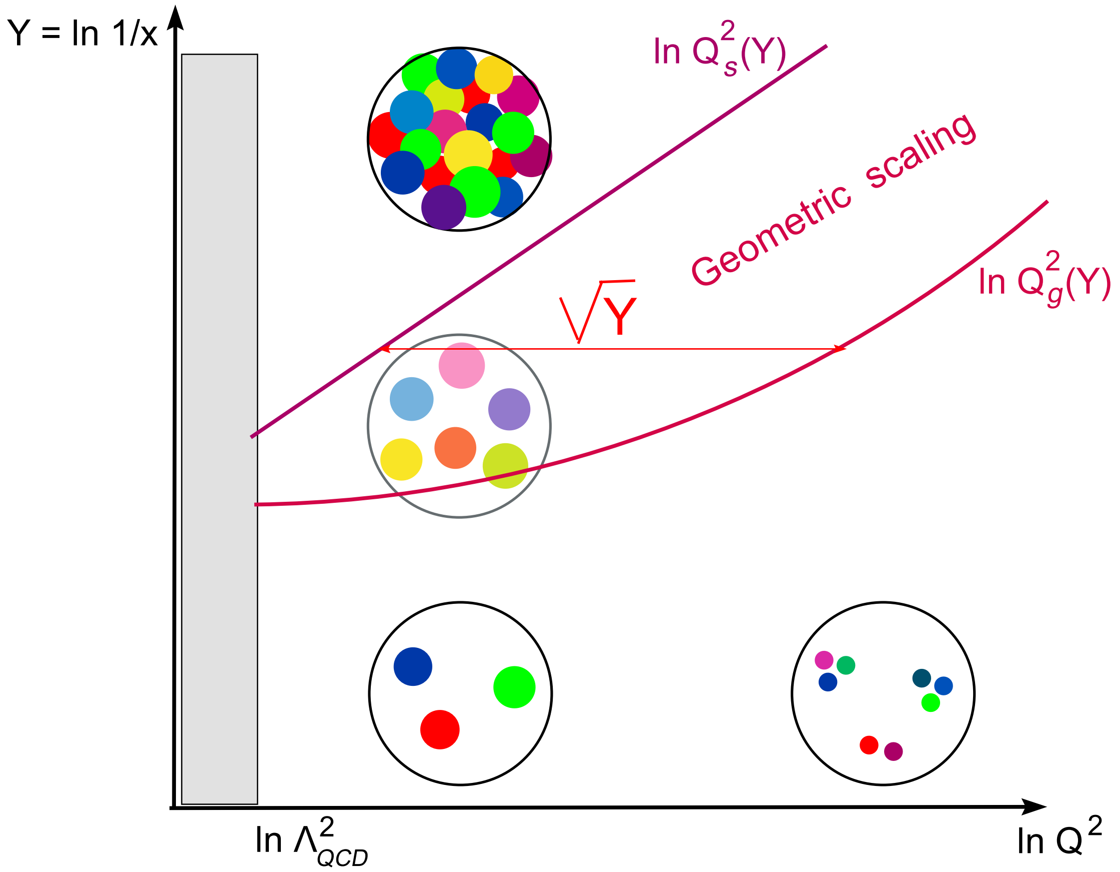
A brief reminder

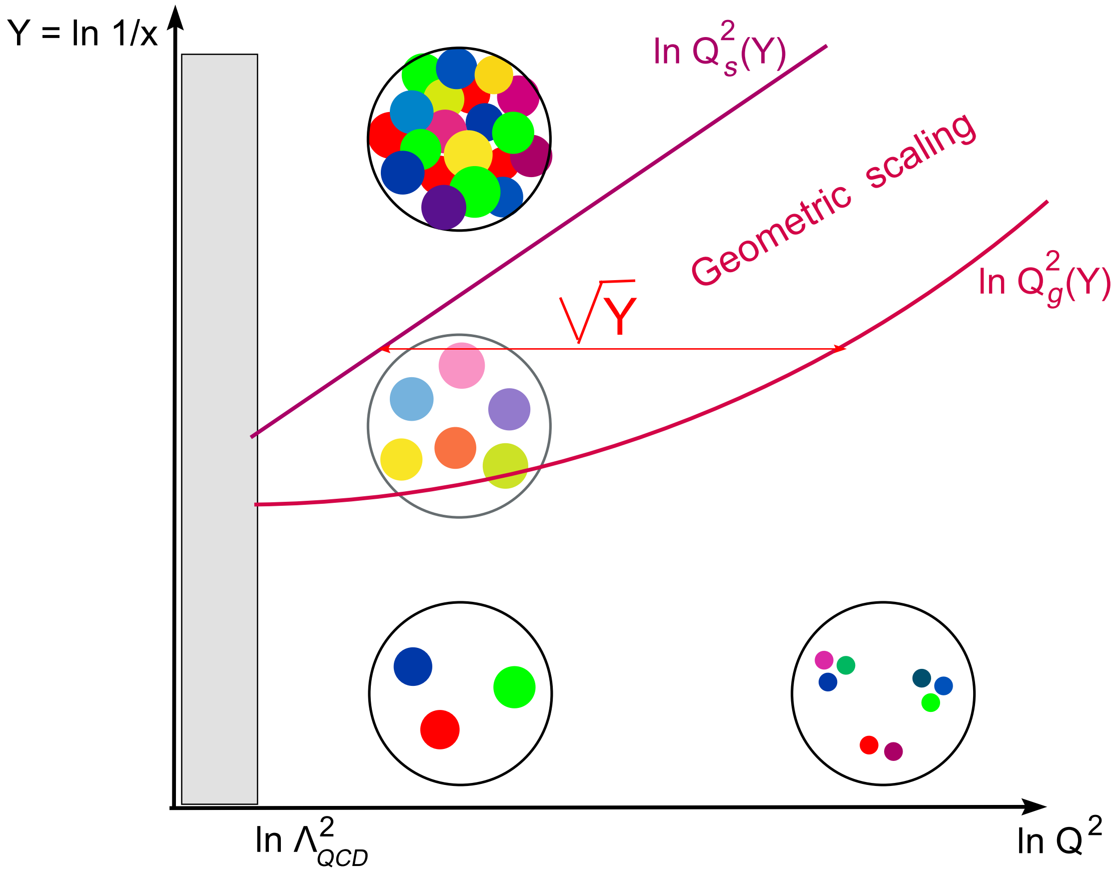
- Previously on “Desperate Housewives”

Unitarization & Geometric scaling



- Saturation line: $\rho_s(Y) = \lambda_s\,\bar{\alpha}_s Y \iff Q_s^2(Y) = Q_0^2\,\mathrm{e}^{\lambda_s \bar{\alpha}_s Y}$
- Leading—order BFKL $\Longrightarrow \lambda_s \approx 4.8$, hence $\lambda_s\bar{\alpha}_s \sim 1$

Unitarization & Geometric scaling



- Geometric scaling window at $1 < \rho - \rho_s \ll \sqrt{2D\bar{\alpha}_s Y}$

$$T(r, Y) \simeq \mathrm{e}^{-\gamma_s(\rho - \rho_s)} = (r^2 Q_s^2)^{\gamma_s}$$

- Saturation makes itself felt in the dilute regime $(Q^2 > Q_s^2)$

Geometric Scaling at HERA

(Staśto, Golec-Biernat and Kwieciński, 2000)

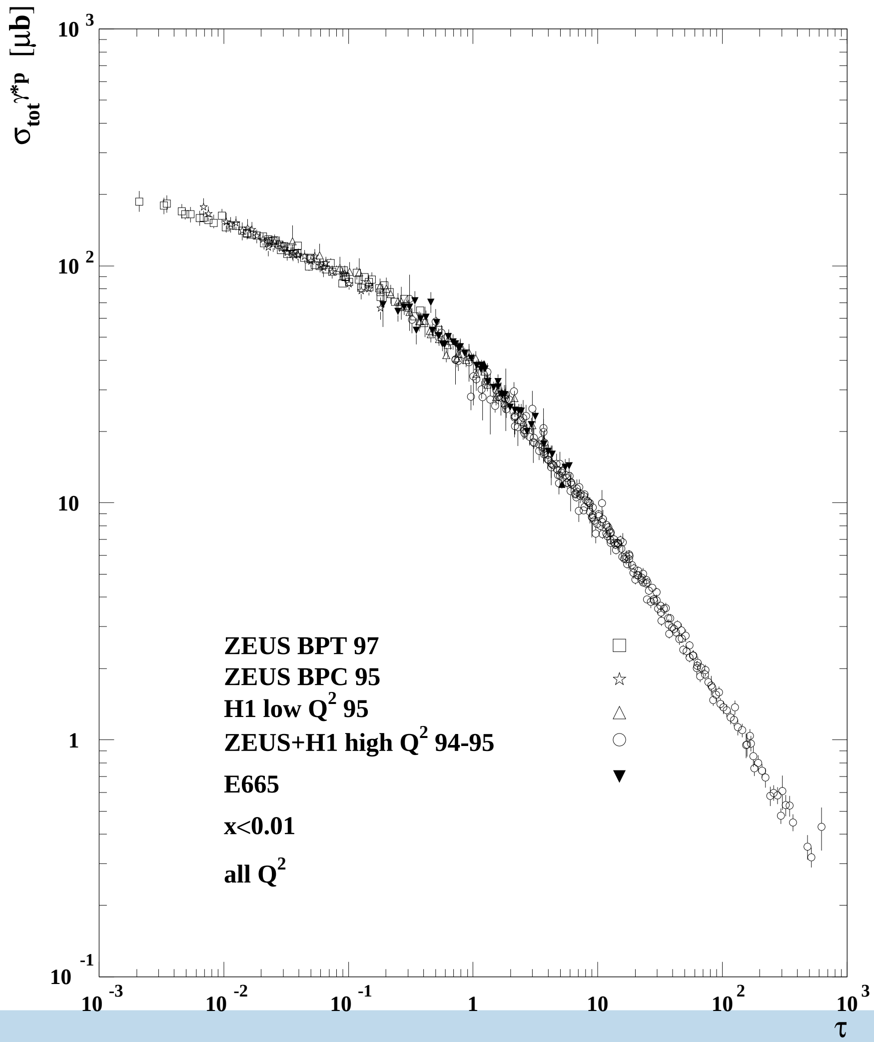
$$\sigma(x,Q^2)\approx\sigma(\tau) \quad \text{with} \quad \tau\equiv Q^2/Q_s^2(x), \quad Q_s^2(x)=(x_0/x)^{\lambda}\,\mathrm{GeV}^2\,, \quad \lambda\simeq 0.3$$

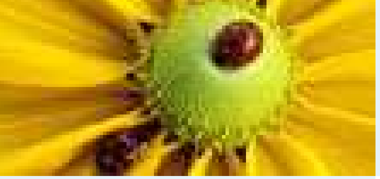
$x\leq 0.01$

$Q^2\leq 450$ GeV^2

$Q_s^2\sim 1$ GeV^2

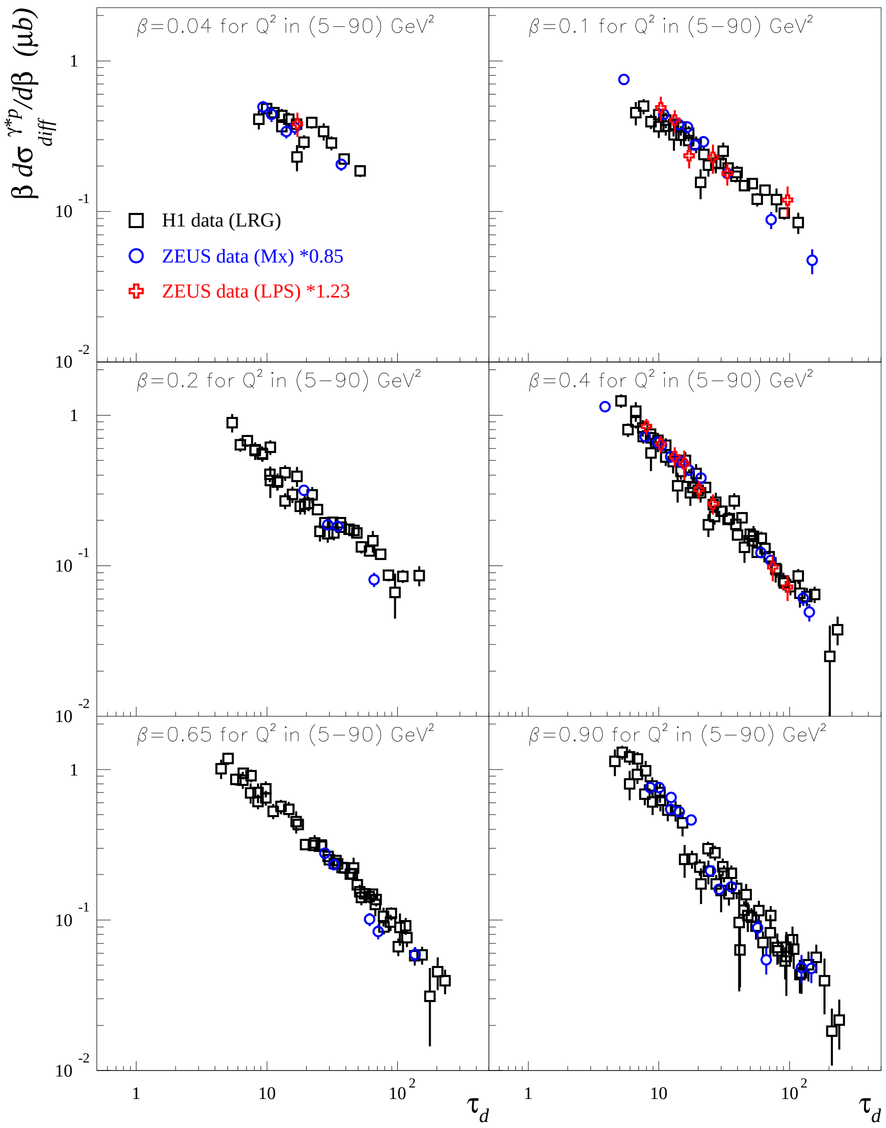
for $x\sim 10^{-4}$

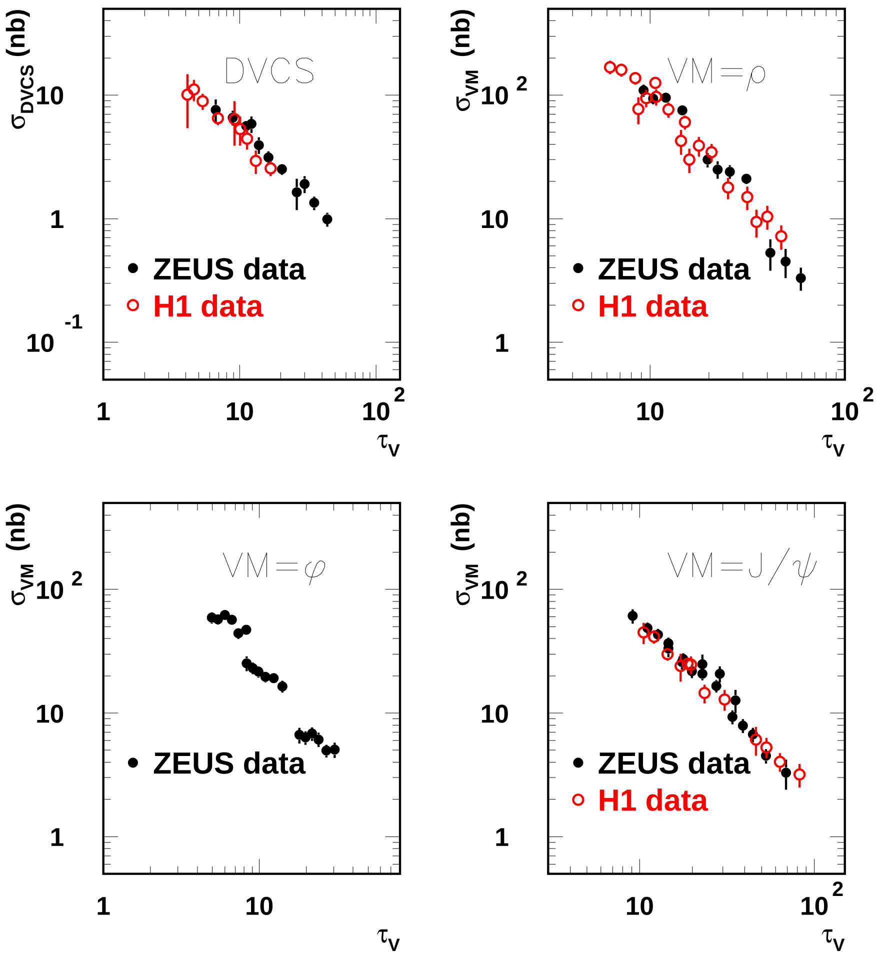


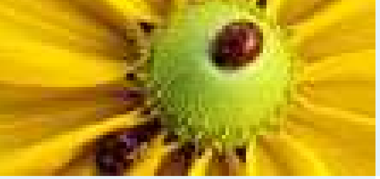


Geometric Scaling at HERA (2)

(Marquet and Schoeffel 2006)

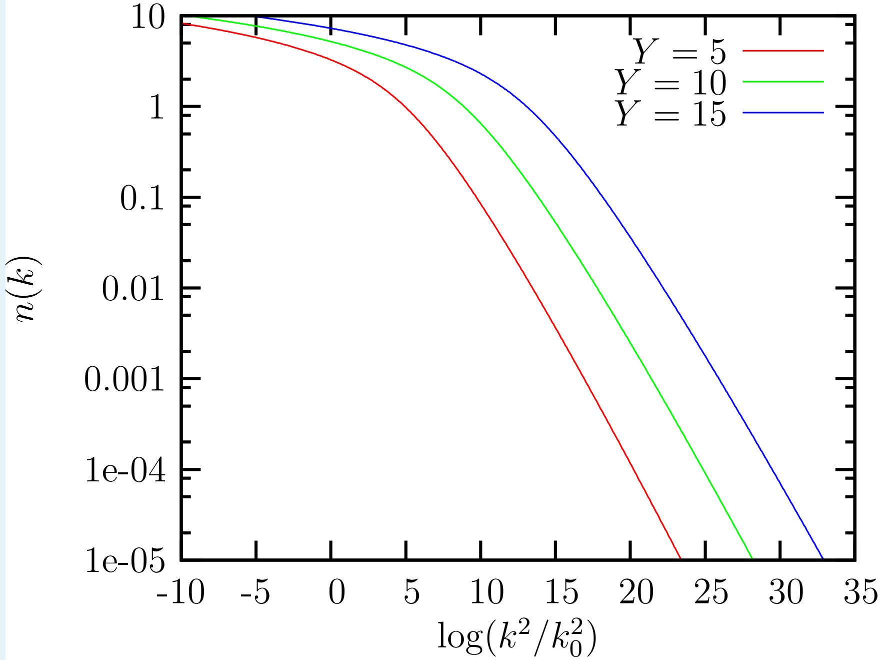


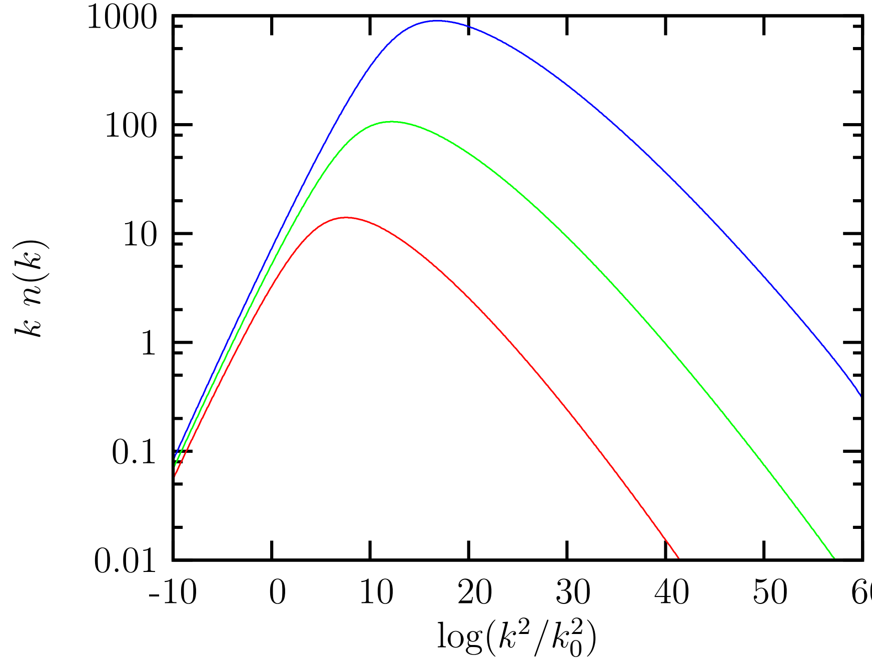




Gluon occupation number

$$xG(x,Q^2) = \int \mathrm{d}^2b \int^Q \mathrm{d}k\, k\, n(x,k)$$





- $Q_s(Y)$: the typical transverse momentum of the gluons
- For $k_\perp \lesssim Q_s(Y)$: Gluon saturation

The Color Glass Condensate

(McLerran, Venugopalan, 1994; E.I., Leonidov, McLerran, 2000)

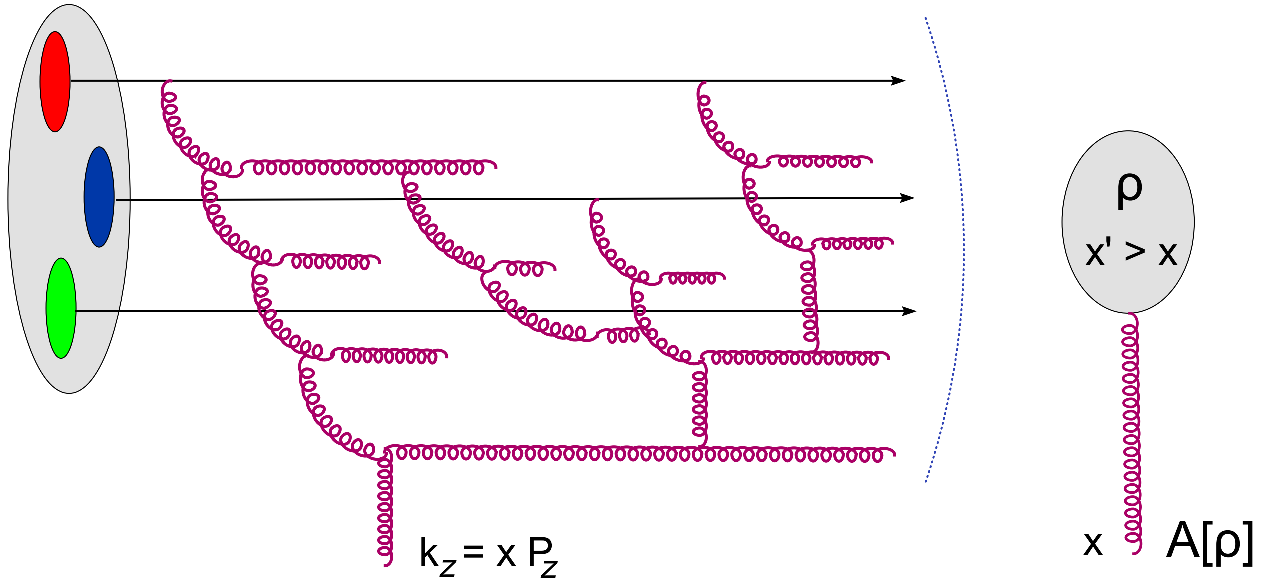
- An effective theory for the small–x gluons in the high–density environment characteristic of saturation
- Large occupation numbers ($n \sim 1/\alpha_s$)
  - The gluons can be described as classical color fields $\checkmark$
  - No fluctuations in the gluon number ($2 \to n$ splitting) $\cancel{\checkmark}$
- Separation of scales (longitudinal momentum/time)
  - The smaller x, the shorter the lifetime of the gluon

$$\Delta t \sim \frac{\hbar}{\Delta E} = \frac{2xp}{k_\perp^2}$$

  - The gluons with $x' \gg x$ are 'frozen' over the typical time scale for the dynamics at x

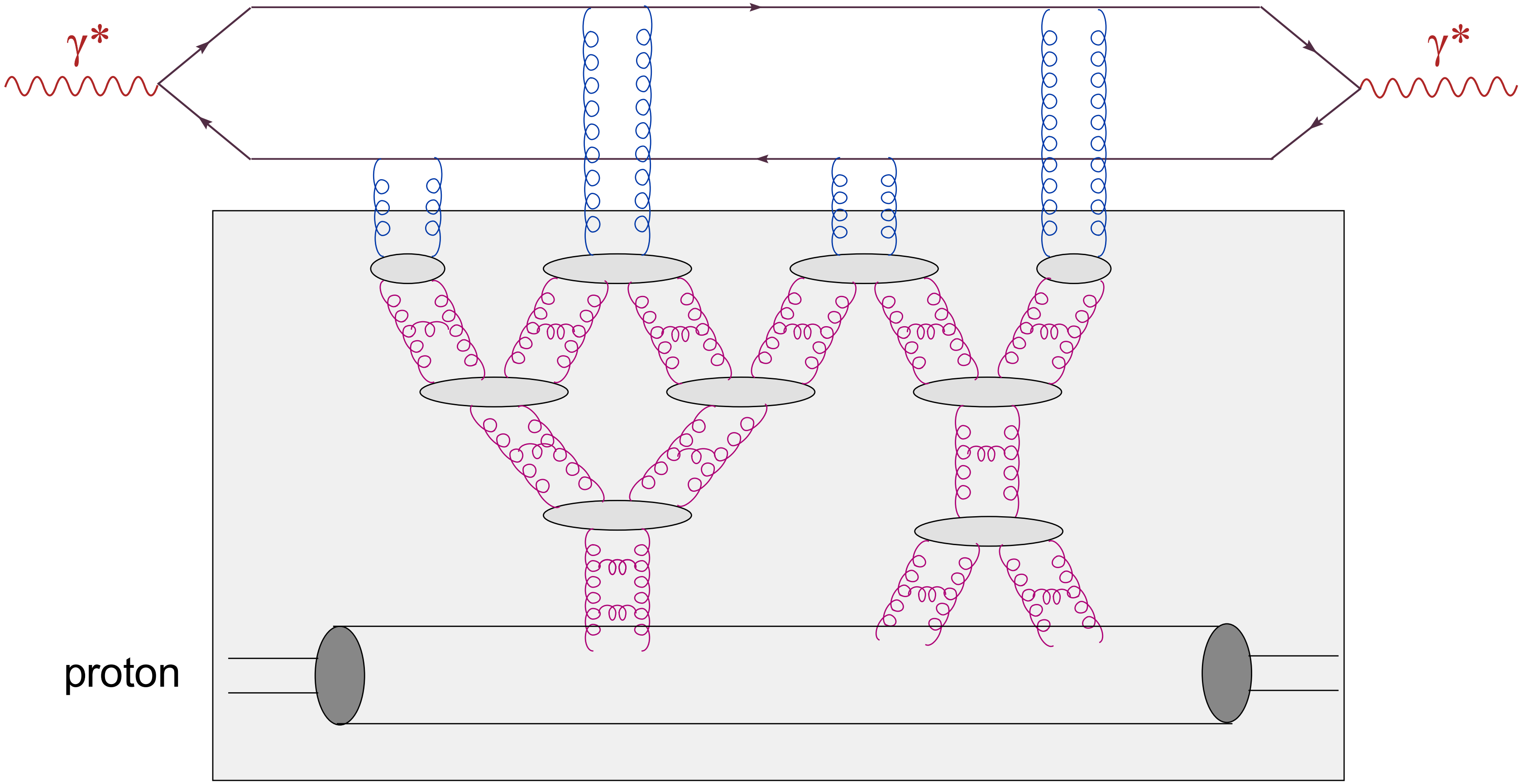
The Color Glass Condensate

- Small–x gluons: Classical color fields radiated by fast color sources ($x' \gg x$) 'frozen' in some random configuration



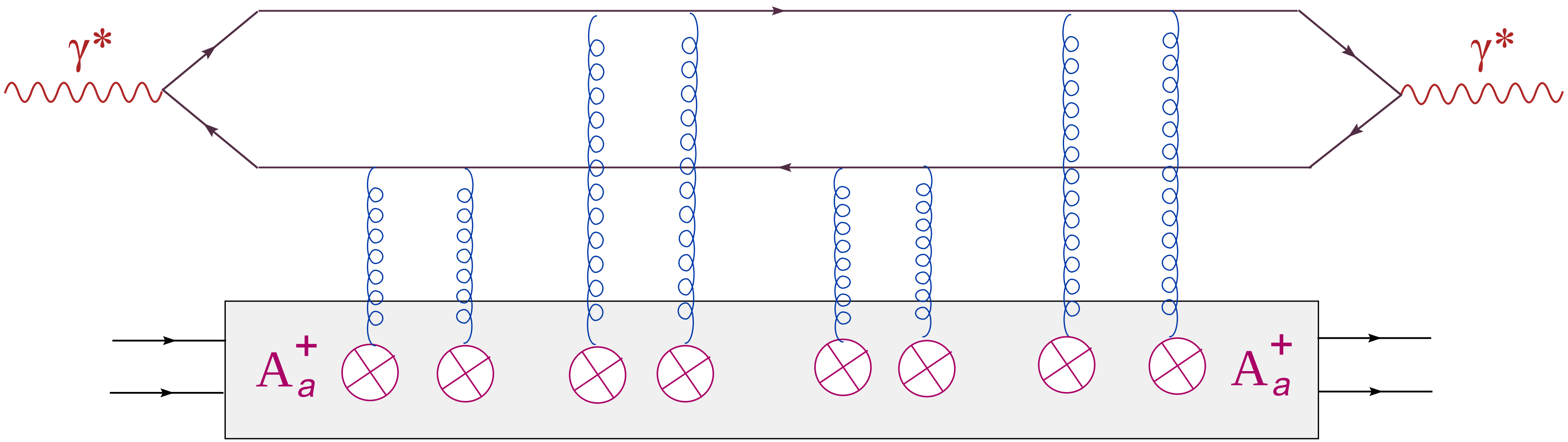
- Classical field equations (Yang–Mills) for the field $A_a^\mu[\rho]$
- Probability distribution for the charge density at Y : $W_Y[\rho]$
- Renormalization group equation for $W_Y[\rho]$: JIMWLK

CGC picture of DIS



- Include all the evolution effects in the target wavefunction
- The target : a random distribution of color charges (ρ_a)

CGC picture of DIS



- The dipole undergoes multiple scattering off the color fields generated by ρ : $S(\boldsymbol{x},\boldsymbol{y})[\rho] =$ the 'event-by-event' S–matrix
- Average over ρ with weight function $W_Y[\rho]$:

$$\langle S(\boldsymbol{x},\boldsymbol{y})\rangle_Y = \int \mathrm{D}[\rho]\; W_Y[\rho]\; S(\boldsymbol{x},\boldsymbol{y})[\rho]$$

- $W_Y[\rho]$: a kind of “super gluon distribution”
  - information about all the n–point correlations of ρ

The Weiszäcker–Williams color field

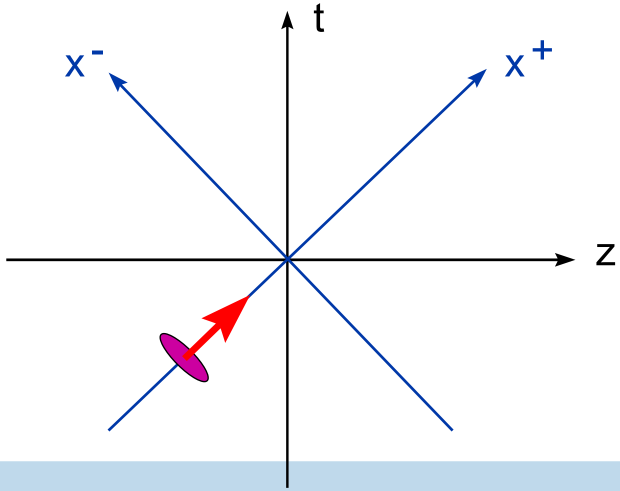
- The Yang–Mills equation:

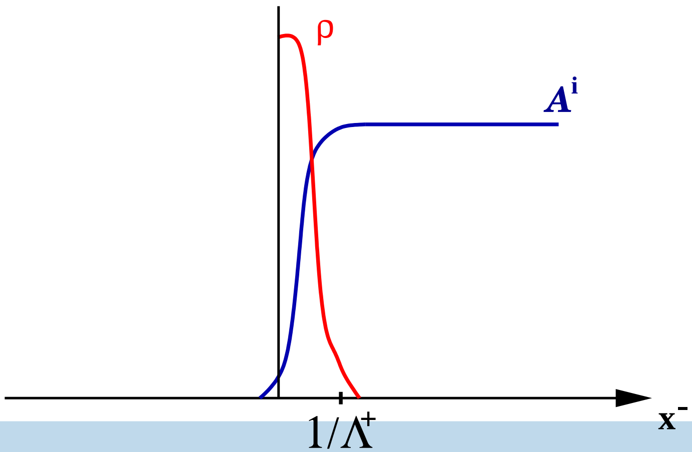
$$\big(D_\nu F^{\nu\mu}\big)_a(x) \;=\; \delta^{\mu+}\rho_a(x^-, x_\perp)$$

- NB : The source ρ_a is
  - independent of the LC time x^+ (Lorentz time dilation)
  - localized near $x^- = 0$ (i.e., $z = t$) within a distance fixed by the uncertainty principle:

$$\Delta x^- \;\sim\; 1/\Lambda^+ \quad \text{with} \quad \Lambda^+ \;=\; \mathrm{xP}^+$$

(the minimal k^+ momentum in the effective theory)





Classical solution : Coulomb gauge

- Exercise: Show that one can choose the solution A_a^μ so that

$$\frac{\partial A_a^\mu}{\partial x^+} = 0, \qquad A_a^- = 0, \qquad F_a^{ij} = 0, \quad i, j = 1, 2$$

After also fixing the gauge : only one independent field

- Coulomb gauge : $\nabla^i A_a^i = 0 \implies A_a^i = 0, \ \ i = 1, 2$

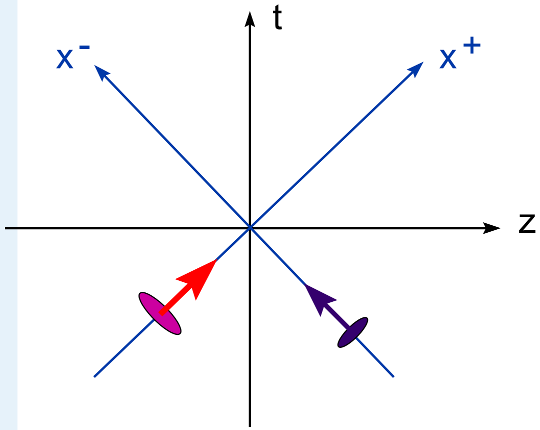
$$-\nabla_\perp^2 A_a^+(x^-, x_\perp) = \rho_a(x^-, x_\perp)$$

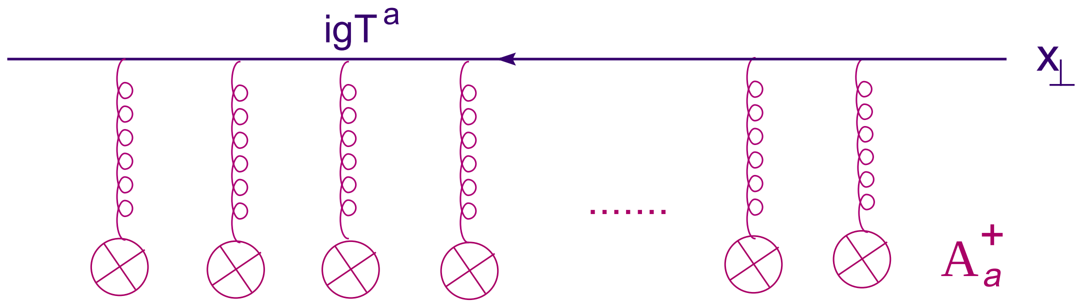
- Exercise: Show that the solution is of the form

$$A_a^+(x^-, \boldsymbol{x}) = \int \mathrm{d}^2\boldsymbol{y} \, \frac{1}{4\pi} \ln \frac{1}{(\boldsymbol{x} - \boldsymbol{y})^2 \mu^2} \, \rho_a(x^-, \boldsymbol{y})$$

NB : Localized near $x^- = 0$, so like the color charge itself.

Eikonal Approximation





- Right moving target (CGC) + Left moving projectile $(q,\ \bar{q},\ g\ldots)$
- Field equation in the background field: $D^\mu = \partial^\mu - igT^a A_a^\mu$

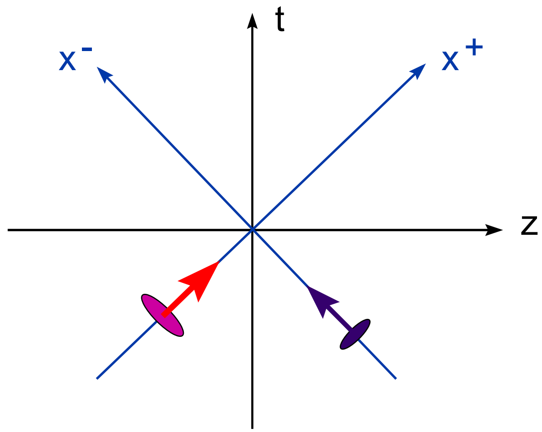
$$D^\mu D_\mu \phi(x) = 0, \qquad \gamma_\mu D^\mu \psi(x) = 0, \qquad \ldots$$

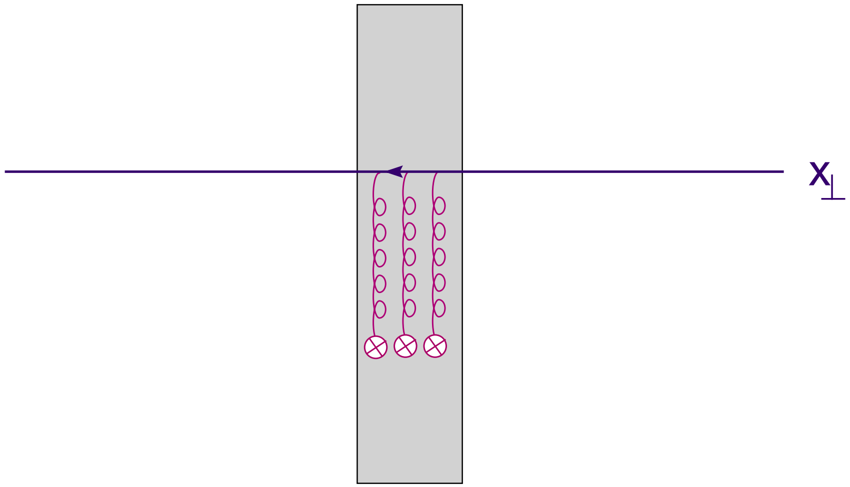
- When $A^\mu = 0$, the projectile has only p^- :

$$\phi_0(x) = \mathrm{e}^{-\mathrm{i}p^- x^+} \implies \phi(x) = \mathrm{e}^{-\mathrm{i}p^- x^+} S(x^-, \boldsymbol{x})$$

$$-\mathrm{i}D^+ S(x^-, \boldsymbol{x}) = \frac{D_\perp^2}{2p^-} S(x^-, \boldsymbol{x})$$

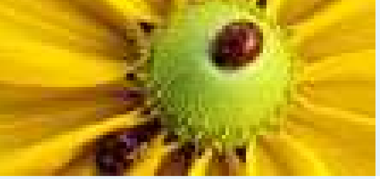
Eikonal Approximation



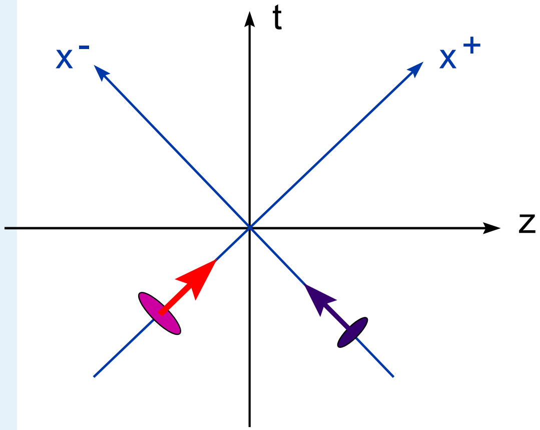


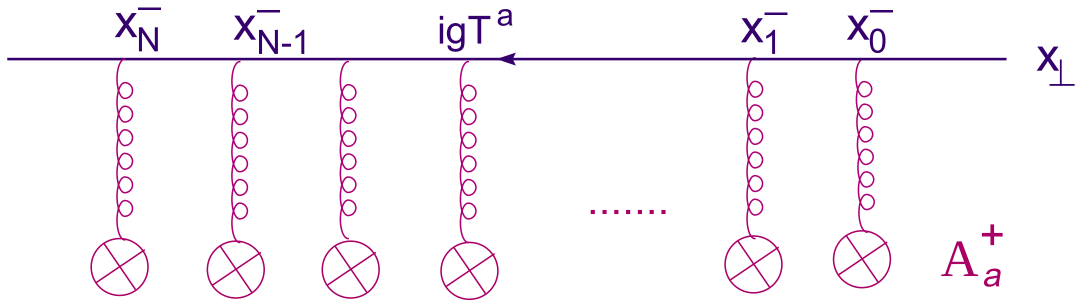
$$-\mathrm{i}D^+ S(x^-, \boldsymbol{x}) = \frac{D_\perp^2}{2p^-} S(x^-, \boldsymbol{x})$$

- High energy limit $\Longrightarrow$ $D^+ S(x^-, \boldsymbol{x}) \approx 0$
  - Obvious in the target rest frame: p^- is very large
  - Also true in the target IMF: $D^+ \sim 1/\Delta x^-$ is very large
- No transverse gradient $\Longrightarrow$ $\boldsymbol{x} = $ const.
  - The particle preserves a straight line trajectory.



Wilson lines *(pour Julie)*

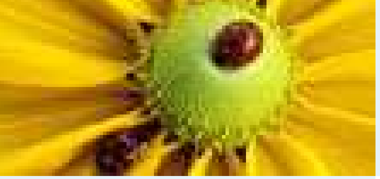




$$D^+ S(x^-, \boldsymbol{x}) \equiv \left(\frac{\partial}{\partial x^-} - igT^a A_a^+\right) S(x^-, \boldsymbol{x}) = 0$$

Abelian case ($gT^a \to e$): $S(x^-, \boldsymbol{x}) = \exp\left\{ \mathrm{i}e \int_{-\infty}^{x^-} \mathrm{d}y^- A^+(y^-, \boldsymbol{x}) \right\}$

- ▷ The integrated field strength along the trajectory: $\mathrm{d}y^- A^+ \leftrightarrow j_\mu A^\mu$
- ▷ The S–matrix is a pure phase $\Longrightarrow$ elastic scattering



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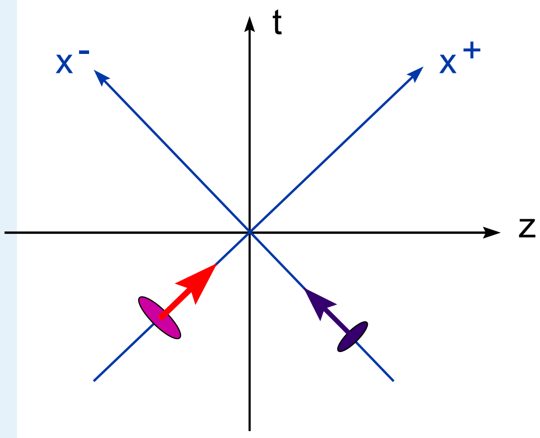
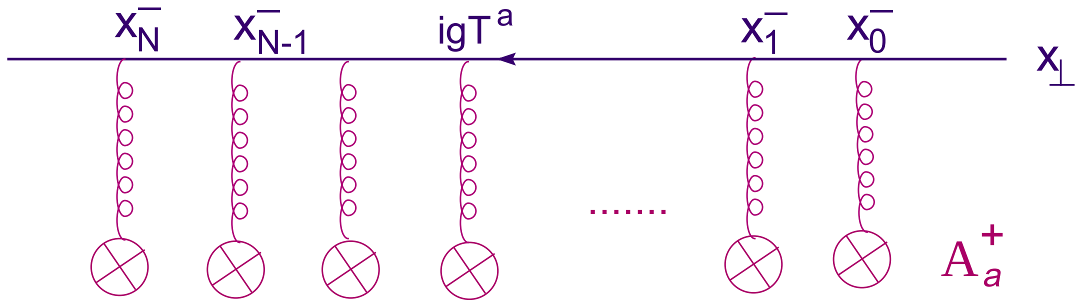




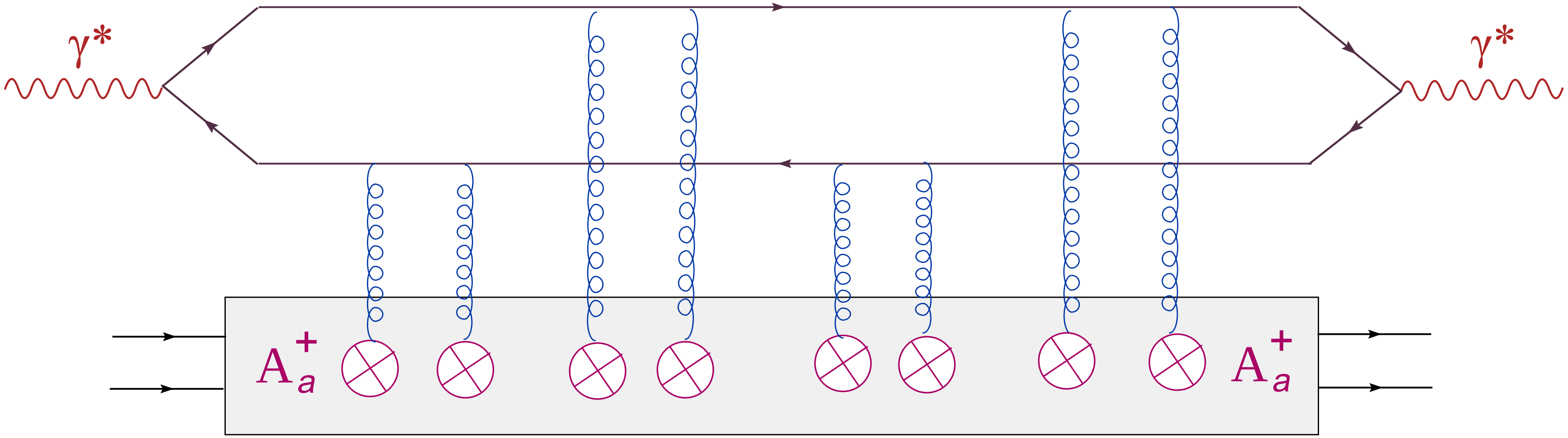
$$D^+ S(x^-, \boldsymbol{x}) \equiv \left(\frac{\partial}{\partial x^-} - igT^a A_a^+ \right) S(x^-, \boldsymbol{x}) = 0$$

non–Abelian case: $$S(x^-, \boldsymbol{x}) = \mathrm{P} \exp\left(\mathrm{i}g \int_{-\infty}^{x^-} \mathrm{d}y^- A_a^+(x^-, \boldsymbol{x}) T^a \right)$$

- Path–ordered exponential (‘Wilson line’) : color rotation

$$S(x^-) = \mathrm{e}^{\mathrm{i}g\epsilon A^+(x_N^-)}\, \mathrm{e}^{\mathrm{i}g\epsilon A^+(x_{N-1}^-)} \dots \mathrm{e}^{\mathrm{i}g\epsilon A^+(x_0^-)}$$

Dipole scattering

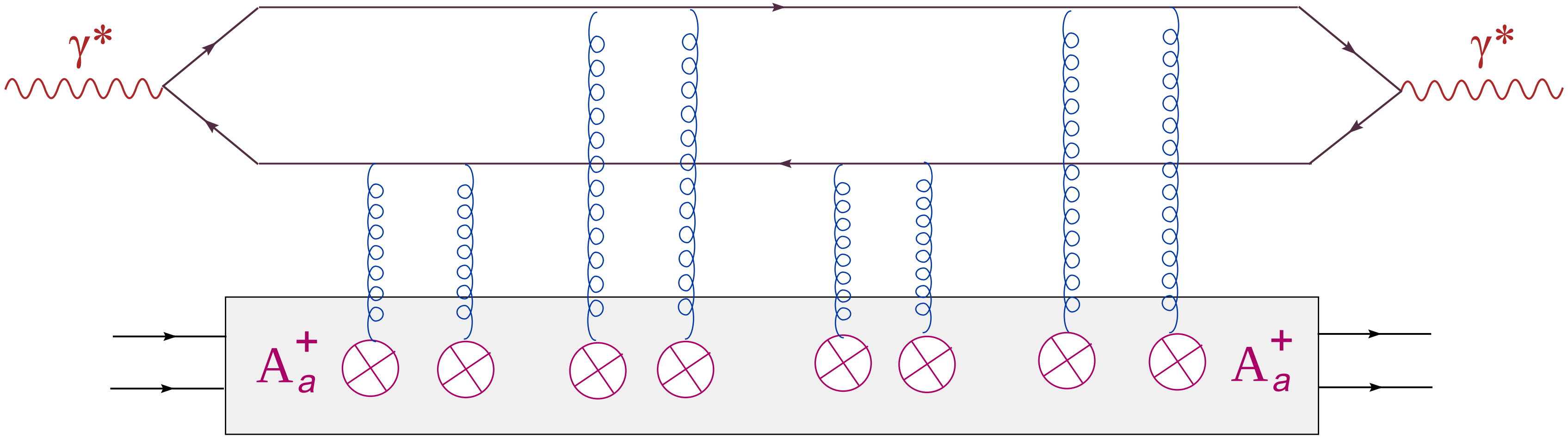


$$S(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{N_c}\mathrm{tr}\Big(V(\boldsymbol{x})\,V^\dagger(\boldsymbol{y})\Big)$$

$$V(\boldsymbol{x}) \equiv \mathrm{Pexp}\Big(\mathrm{i}g\int \mathrm{d}x^- A^+_a(x^-,\boldsymbol{x})t^a\Big) \;:\; \in \mathrm{SU}(N_c)$$

- Color trace: the dipole is color neutral
- Color transparency: when $\boldsymbol{x}\to\boldsymbol{y}$, $S\to 1$
- Unitarity manifest: $|S|\leq 1$ (multiple scattering)

The weight function $W_Y[\rho]$



- $S(\boldsymbol{x},\boldsymbol{y})[\rho] \;=\;$ the 'event-by-event' S–matrix :

  a given configuration of the color sources in the target (ρ_a)

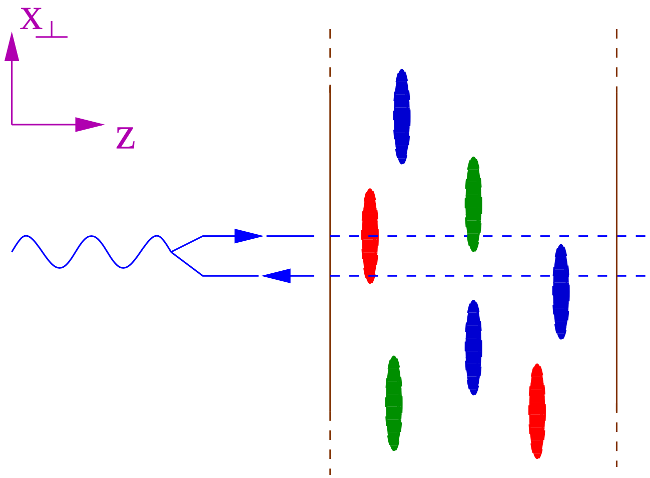
- The physical amplitude: average over all configurations

  average over ρ with weight function $W_Y[\rho]$:

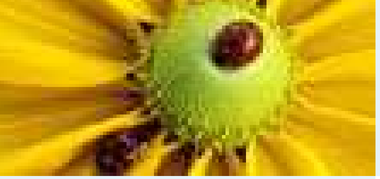
$$\langle S(\boldsymbol{x},\boldsymbol{y})\rangle_Y \;=\; \int \mathrm{D}[\rho]\;W_Y[\rho]\;S(\boldsymbol{x},\boldsymbol{y})[\rho]$$

- Computing $W_Y[\rho]$: the main issue in the CGC formalism

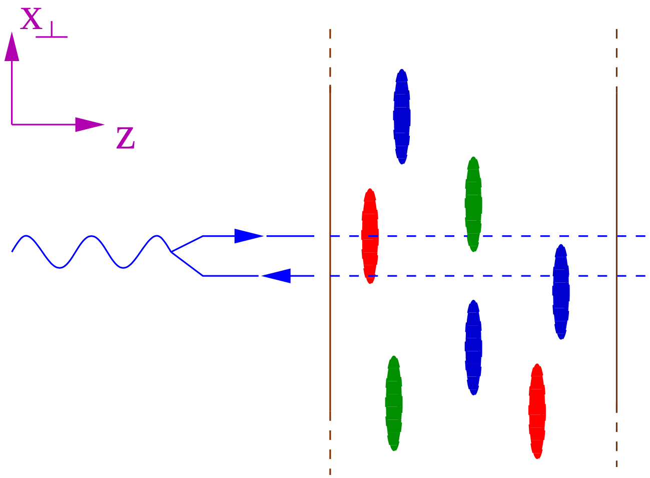
The McLerran–Venugopalan model



- The gluon distribution of a large nucleus $(A \gg 1)$ at not so large energy : $\alpha_s Y \ll 1$ (say, $x = 10^{-1} \div 10^{-2}$)



The McLerran–Venugopalan model



- The 'color sources': randomly distributed, uncorrelated, valence quarks $\Longrightarrow$ a Gaussian weight function

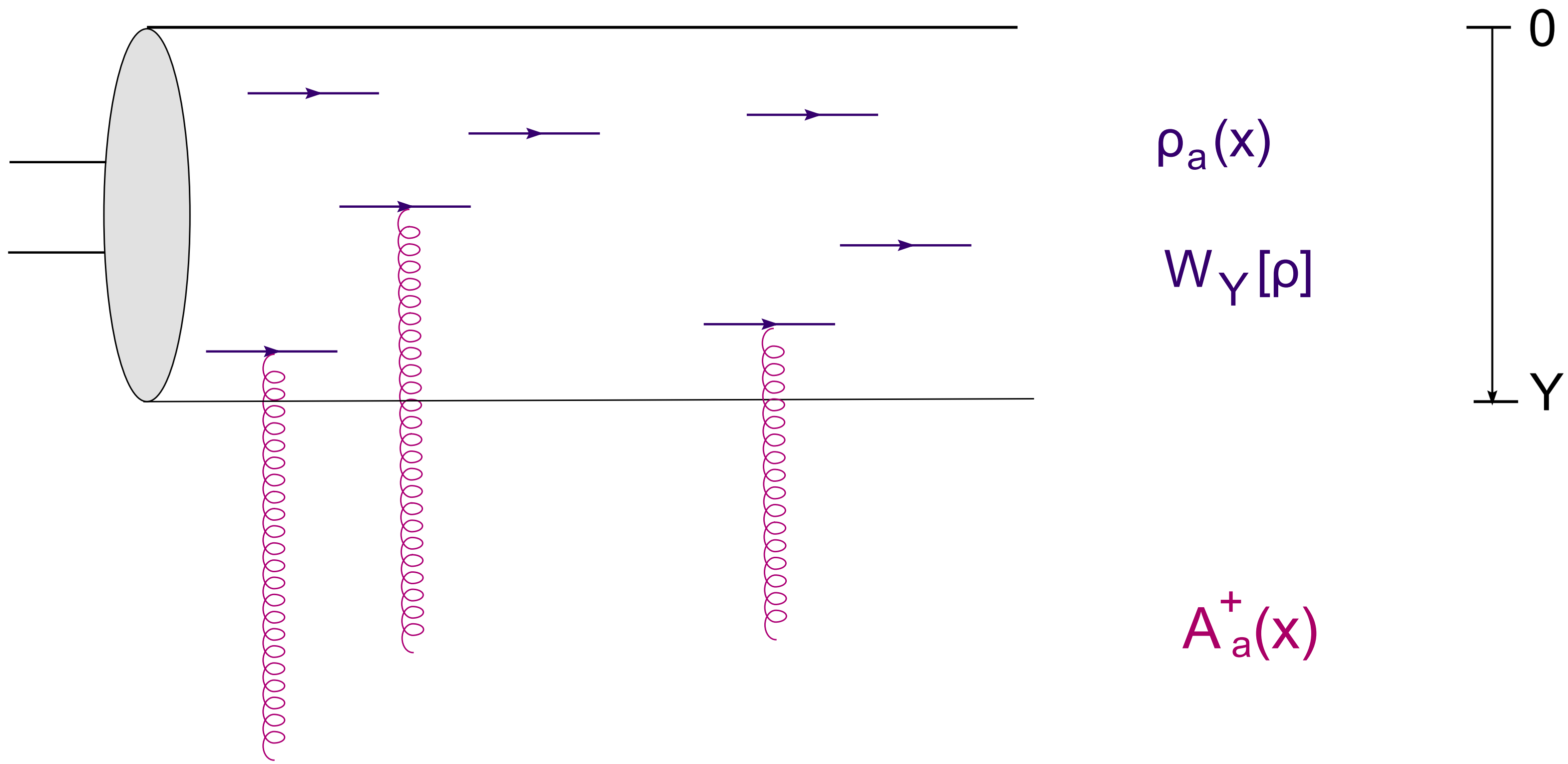
$$W_A[\rho] \,=\, \mathcal{N} \exp\left\{-\,\frac{1}{2}\int \mathrm{d}x^- \mathrm{d}^2\boldsymbol{x}\;\frac{\rho_a(x^-,\boldsymbol{x})\rho_a(x^-,\boldsymbol{x})}{\mu_A(x^-)}\right\}$$

- Exercice: Perform the average over ρ_a to show that:

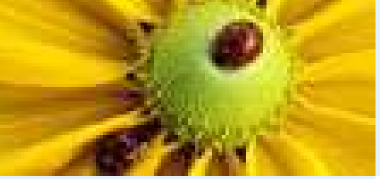
$$S(r) \,\simeq\, \mathrm{e}^{-r^2 Q_s^2(A)} \quad \text{with} \quad Q_s^2(A) \,\equiv\, \alpha_s N_c \int \mathrm{d}x^- \mu_A(x^-) \,\propto\, A^{1/3}$$

$\triangleright$ multiple scattering (Glauber–like), unitarity, absorbtion

Renormalization group at small x



- Rapidity Y : All the information about the gluon distribution has been included in the weight function $W_Y[\rho]$
- Alternatively: $W_Y[A^+]$ (a simple change of variables)



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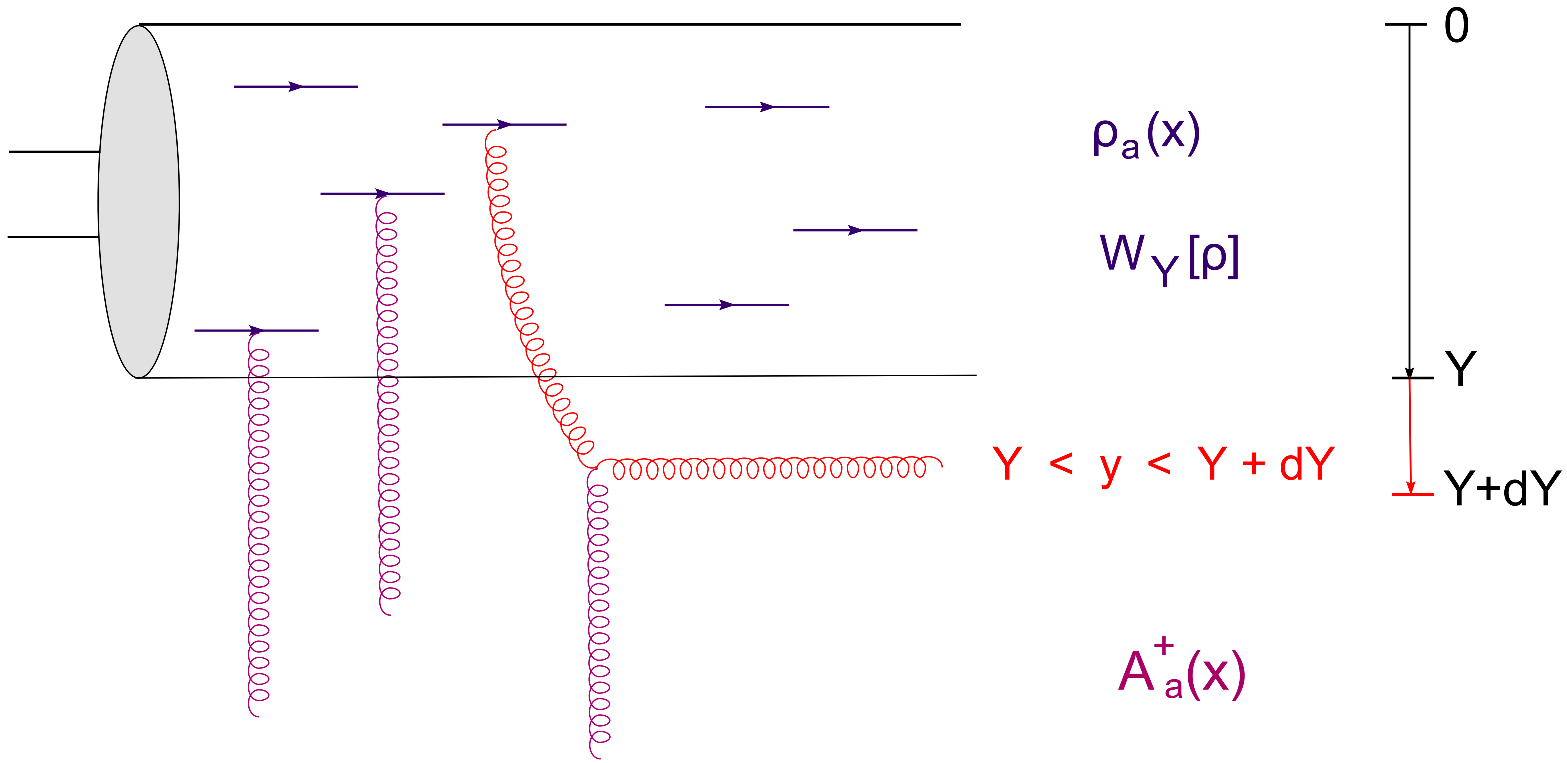
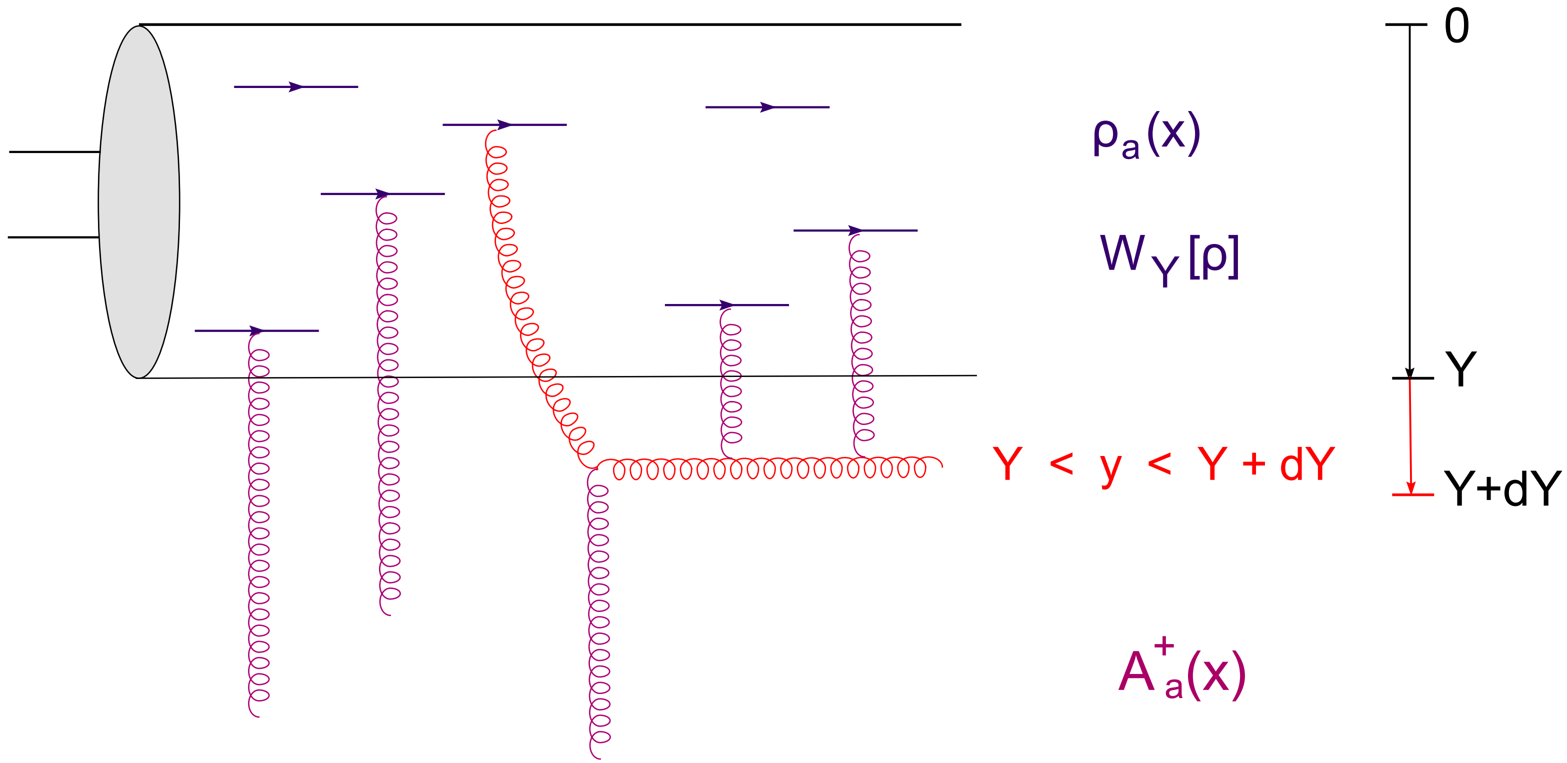
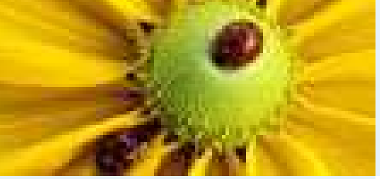


- Rapidity $Y + \mathrm{d}Y$: One addition ‘color source’ (gluon) is being radiated, from one of the previous color sources
- Low density/energy : The new gluon is incoherently produced from any of the previous sources: $\delta\rho \propto \rho$

Renormalization group at small x



- **High density/energy** : The new gluon can rescatter off the color field produced by other sources: $\delta\rho =$ non–linear in ρ
- New correlations among the color fields A_a^+



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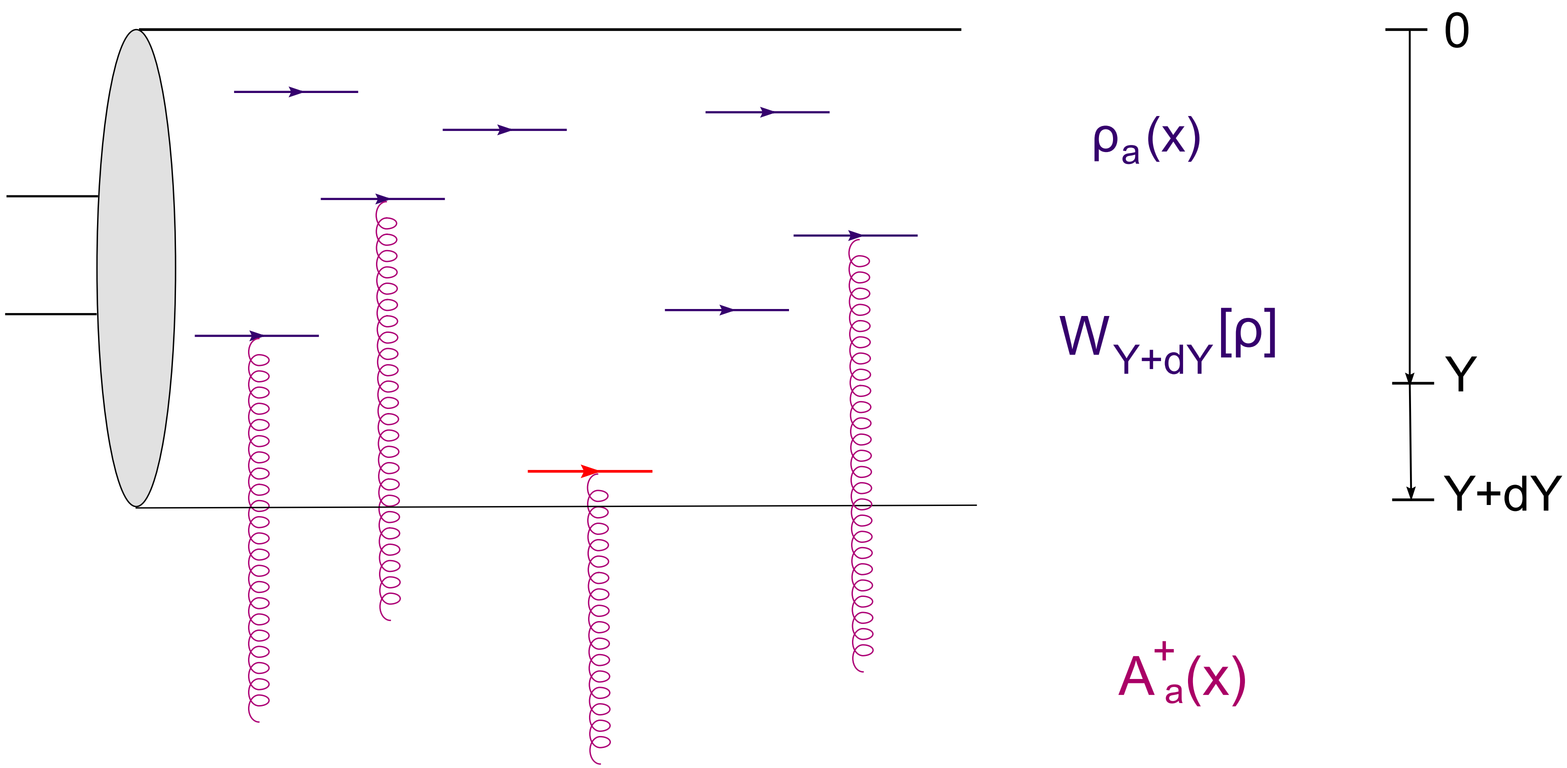
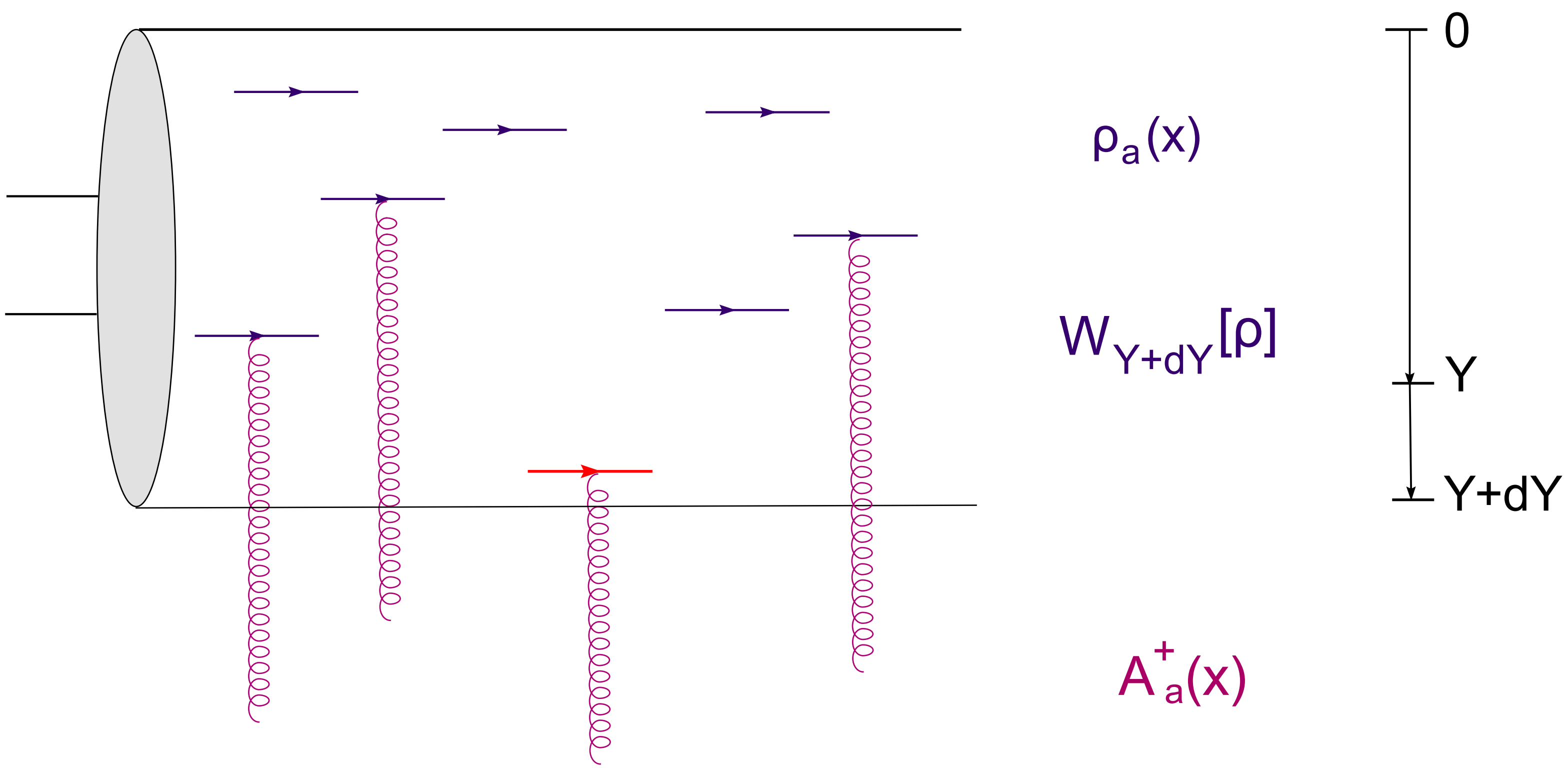


- Absorb the change in ρ and in the correlations into a change of the weight function: $W_Y[\rho] \longrightarrow W_{Y+\mathrm{d}Y}[\rho]$
- Evolution equation for $W_Y[\rho]$ ('JIMWLK')

$$\frac{\partial W_Y[\rho]}{\partial Y} = H\left[\rho, \frac{\delta}{\delta\rho}\right] W_Y[\rho]$$

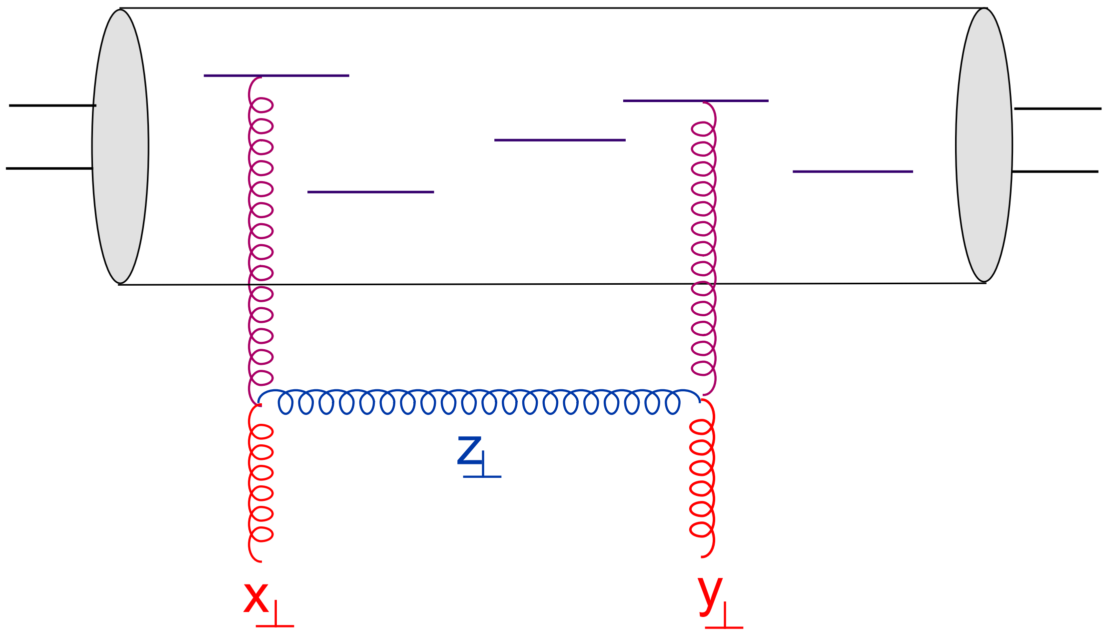
Renormalization group at small x



- Absorb the change in ρ and in the correlations into a change of the weight function: $W_Y[\rho] \longrightarrow W_{Y+\mathrm{d}Y}[\rho]$
- Evolution equation for $W_Y[\rho]$ (read 'JIMWaLK')

  (Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, and Kovner, 97–00)

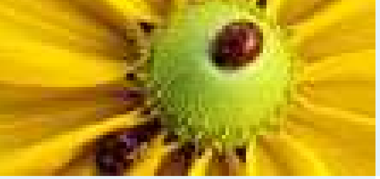
JIMWLK Hamiltonian



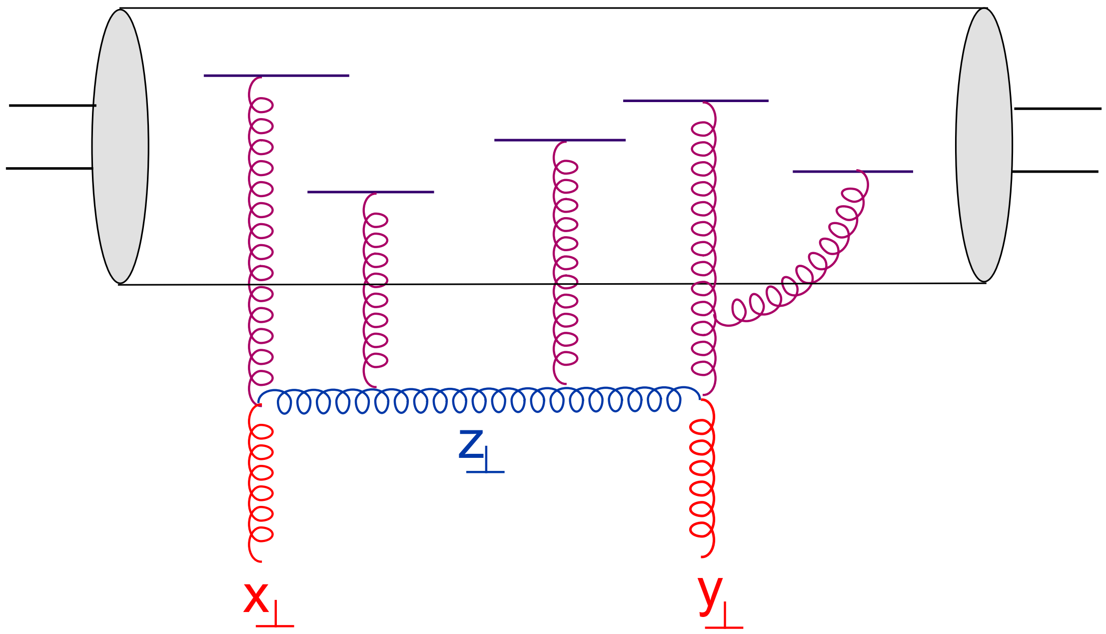
- A ‘cut’ diagram : amplitude $\times$ complex conjugate amplitude

factorized structure: $$H = \int \frac{d^2 z_\perp}{2\pi} J_a^i(z_\perp) J_a^i(z_\perp)$$

- The ‘current’ $J_a^i(z_\perp)$ is real $\Longrightarrow$ positivity of H
- The diagram above: linear approximation $\Longleftrightarrow$ BFKL



JIMWLK Hamiltonian

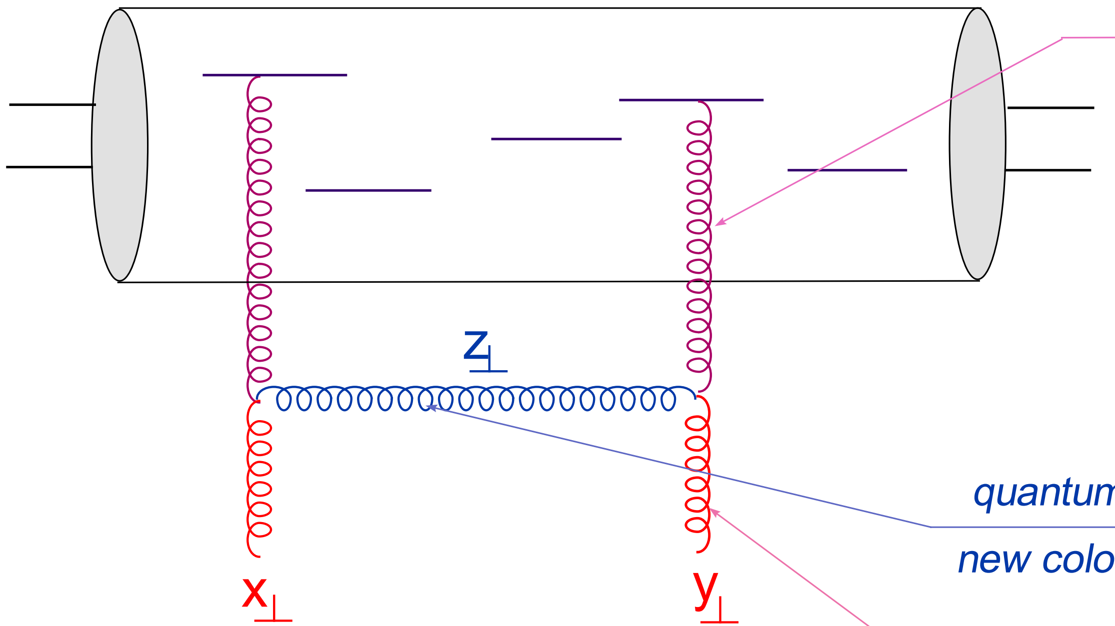


- A 'cut' diagram : amplitude $\times$ complex conjugate amplitude

factorized structure: $$H = \int \frac{\mathrm{d}^2 z_\perp}{2\pi} J_a^i(z_\perp) J_a^i(z_\perp)$$

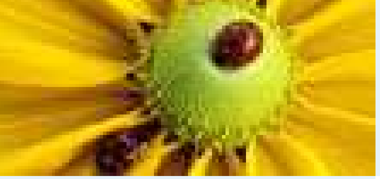
- The 'current' $J_a^i(z_\perp)$ is real $\Longrightarrow$ positivity of H
- But the same factorized structure holds in the general case.

The linear approximation: Recovering BFKL

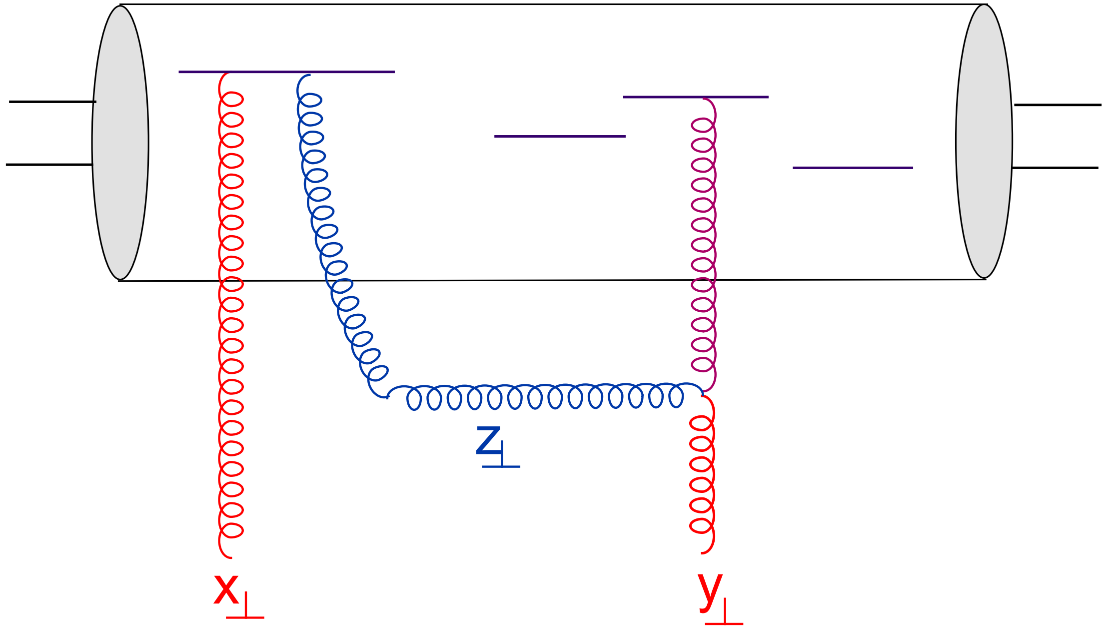


$$J_a^i(\boldsymbol{z}) \;=\; \mathrm{i}g\,T^c_{ab}\int\frac{\mathrm{d}^2\boldsymbol{x}}{2\pi}\,\frac{z^i-x^i}{(\boldsymbol{z}-\boldsymbol{x})^2}\left(A_c^+(\boldsymbol{z})-A_c^+(\boldsymbol{x})\right)\frac{\mathrm{i}\delta}{\delta A_b^+(\boldsymbol{x})}$$

- The amplitude for producing a gluon at $\boldsymbol{x}$ out of a source at $\boldsymbol{z}$
- Valid so long as the fields are weak, such that $gA^+ \ll 1$



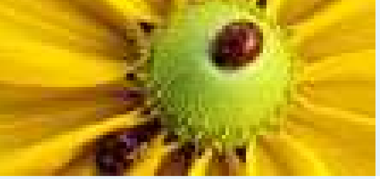
The linear approximation: Recovering BFKL



$$J_a^i(\boldsymbol{z}) \;=\; \mathrm{i}g\,T_{ab}^c \int \frac{\mathrm{d}^2\boldsymbol{x}}{2\pi}\,\frac{z^i - x^i}{(\boldsymbol{z}-\boldsymbol{x})^2}\left(A_c^+(\boldsymbol{z}) - A_c^+(\boldsymbol{x})\right)\frac{\mathrm{i}\delta}{\delta A_b^+(\boldsymbol{x})}$$

- The amplitude for producing a new gluon is proportional to the strength A of the field created in the previous steps

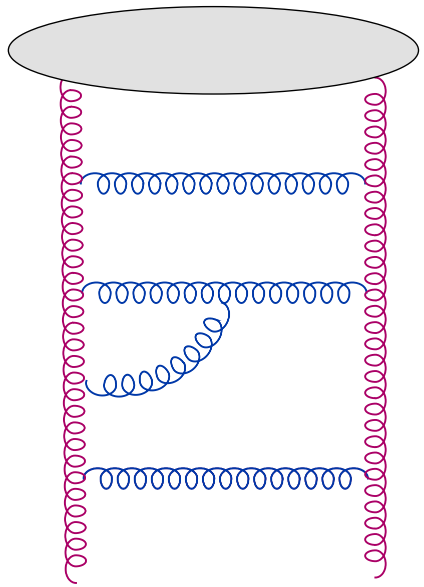
  $\Longrightarrow$ exponential amplification

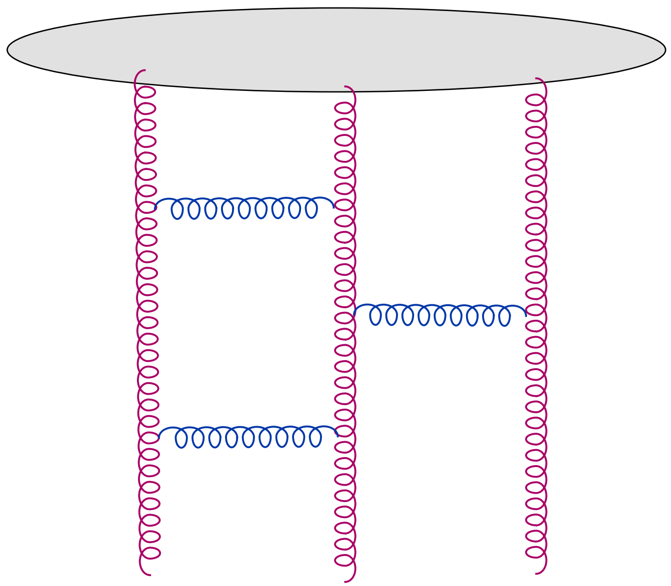


The linear approximation: Recovering BFKL

- The weak–field (BFKL) limit of the JIMWLK Hamiltonian

$$\frac{\partial W_Y[A]}{\partial Y} \approx \frac{1}{2}\,\frac{\delta}{\delta A}\,A\,A\,\frac{\delta}{\delta A}\,W_Y \;\equiv\; H_{\rm BFKL}\,W_Y[A]$$



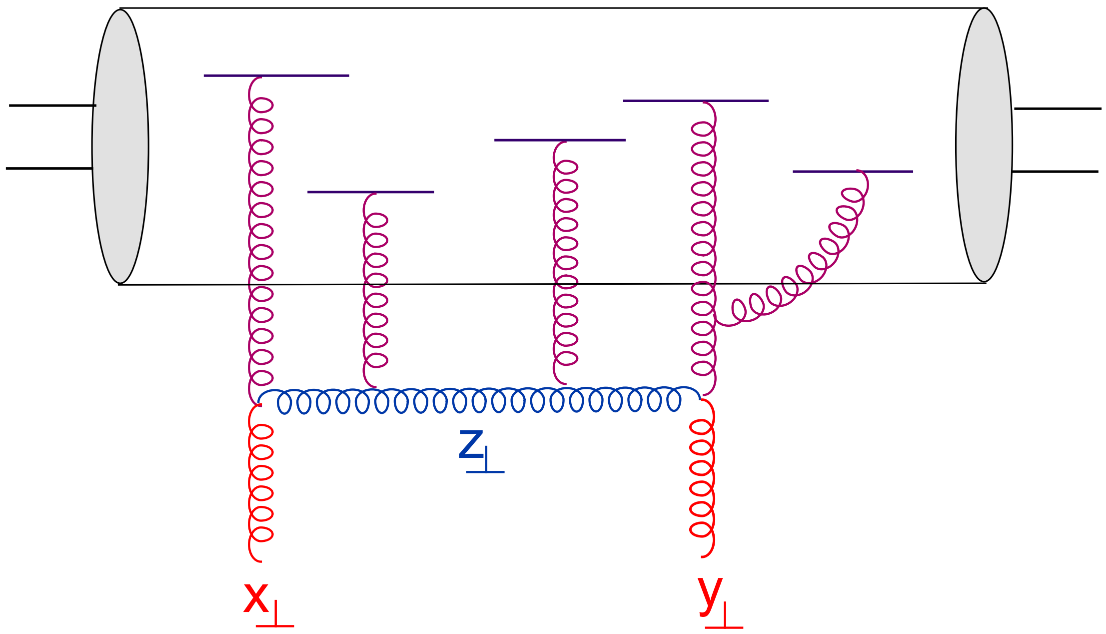


- Each step in the evolution : $2 \to 2$ gluon vertex

  Insert a 'BFKL exchange' in between each pair of fields

- All gluon correlations rise exponentially with Y

The general case: JIMWLK equation

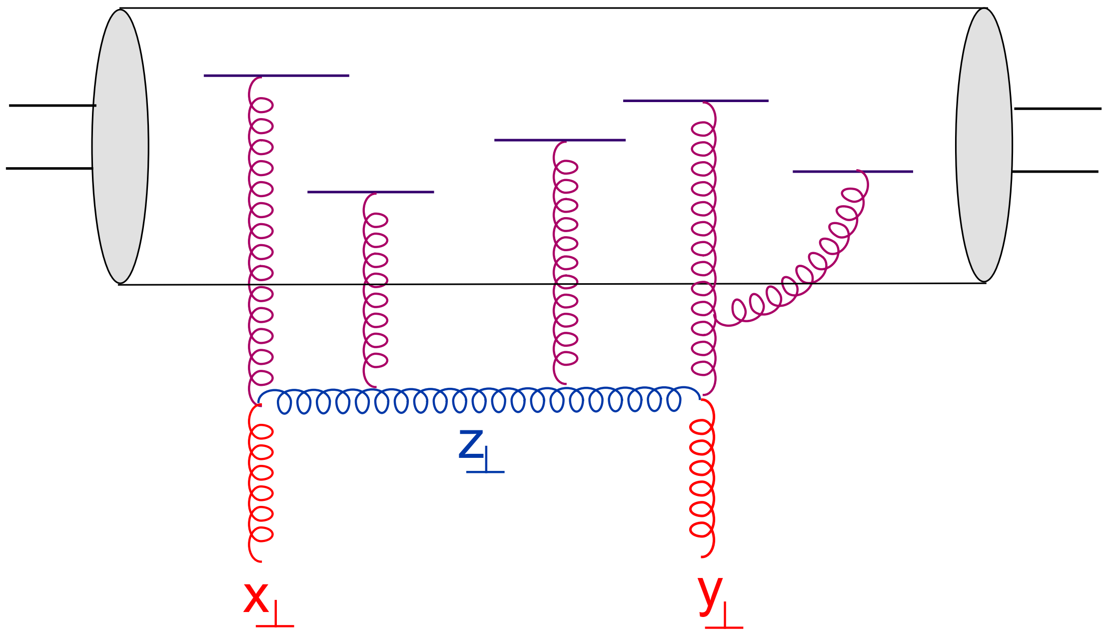


- Strong fields $gA^+ \sim \mathcal{O}(1)$: The quantum gluon rescatters of the background field in the eikonal approximation

$$J_a^i(\boldsymbol{z}) \;=\; \int \frac{\mathrm{d}^2\boldsymbol{x}}{2\pi}\,\frac{z^i - x^i}{(\boldsymbol{z}-\boldsymbol{x})^2}\,\big(1 - V^\dagger(\boldsymbol{z})V(\boldsymbol{x})\big)_{ab}\,\frac{\mathrm{i}\delta}{\delta A_b^+(\boldsymbol{x})}$$

- The induced color field is coherently produced out of all the preexisting color sources

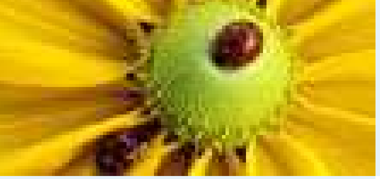
The general case: JIMWLK equation



- The Wilson lines $V \sim \mathrm{e}^{\mathrm{i}gA}$ rapidly oscillate and their products self–average to zero

$$J_a^i(\boldsymbol{z}) \approx \int \frac{\mathrm{d}^2\boldsymbol{x}}{2\pi} \frac{z^i - x^i}{(\boldsymbol{z}-\boldsymbol{x})^2} [1] \frac{\mathrm{i}\delta}{\delta A_a^+(\boldsymbol{x})}$$

- The gluon emission rate saturates at a value independent of the background field ! $\Longrightarrow$ “gluon saturation”

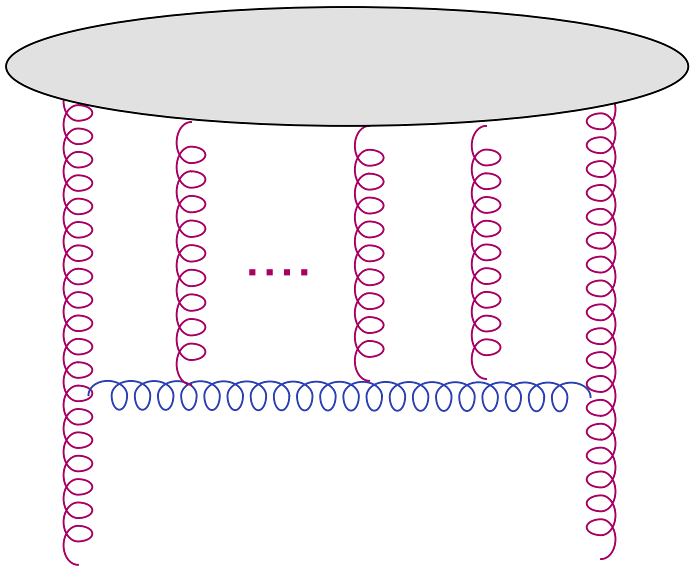


The general case: JIMWLK equation

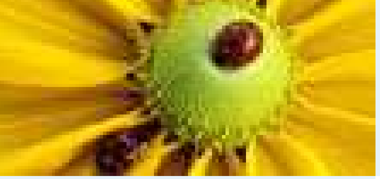
- The JIMWLK equation in compact notations (with $A_a \equiv A_a^+$)

$$\frac{\partial W_Y[A]}{\partial Y} = \frac{1}{2} \frac{\delta}{\delta A^a} \chi_{ab}[A] \frac{\delta W_Y}{\delta A^b}$$

where $\chi^{ab}[A]$ is non–linear in A to all orders (Wilson lines)



- $n \rightarrow 2$ gluon vertices with arbitrary n (non–linear evolution)
- Infinite sum of diagrams in coordinate space



The essence of saturation

A brief reminder

Color Glass Condensate

JIMWLK

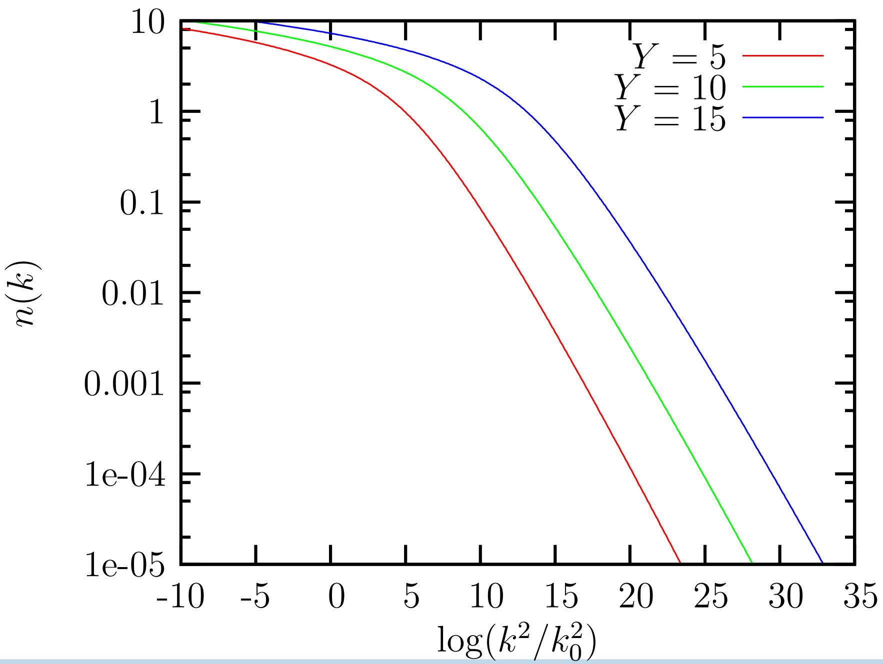
Gluon saturation
- Saturation
- Saturation
- Color neutrality
- Observables
- Balitsky eqs.
- Balitsky eqs.
- Hierarchy
- Ploops
- Front diffusion
- High–energy ‘phase-diagram’

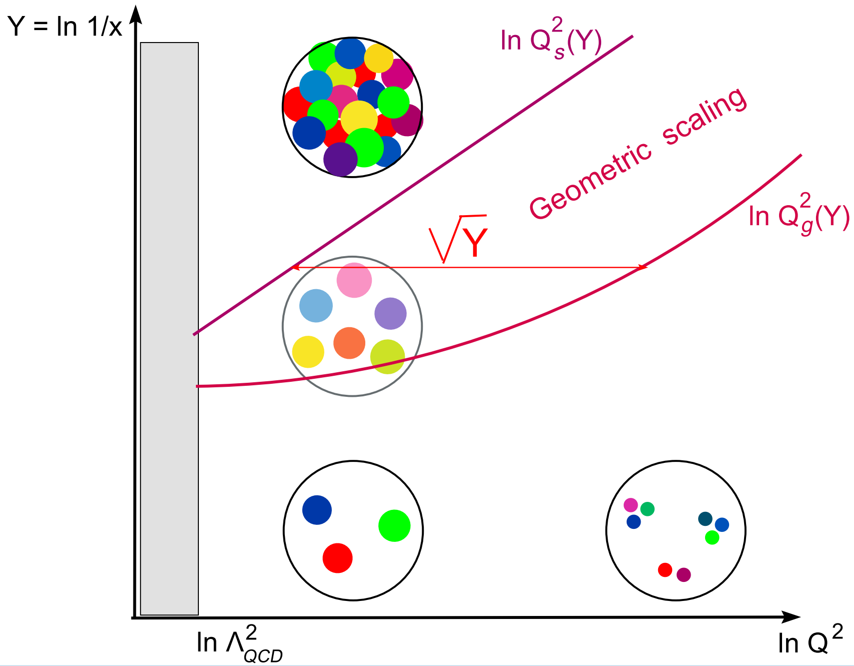
$$J_a^i(\boldsymbol{z}) \;=\; \int \frac{\mathrm{d}^2\boldsymbol{x}}{2\pi}\,\frac{z^i - x^i}{(\boldsymbol{z}-\boldsymbol{x})^2}\,\big(1 - V^\dagger(\boldsymbol{z})V(\boldsymbol{x})\big)_{ab}\,\frac{\mathrm{i}\delta}{\delta A_b^+(\boldsymbol{x})}$$

- The Wilson line bilinear $V^\dagger(\boldsymbol{z})V(\boldsymbol{x})$ self–averages to zero.
- An ‘all orders’/strong field effect: $A \sim 1/g$
  (semiclassical: like a Laser, or a Bose Condensate)
- This only happens on sufficiently large distances !
  ($V^\dagger(\boldsymbol{z})V(\boldsymbol{x}) \,\to\, 1$ as $\boldsymbol{z} \,\to\, \boldsymbol{x}$!)
- Remember: $V^\dagger(\boldsymbol{z})V(\boldsymbol{x})$ = the S–matrix for a color dipole
  $V^\dagger(\boldsymbol{z})V(\boldsymbol{x}) \,\to\, 0$ over distances $|\boldsymbol{z}-\boldsymbol{x}| \gtrsim 1/Q_s(Y)$
- The connection saturation $\leftrightarrow$ dipole unitarity is manifest!
- This happens on relatively large time scales : “Glass”

Gluon Saturation

- Large distances $r \gtrsim 1/Q_s(Y) \quad \longleftrightarrow \quad k_\perp \lesssim Q_s(Y)$
- For $k_\perp \gg Q_s(Y)$, the gluons obey BFKL/DGLAP evolution,
  ... but with saturation boundary condition:
  - $n(Y, k_\perp) \propto 1/k_\perp^2$ for $k_\perp \ggg Q_s(Y)$
  - ... but $n(Y, k_\perp) \approx \left(Q_s^2(Y)/k_\perp^2\right)^{\gamma_s}$ at 'geometric scaling'

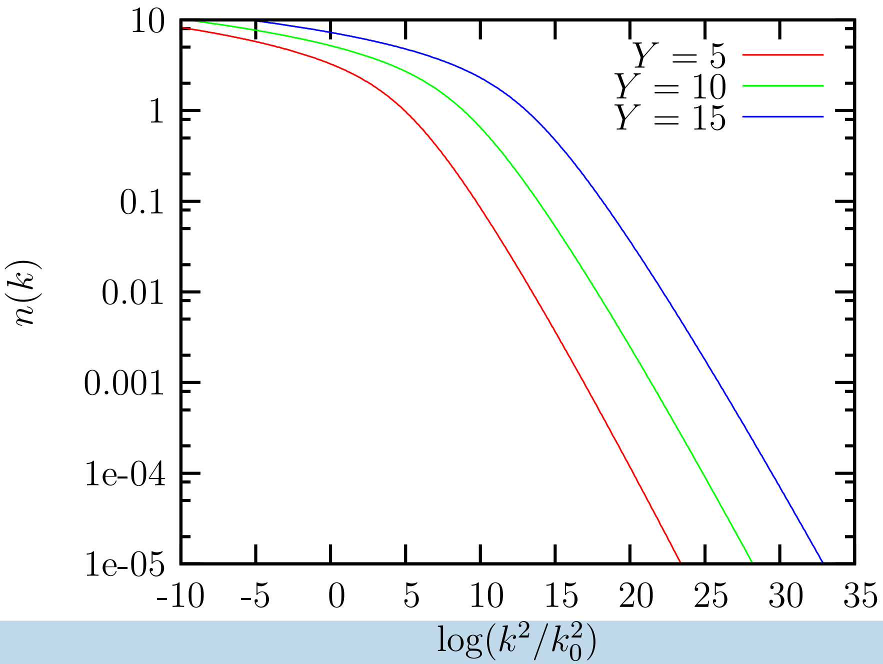


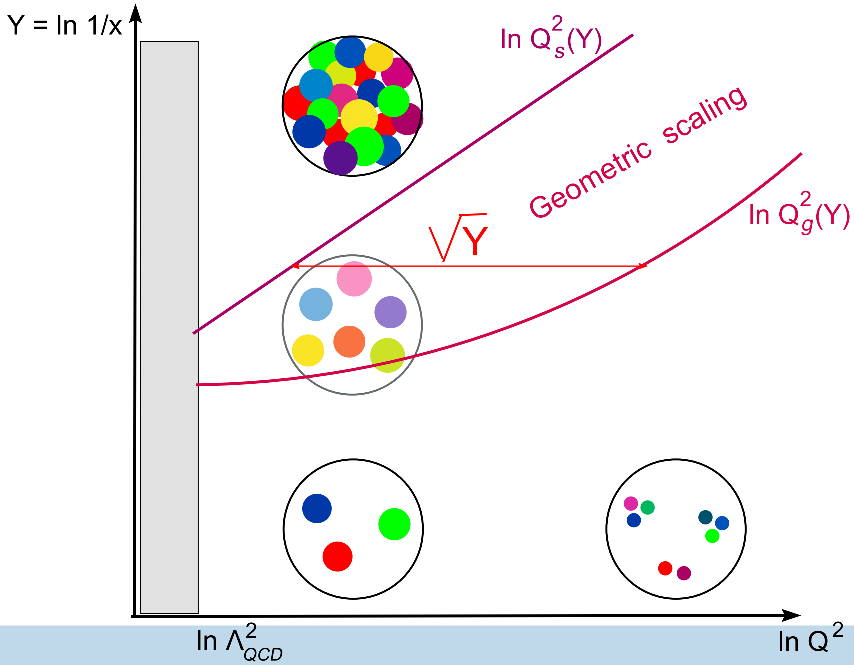


Gluon Saturation

- Large distances $r \gtrsim 1/Q_s(Y) \quad \longleftrightarrow \quad k_\perp \lesssim Q_s(Y)$
- For $k_\perp \lesssim Q_s(Y)$, the gluon occupation number "saturates" :
  - $n(Y, k_\perp)$ rises linearly with Y (rather than exponentially)
  - ... and logarithmically with $1/k_\perp^2$:

$$n(Y, k_\perp) \approx \frac{1}{\alpha_s N_c} \ln \frac{Q_s^2(Y)}{k_\perp^2} \propto Y$$



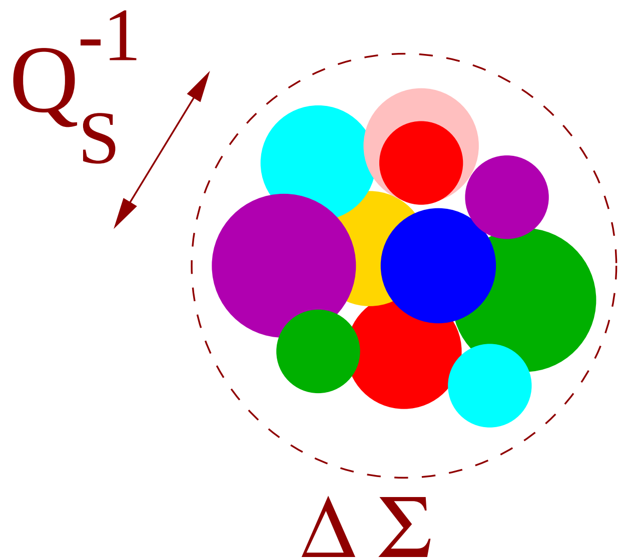


Color neutrality at saturation

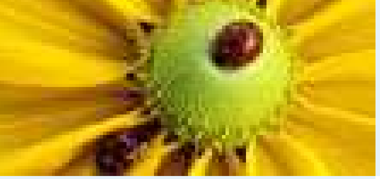
- In a low energy hadron, color is screened by confinement:

  $r \sim \Lambda_{\rm QCD}^{-1} \sim 1 \text{ fm}$

- At high energy, the densely packed gluons screen each other, in such a way that color neutralization occurs already at the perturbative scale $Q_s^{-1} \ll \Lambda_{\rm QCD}^{-1}$



- $Q_s(Y)$ is the “infrared cutoff” at high energy, and is hard !



Evolution of observables

- Recall: Observables are obtained by averaging over ρ, or A :

$$\langle \mathcal{O}[A] \rangle_Y = \int \mathrm{D}[A] \; W_Y[A] \; \mathcal{O}[A]$$

- Examples:
  - The dipole S–matrix: $S(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{N_c} \mathrm{tr}\big(V(\boldsymbol{x})\, V^\dagger(\boldsymbol{y})\big)$
  - The unintegrated gluon distribution: $n(Y, k_\perp) \leftrightarrow E_a^i E_a^i$

- Differentiate w.r.t. Y, use JIMWLK, and integrate by parts:

$$\begin{aligned}
\partial_Y \langle \mathcal{O}[A] \rangle_Y &= \int \mathrm{D}[A] \, (\partial_Y W_Y) \; \mathcal{O}[A] \\
&= \int \mathrm{D}[A] \, \frac{1}{2} \left(\frac{\delta}{\delta A_a} \chi_{ab} \frac{\delta \, W_Y}{\delta A_b} \right) \mathcal{O}[A] \\
&= \int \mathrm{D}[A] \, W_Y[A] \, \frac{1}{2} \, \frac{\delta}{\delta A_a} \chi_{ab} \frac{\delta}{\delta A_b} \; \mathcal{O}[A]
\end{aligned}$$

Evolution of observables

- Recall: Observables are obtained by averaging over ρ, or A :

$$\langle \mathcal{O}[A] \rangle_Y = \int \mathrm{D}[A] \; W_Y[A] \; \mathcal{O}[A]$$

- Examples:
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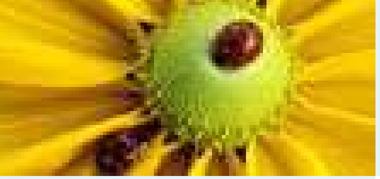
  - The unintegrated gluon distribution: $n(Y,k_\perp) \leftrightarrow E_a^i E_a^i$
- Differentiate w.r.t. Y, use JIMWLK, and integrate by parts:
- ... or, simply,

$$\partial_Y \langle \mathcal{O}[A] \rangle_Y = \langle H_{\mathrm{JIMWLK}}\, \mathcal{O}[A] \rangle_Y$$

- One uses the functional derivatives to act on the observable



Recovering Balitsky equations

- Exercise:
  By using JIMWLK and $S(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{N_c}\mathrm{tr}\big(V(\boldsymbol{x})\,V^\dagger(\boldsymbol{y})\big)$, deduce

$$\frac{\partial}{\partial Y}\langle S(\boldsymbol{x},\boldsymbol{y})\rangle_Y = \frac{\bar{\alpha}_s}{2\pi}\int_{\boldsymbol{z}} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2}\Big\{-\langle S(\boldsymbol{x},\boldsymbol{y})\rangle_Y + \langle S(\boldsymbol{x},\boldsymbol{z})S(\boldsymbol{z},\boldsymbol{y})\rangle_Y\Big\}$$

- The first Balitsky equation !
- The action of the functional derivatives on Wilson lines

$$V(\boldsymbol{x}) \equiv \mathrm{P}\exp\Big(\mathrm{i}g\int_{-\infty}^{\infty}\mathrm{d}x^- A_a^+(x^-,\boldsymbol{x})T^a\Big)$$

$$\frac{\delta}{\delta A_a^+(\boldsymbol{x})}V(\boldsymbol{x}) \equiv \frac{\delta}{\delta A_a^+(x^-\to\infty,\boldsymbol{x})}V(\boldsymbol{x}) = \mathrm{i}gT^a\,V(\boldsymbol{x})$$

- The derivative w.r.t. the field at $x^- = \infty$! (“Lie derivative”)
- The CGC color source/field gets built in layers of x^-

Recovering Balitsky equations

- Exercise:

  By using JIMWLK and $S(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{N_c}\mathrm{tr}\big(V(\boldsymbol{x})\,V^\dagger(\boldsymbol{y})\big)$, deduce

$$\frac{\partial}{\partial Y}\langle S(\boldsymbol{x},\boldsymbol{y})\rangle_Y = \frac{\bar{\alpha}_s}{2\pi}\int_{\boldsymbol{z}} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2}\Big\{-\langle S(\boldsymbol{x},\boldsymbol{y})\rangle_Y + \langle S(\boldsymbol{x},\boldsymbol{z})S(\boldsymbol{z},\boldsymbol{y})\rangle_Y\Big\}$$

- The first Balitsky equation !
- Not a closed equation !

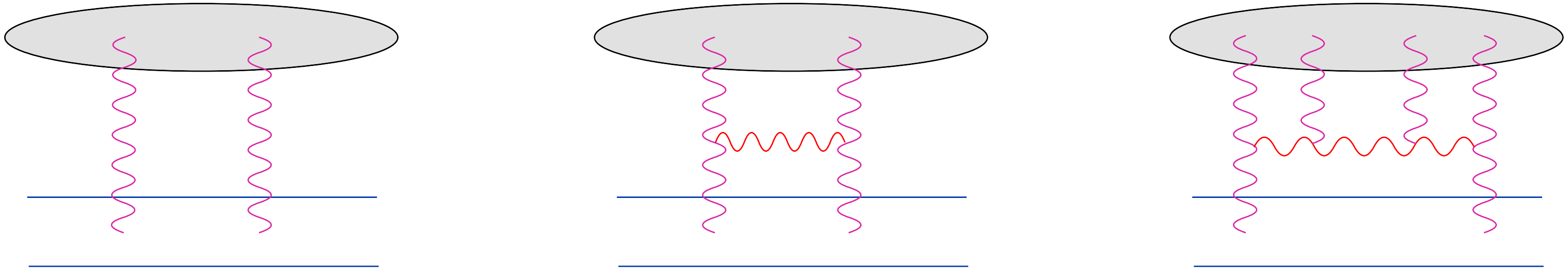
  We also need the equation for the 2–dipole projectile, etc.

$$\frac{\partial}{\partial Y}\langle S(\boldsymbol{x}_1,\boldsymbol{y}_1)\; S(\boldsymbol{x}_2,\boldsymbol{y}_2)\rangle_Y \;=\; \cdots$$

- The first equation in an infinite hierarchy !

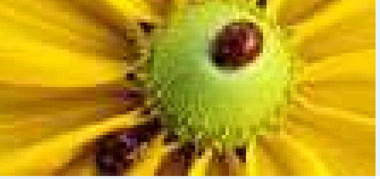
Balitsky hierarchy ... and beyond (1)

- Diagrammatic summary of the equations



$$\frac{\partial \langle T \rangle}{\partial t} \simeq \langle T \rangle - \langle T\,T \rangle$$

- The first Balitsky equation in schematic form (with $t = \bar{\alpha}_s Y$)

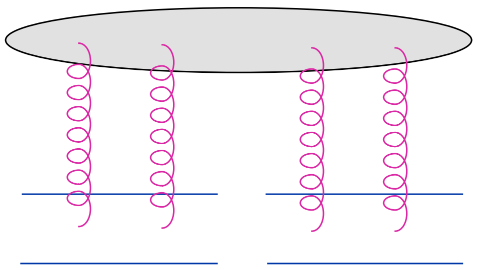


Balitsky hierarchy ... and beyond (2)

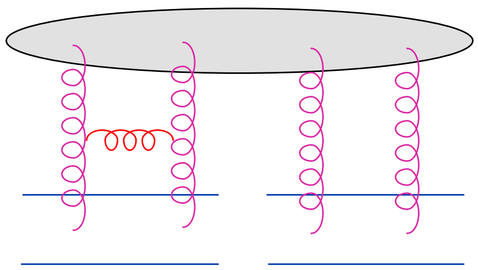
- Diagrammatic summary of the equations (with $t = \bar{\alpha}_s Y$)

$$\frac{\partial \langle T \rangle}{\partial t} \simeq \langle T \rangle - \langle T\,T \rangle$$

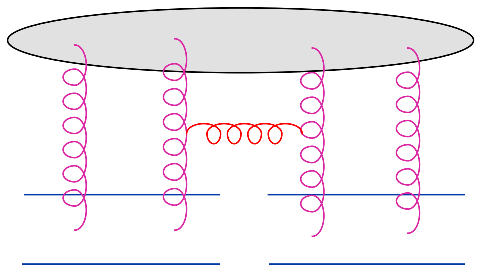
$$\frac{\partial \langle T\,T \rangle}{\partial t} \simeq 2\langle T\,T \rangle - 2\,\langle T\,T\,T \rangle + \mathcal{O}(1/N_c^2)$$



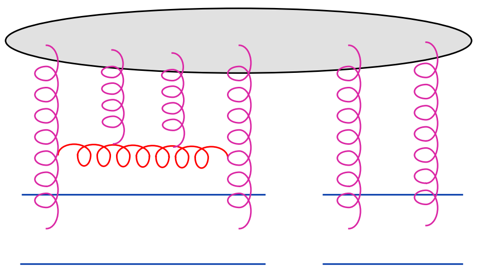
(a)



(b)



(c)



(d)

Balitsky hierarchy ... and beyond (3)

A brief reminder
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- Balitsky eqs.
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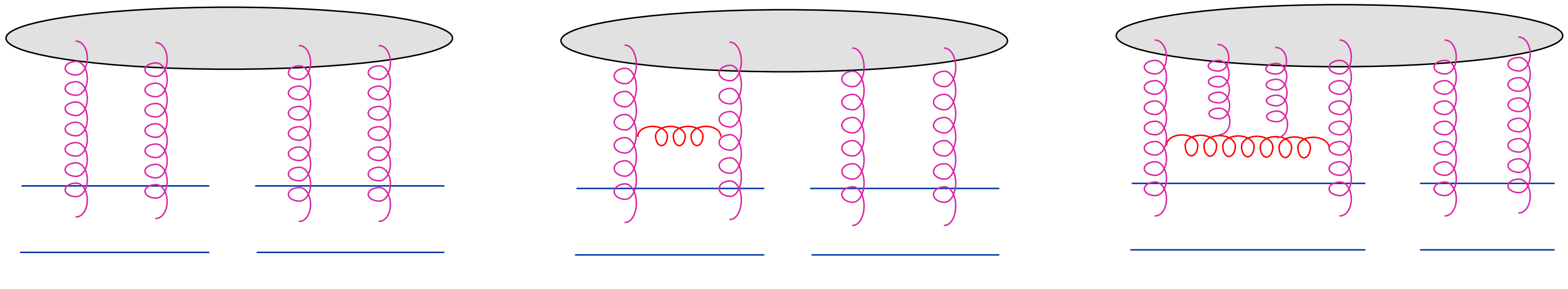
- Ploops
- Front diffusion
- High–energy ‘phase-diagram’

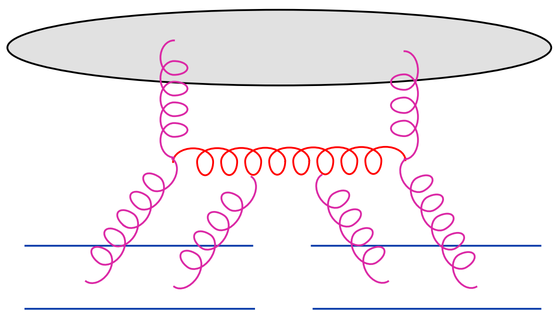
- Diagrammatic summary of the equations (with $t = \bar{\alpha}_s Y$)

$$\frac{\partial \langle T\,T \rangle}{\partial t} \simeq 2\langle T\,T \rangle - 2\,\langle T\,T\,T \rangle + \alpha_s^2 \langle T \rangle + \mathcal{O}(1/N_c^2)$$



(a) (b) (d)

- Additional, ‘fluctuation’, term from the $2 \to 4$ splitting vertex



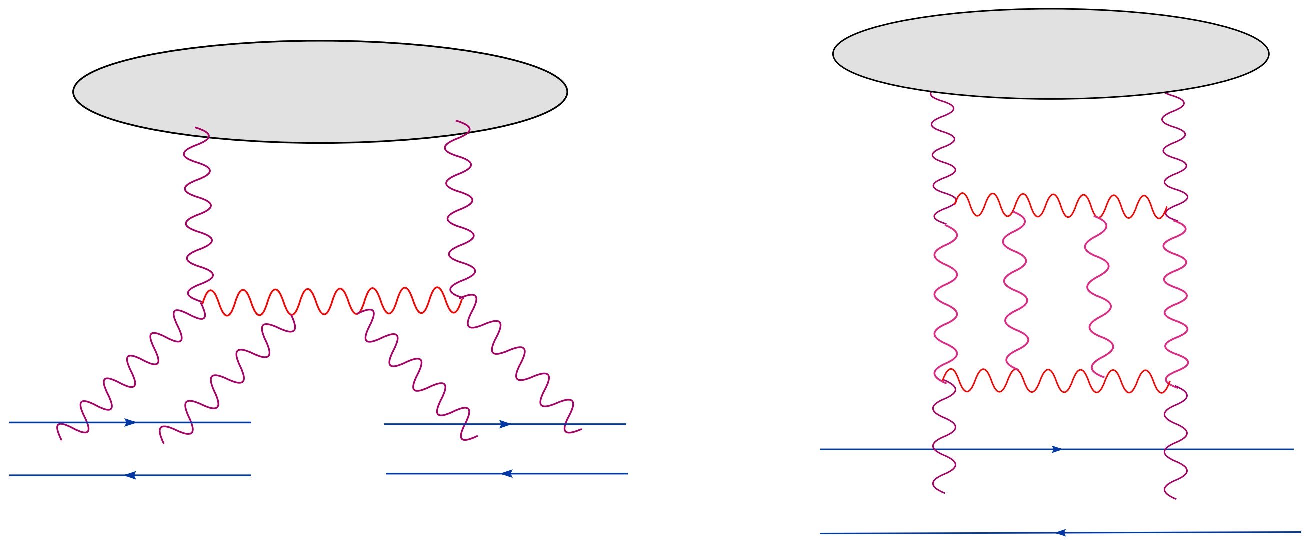
(e)

The Pomeron loop hierarchy

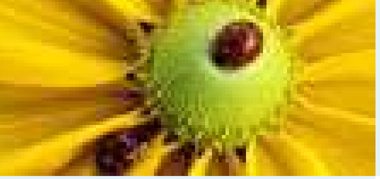
- Schematic structure of the evolution equations (large N_c)

$$\frac{\partial\langle T\rangle}{\partial t} \simeq \langle T\rangle - \langle T\,T\rangle$$

$$\frac{\partial\langle T\,T\rangle}{\partial t} \simeq 2\langle T\,T\rangle - 2\,\langle T\,T\,T\rangle + \alpha_s^2\langle T\rangle$$



- Cartoon versions of the ‘Pomeron loop’ equations in QCD at large N_c *(E.I. and D. Triantafyllopoulos, 04)*



The Pomeron loop hierarchy

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$$\frac{\partial \langle T \rangle}{\partial t} \simeq \langle T \rangle - \langle T\,T \rangle$$

$$\frac{\partial \langle T\,T \rangle}{\partial t} \simeq 2\langle T\,T \rangle - 2\,\langle T\,T\,T \rangle + \alpha_s^2 \langle T \rangle$$

- The fluctuation term dominates when $\langle T \rangle \lesssim \alpha_s^2$ (dilute tail)
- Remember: The evolution is driven by the dynamics in the dilute tail $\Longrightarrow$ strong sensitivity to fluctuations !

  (Mueller, Shoshi; E.I., Mueller, Munier, 2004)

- Infinite hierarchy for $\langle TT \cdots \rangle \Longleftrightarrow$ Single Langevin equation:

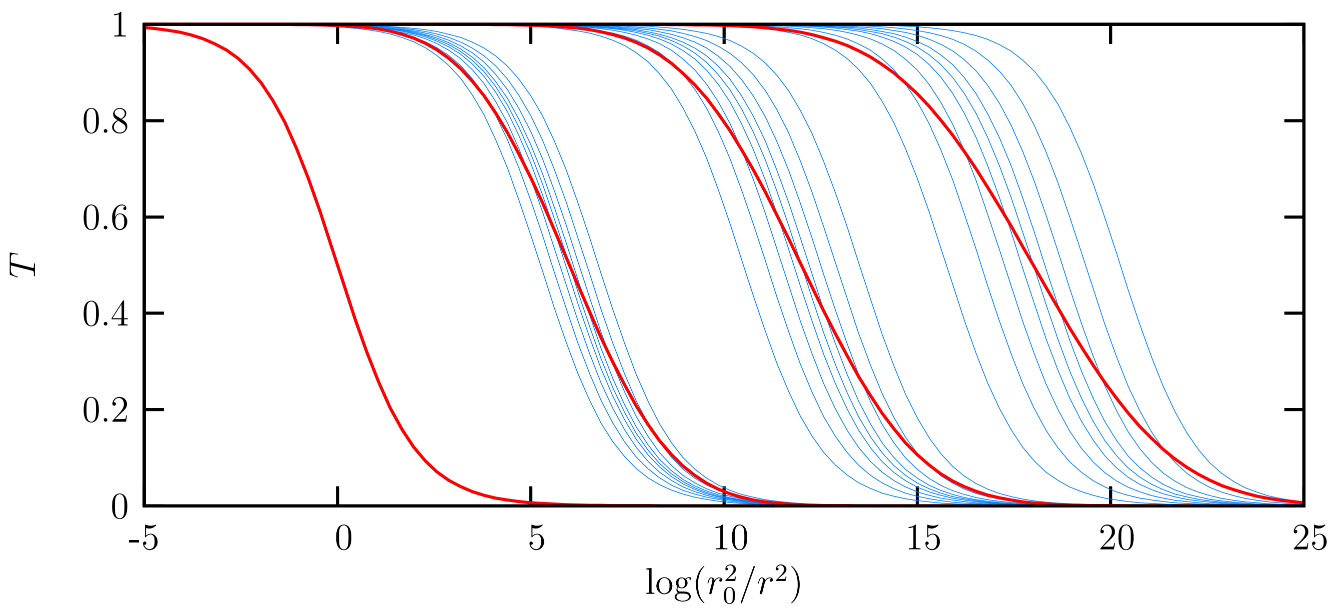
$$\frac{\partial T}{\partial t} = T - T^2 + \sqrt{\alpha_s^2 T}\,\nu\,, \quad \langle \nu(t_1)\nu(t_2) \rangle = \delta(t_1 - t_2)$$

- Stochastic FKPP equation $\Longrightarrow$ reaction–diffusion process

Front diffusion through fluctuations

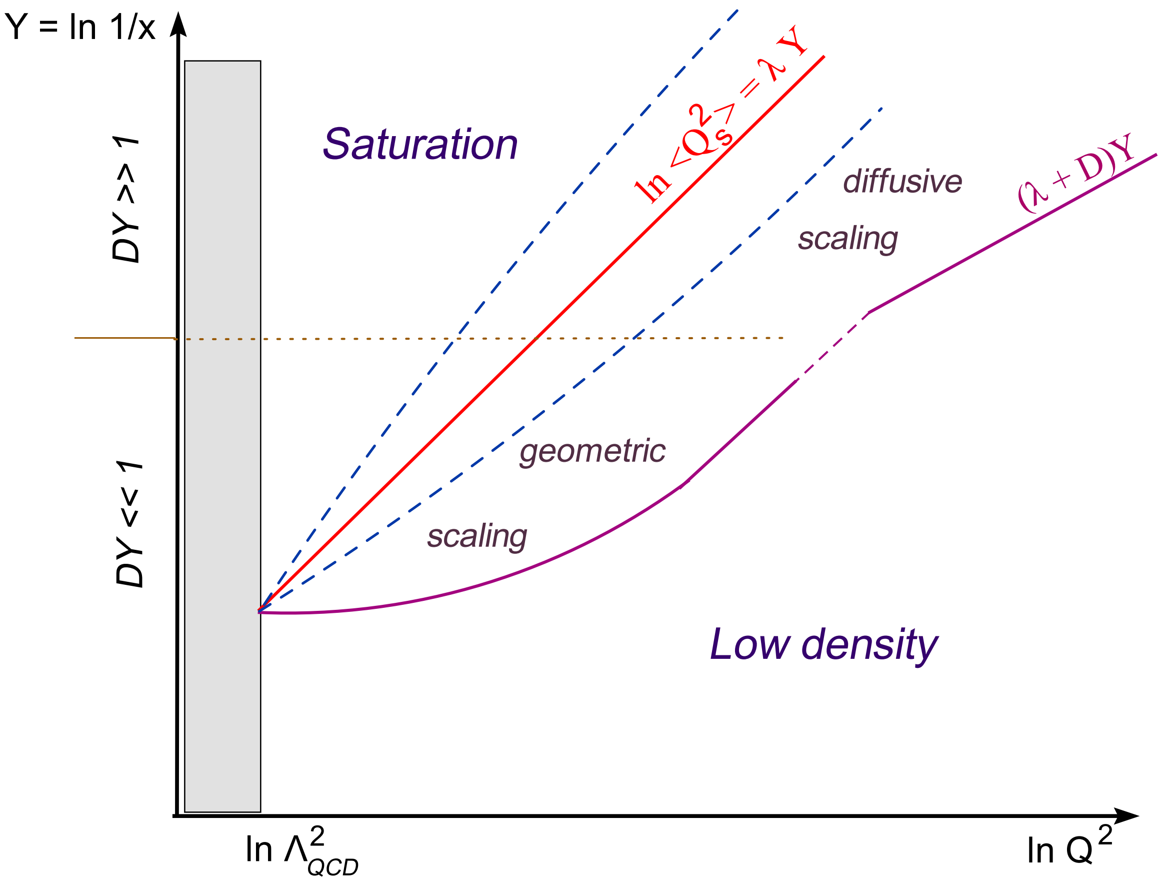
- The stochastic evolution generates un ensemble of fronts which differ by their saturation momentum $\rho_s \equiv \ln Q_s^2$

$$\langle \rho_s(Y) \rangle = \lambda Y, \qquad \langle \rho_s^2 \rangle - \langle \rho_s \rangle^2 = DY, \qquad D \sim \frac{1}{\ln^3(1/\alpha_s)}$$



- With increasing energy, the fronts spread from each other $\Longrightarrow$ geometric scaling is progressively washed out !

The “phase–diagram” revisited



- HERA, RHIC ≈ intermediate energies (most likely)
- LHC ? ... We don’t really know ! *(large theoretical uncertainties)*
- Compare theoretical expectations with the data !