NLO BFKL at work: the electroproduction of two light vector mesons

A. Papa

Università della Calabria & INFN - Cosenza

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Bibliography:

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- D.Yu. Ivanov and A. P., Nucl. Phys. B **732** (2006) 183 [arXiv:hep-ph/0508162]
- D.Yu. Ivanov and A. P., Eur. Phys. J. C **49** (2007) 947 [arXiv:hep-ph/0610042]
- F. Caporale, A. P. and A. Sabio Vera, work in progress.

Outline

1. The $\gamma^*\gamma^* \to V\,V$ forward amplitude
   - Introduction and Motivations
   - Kinematics and BFKL amplitude
2. Part 1 - The $\gamma^* \to V$ impact factor
   - Definition and general structure
   - Impact factor in the LLA
   - Impact factor in the NLA
   - Final result for the NLA impact factor
   - Cancellation of the leading energy scale uncertainty
3. Part 2 - Building up the amplitude
   - The $\gamma^* \to V$ impact factor - Reminder
   - The Green's function
   - Series representation of the amplitude
   - Numerical analysis - The "pure" BFKL regime
   - "Collinear factorization" regime
   - Analysis of systematic effects
   - Collinear improvement of the BFKL kernel
4. Conclusions

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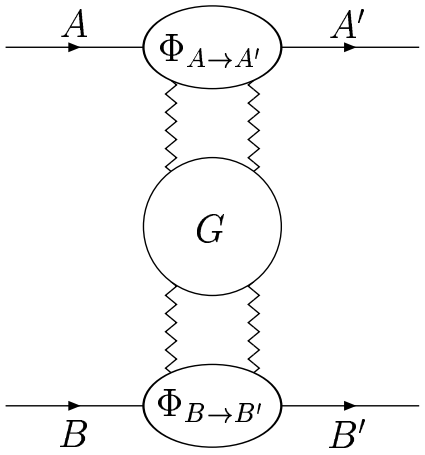
Introduction and Motivations

Scattering $A + B \longrightarrow A' + B'$ in the Regge kinematical region $s \to \infty$, t fixed

- BFKL approach: convolution of the Green's function of two interacting Reggeized gluons and of the impact factors of the colliding particles.



- Valid both in
  LLA (resummation of all terms $(\alpha_s \ln s)^n$)
  NLA (resummation of all terms $\alpha_s(\alpha_s \ln s)^n$).
- The Green's function is determined through the BFKL equation.
  [Ya.Ya. Balitsky, V.S. Fadin, E.A. Kuraev, L.N. Lipatov (1975)]
- The kernel of the BFKL equation is completely known in the NLA for the forward ($t = 0$) case ... [V.S. Fadin, L.N. Lipatov (1998)] [G. Camici, M. Ciafaloni (1998)]
- ... and for the non-forward ($t \neq 0$) case [V.S. Fadin, R. Fiore (2005)]

- Impact factors have been calculated in the NLA for
  colliding partons [V.S. Fadin, R. Fiore, M.I. Kotsky, A.P. (2000)]
  [M. Ciafaloni and G. Rodrigo (2000)]
  forward jet production [J. Bartels, D. Colferai, G.P. Vacca (2003)]

- Colorless NLA impact factors
  - $\gamma^* \to \gamma^*$
    [J. Bartels, D. Colferai, S. Gieseke, A. Kyrieleis (2002)]
    [V.S. Fadin, D.Yu. Ivanov, M.I. Kotsky (2003)]
    [J. Bartels, A. Kyrieleis (2004)]
  - $\gamma^* \to V$, with $V = \rho^0, \omega, \phi$, forward case
    [D.Yu. Ivanov, M.I. Kotsky, A. P. (2004)]

The $\gamma^* \to V$ impact factor can be used to build the (forward) $\gamma^*\gamma^* \to V\,V$ amplitude; this is the first amplitude of a physical process completely calculable within perturbative QCD in the NLA.

(For the Born non-forward case, see [B. Pire, L. Szymanowsky, S. Wallon (2004); B. Pire, M. Segond, L. Szymanowsky, S. Wallon (2006)]; for the LLA case and for an estimated NLA result, see [R. Enberg, B. Pire, L. Szymanowsky, S. Wallon (2005)])

- Theoretical importance:
  - possibility to understand the role and the optimal choice of energy scales in the BFKL approach
  - comparison between different approaches (BFKL *vs.* DGLAP, etc.)
- Phenomenological interest:
  - first step toward the application of the BFKL approach to the description of
    - $\gamma^* p \to Vp$, at HERA
    - $\gamma^*\gamma^* \to V\,V$ or $\gamma^*\gamma \to VJ/\Psi$, at high-energy e^+e^- and $e\gamma$ colliders

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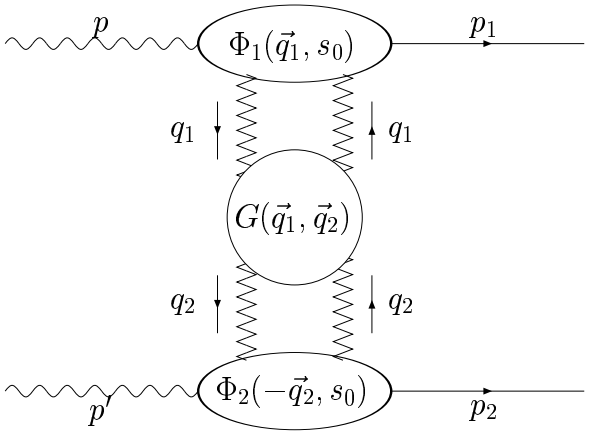
Kinematics and BFKL amplitude

$\gamma^*(p)\gamma^*(p') \to V(p_1)V(p_2)$

$p_1^2 = p_2^2 = 0\ , \qquad 2(p_1 p_2) = s$
(p_1 and p_2 Sudakov vectors)



virtual photons' momenta:

$$p = \alpha p_1 - \frac{Q_1^2}{\alpha s} p_2\ , \qquad p' = \alpha' p_2 - \frac{Q_2^2}{\alpha' s} p_1$$

$$s \gg Q_{1,2}^2 \gg \Lambda_{QCD}^2$$

$$p \simeq p_1 - \frac{Q_1^2}{s} p_2\ , \qquad p' \simeq p_2 - \frac{Q_2^2}{s} p_1$$

- Longitudinally polarized vector mesons are produced by longitudinally polarized photons; other helicity amplitudes power suppressed by $\sim m_\rho / Q_{1,2}$;
  [D.Yu. Ivanov, M.I. Kotsky, A. P. (2004)]
- forward scattering, i.e. zero transverse momenta of the produced mesons

$$\mathcal{I}m_s\,(\mathcal{A}) = \frac{s}{(2\pi)^2} \int \frac{d^2\vec{q}_1}{\vec{q}_1^{\,2}} \Phi_1(\vec{q}_1, s_0) \int \frac{d^2\vec{q}_2}{\vec{q}_2^{\,2}} \Phi_2(-\vec{q}_2, s_0) \int_{\delta - i\infty}^{\delta + i\infty} \frac{d\omega}{2\pi i} \left(\frac{s}{s_0}\right)^{\omega} G_\omega(\vec{q}_1, \vec{q}_2)$$

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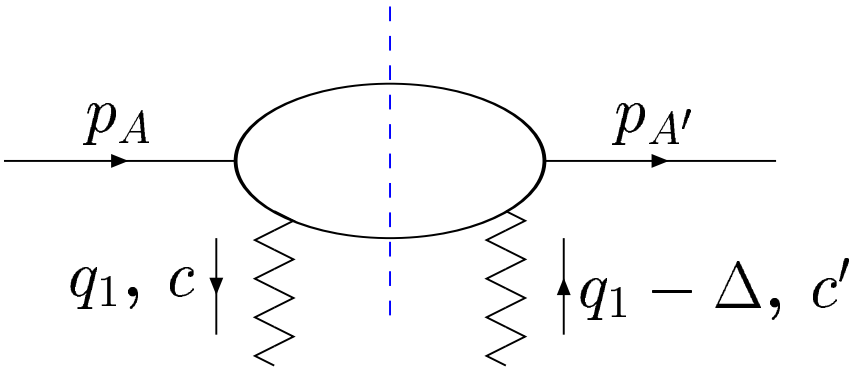
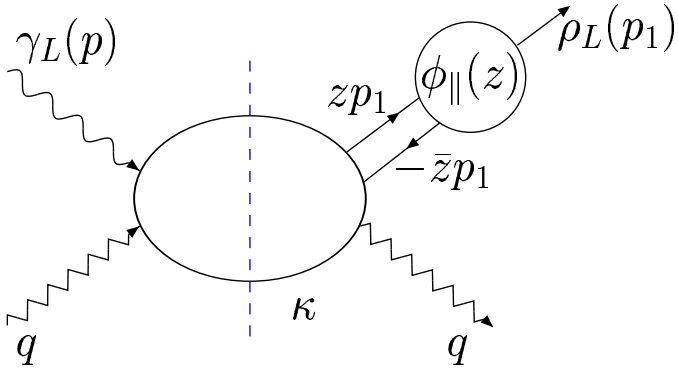
Definition and general structure



$$\Phi_{A\to A'}(\vec{q}_1,\vec{\Delta},s_0)=\frac{\delta^{cc'}}{\sqrt{N_c^2-1}}\left[\left(\frac{s_0}{\vec{q}_1^{\,2}}\right)^{\frac{1}{2}\omega(-\vec{q}_1^{\,2})}\left(\frac{s_0}{(\vec{q}_1-\vec{\Delta})^2}\right)^{\frac{1}{2}\omega(-(\vec{q}_1-\vec{\Delta})^2)}\right.$$
$$\left.\times\sum_{\{f\}}\int\frac{d\kappa}{2\pi}\theta(s_\Lambda-\kappa)d\rho_f\Gamma^c_{A\{f\}}\left(\Gamma^{c'}_{A'\{f\}}\right)^*\right]$$
$$-\frac{1}{2}\int\frac{d^{D-2}k}{\vec{k}^2(\vec{k}-\vec{\Delta})^2}\Phi^{Born}_{A\to A'}(\vec{k},\vec{\Delta},s_0)\mathcal{K}^{Born}_r(\vec{k},\vec{q}_1,\vec{\Delta})\ln\left(\frac{s_\Lambda^2}{s_0(\vec{k}-\vec{q}_1)^2}\right)$$

$\Gamma^c_{A\{f\}}$ - effective vertex for the process A+Reggeon$\to\{f\}$



$$\Phi_{\gamma^* \to V}\ , \qquad V = \rho^0, \omega, \phi \qquad \text{forward case } (\vec{\Delta} = 0)$$

Sudakov vectors: $p_1, p_2, \qquad p_1^2 = p_2^2 = 0, \qquad 2(p_1 p_2) = s$

$$p = p_1 - \frac{Q^2}{s} p_2$$

$$q = \frac{\kappa + Q^2 + \vec{q}^{\,2}}{s} p_2 + q_\perp\ , \quad q^2 = q_\perp^2 = -\vec{q}^{\,2}$$

$$q' = \frac{\kappa + \vec{q}^{\,2}}{s} p_2 + q_\perp\ ,$$

$\vec{q}^{\,2}, Q^2 \gg$ any hadronic scale; neglected power suppressed contributions

In this kinematics the impact factor can be calculated in the collinear factorization framework.

[V.L. Chernyak, A.R. Zhitnitsky (1977, 1980)]
[V.L. Chernyak, V.G. Serbo, A.R. Zhitnitsky (1977, 1980)]
[G.P. Lepage and S.J. Brodsky (1979, 1980)]
[S.J. Brodsky, G.P. Lepage, A.A. Zaidi (1981)]
[A.V. Efremov, A.V. Radyushkin (1980)]

The dominant helicity amplitude is $\gamma_L^* \to V_L$.

Factorization, both in LLA and in NLA, in the convolution (here, $\Phi_{\gamma_L^* \to V_L}$ is the unprojected impact factor)

$$\Phi_{\gamma_L^* \to V_L}(\alpha, s_0) = -\frac{4\pi e_q f_V \delta^{cc'}}{N_c Q} \int_0^1 dz\, T_H(z, \alpha, s_0, \mu_F, \mu_R)\, \phi_{||}(z, \mu_F)$$

$$f_V \simeq 200\,\mathrm{MeV}\,, \qquad \alpha = \frac{\vec{q}^{\,2}}{Q^2}$$

T_H hard-scattering amplitude, perturbative series in $\alpha_S(\mu_R)$, $\mu_R^2 \sim Q^2, \vec{q}^{\,2}$

$\phi_{\|}(z)$ meson twist-2 distribution amplitude

$$\langle 0|\bar{\Psi}(0)\gamma^{\mu}\Psi(y)|V_L(p_1)\rangle_{y^2\to 0} = f_V\, p_1^{\mu} \int_0^1 dz\, e^{-iz(p_1 y)}\, \phi_{\|}(z,\mu_F)$$

(path-ordered gauge factor between quark operators implied)

The distribution amplitude has the meaning of probability amplitude to find a meson in a state with minimal number of constituents – quark and antiquark – when they are separated by small transverse distances, $r_{\perp} \sim 1/\mu_F$

$\mu_F^2 \sim Q^2, \vec{q}^{\,2}$ factorization scale

$z, \bar{z} \equiv 1 - z$ longitudinal momentum fractions carried by quark and antiquark

	$\lvert\rho^0\rangle = \frac{1}{\sqrt{2}}\left(\lvert\bar{u}u\rangle - \lvert\bar{d}d\rangle\right)$	$\lvert\omega\rangle = \frac{1}{\sqrt{2}}\left(\lvert\bar{u}u\rangle + \lvert\bar{d}d\rangle\right)$	$\lvert\phi\rangle = \lvert\bar{s}s\rangle$
$e_q \longrightarrow$	$\frac{e}{\sqrt{2}}$	$\frac{e}{3\sqrt{2}}$	$-\frac{e}{3}$

The distribution amplitude defined by

$$\langle 0|\bar{\Psi}(0)\gamma^\mu\Psi(y)|V_L(p_1)\rangle_{y^2\to 0} = f_V\, p_1^\mu \int\limits_0^1 dz\, e^{-iz(p_1 y)}\, \phi_{||}(z,\mu_F)$$

is a non-perturbative function, but its dependence on μ_F is perturbative

$$\mu_F^2 \frac{d\phi_{||}(z,\mu_F)}{d\mu_F^2} = \int\limits_0^1 V(z,z')\phi_{||}(z',\mu_F)dz' \ .$$

Local limit $y \to 0$, $\bar{\Psi}\gamma^\mu\Psi$ conserved current $\longrightarrow \int\limits_0^1 dz\, \phi_{||}(z,\mu_F)$ renorm-invariant;

$\int\limits_0^1 dz\, \phi_{||}(z,\mu_F) = 1$ is our convention for f_V.

Evolution kernel known up to the second order:

$$V(z,z') = \frac{\alpha_s(\mu_F)}{2\pi} V^{(1)}(z,z') + \left(\frac{\alpha_s(\mu_F)}{2\pi}\right)^2 V^{(2)}(z,z') + \ldots$$

$$V^{(1)}(z,z') = C_F \left[\frac{1-z}{1-z'}\left(1+\frac{1}{z-z'}\right)\theta(z-z') + \frac{z}{z'}\left(1+\frac{1}{z'-z}\right)\theta(z'-z)\right]_+$$

$$[f(z,z')]_+ \equiv f(z,z') - \delta(z-z')\int_0^1 dt\, f(t,z)$$

Cancellation of divergences:
T_H calculated for bare α_S and meson distribution amplitude $\phi_{\|}^{(0)}(z)$
$\longrightarrow$ UV and IR divergences, common parameter ε

UV divergences disappear after coupling constant renormalization ($\overline{\mathrm{MS}}$)

$$\alpha_s = \alpha_s(\mu_R)\left[1 + \frac{\alpha_s(\mu_R)}{4\pi}\beta_0\left(\frac{1}{\hat{\varepsilon}} + \ln\left(\frac{\mu_R^2}{\mu^2}\right)\right)\right]$$

$$\beta_0 = \frac{11\,N_c}{3} - \frac{2n_f}{3}\,, \qquad \frac{1}{\hat{\varepsilon}} = \frac{1}{\varepsilon} + \gamma_E - \ln(4\pi)$$

IR divergences
soft: cancel in the sum of “virtual” and “real” parts of radiative corrections
collinear: absorbed into the definition of the distribution amplitude ($\overline{\mathrm{MS}}$)

$$\phi_{\|}^{(0)}(z) \to \phi_{\|}(z,\mu_F) - \frac{\alpha_s(\mu_F)}{2\pi}\left(\frac{1}{\hat{\varepsilon}} + \ln\left(\frac{\mu_F^2}{\mu^2}\right)\right)\int_0^1 V^{(1)}(z,z')\phi_{\|}(z',\mu_F)dz'$$

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Impact factor in the LLA

$$\Phi_{\gamma^* \to V} = -i \int \frac{d\kappa}{2\pi} \, Disc_\kappa \, \mathcal{A}_{\gamma^* R \to V R'} \, ,$$

$$(2\pi)^4 \delta^4(p + q - p_1 - q') \mathcal{A}_{\gamma^* R \to V R'} = \int d^4x \, e^{-i(px)} e_\mu \langle V(p_1) R' | J^\mu_{\rm em}(x) | R \rangle$$

$$J^\mu_{\rm em} = -e_q \bar{\Psi} \gamma^\mu \Psi$$

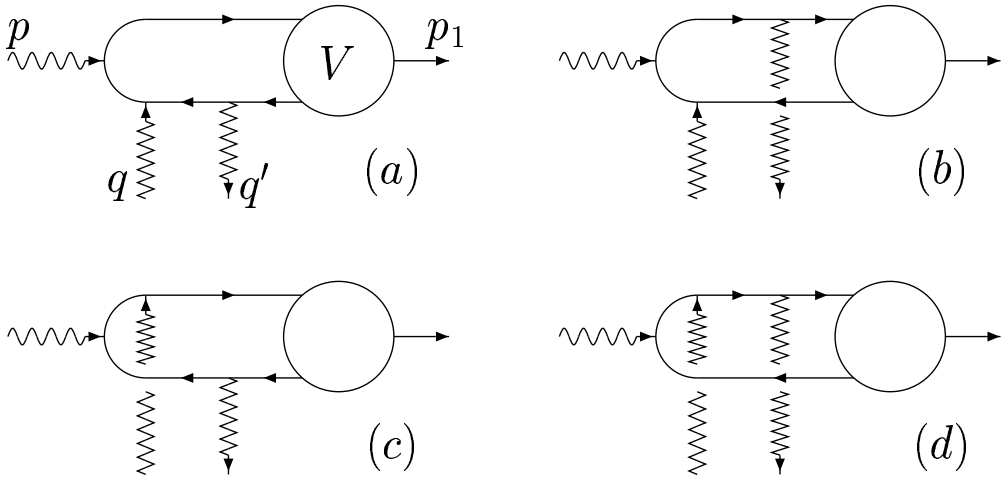

Diagram (a):

$$(2\pi)^4 \delta^4(p + q - p_1 - q') \mathcal{A}^{(a)}_{\gamma^* R \to V R'} = -e_q (ig)^2 \int d^4x d^4y d^4y_1 e^{-i(px) - i(qy_1) + i(q'y)}$$

$$\times \left(t^c t^{c'} \right)_{ij} e_\mu \langle V(p_1) | T \left[\overline{\Psi}^i(x) \gamma^\mu \, \overbracket{\Psi(x) \overline{\Psi}(y_1)} \, \frac{\not{p}_2}{s} \, \overbracket{\Psi(y_1) \overline{\Psi}(y)} \, \frac{\not{p}_2}{s} \Psi^j(y) \right] | 0 \rangle$$

Translational invariance, color neutrality of the meson, Fierz identity

$$\longrightarrow \langle V(p_1)|\overline{\Psi}^i_\alpha(x)\Psi^j_\beta(y)|0\rangle = e^{i(p_1 y)}\langle V(p_1)|\overline{\Psi}^i_\alpha(x-y)\Psi^j_\beta(0)|0\rangle$$
$$= e^{i(p_1 y)}\frac{\delta^{ij}}{N_c}\frac{1}{4}\left\{(\gamma^\mu)_{\beta\alpha}\langle V(p_1)|\overline{\Psi}(x-y)\gamma_\mu\Psi(0)|0\rangle + \ldots\right\}$$

Other Fierz structures:

- even number of γ's $\longrightarrow$ chiral-odd quark-pair $\longrightarrow$ no contribution (chirality conserved in massless limit)
- $\gamma_\mu\gamma_5$ $\longrightarrow$ twist-3 (important for transverse VM production, suppressed by m_V/Q)
  [P. Ball, V.M. Braun, Y. Koike and K. Tanaka (1998)]

$$\mathcal{A}^{(a)}_{\gamma^* R\to V_L R'} = -\frac{e_q g^2\delta^{cc'}}{8N_c}\int\frac{d^4l_1 d^4l_2 d^4(x-y)}{(2\pi)^4}\delta^4(l_1-l_2-q)e^{-i((p+l_1)(x-y))}$$
$$\times \quad \mathrm{Sp}\left[\not{e}\frac{\not{l}_1}{l_1^2+i\epsilon}\frac{\not{p}_2}{s}\frac{\not{l}_2}{l_2^2+i\epsilon}\frac{\not{p}_2}{s}\gamma^\mu\right]\langle V_L(p_1)|\overline{\Psi}(x-y)\gamma_\mu\Psi(0)|0\rangle\,.$$

Leading power asymptotics from the region $(x-y)^2 \to 0$, where

$$\langle V_L(p_1)|\overline{\Psi}(y)\gamma^\mu\Psi(0)|0\rangle_{y^2\to 0} = f_V p_1^\mu \int\limits_0^1 dz\, e^{iz(p_1 y)}\phi_{||}(z)$$

$$\begin{aligned}\longrightarrow \mathcal{A}^{(a)}_{\gamma^* R\to V_L R'} &= -\frac{e_q g^2 f_V \delta^{cc'}}{8N_c}\int\limits_0^1 dz\phi_{||}(z)\int d^4 l_1 \delta^4(p_1 z - p - l_1)\\ &\times \quad \mathrm{Sp}\left[\not{e}\frac{\not{l}_1}{l_1^2+i\epsilon}\frac{\not{p}_2}{s}\frac{\not{l}_1-\not{q}}{(l_1-q)^2+i\epsilon}\frac{\not{p}_2}{s}\not{p}_1\right].\end{aligned}$$

The hard-scattering amplitude is calculated for on-shell quark and antiquark momenta, $p_1 z$ and $p_1\bar{z}$.

The κ-channel discontinuity comes only from the antiquark cut

$$\frac{i}{2\pi}Disc_\kappa\frac{1}{(l_1-q)^2+i\epsilon} = \delta((l_1-q)^2) = \delta((p_1 z - p - q)^2) = \delta(\bar{z}\kappa - z\vec{q}^{\,2})$$

$$\Phi^{(a)}_{\gamma_L^*\to V_L} = -\frac{e_q g^2 f_V \delta^{cc'}(ep_1)}{2N_c Q^2}\int\limits_0^1 dz\,\phi_{||}(z)$$

only longitudinal photon polarization $\qquad e_L = \frac{1}{Q}p_1 + \frac{Q}{s}p_2 = \frac{1}{Q}p + 2\frac{Q}{s}p_2 \to 2\frac{Q}{s}p_2$

Taking into account also diagrams (b), (c), (d), one gets

$$T_H^{(0)}(z, \alpha, s_0, \mu_F, \mu_R) = \alpha_S \frac{\alpha}{\alpha + z\bar{z}} \qquad \alpha = \frac{\vec{q}^{\,2}}{Q^2}$$

By collinear factorization, some fermion lines are effectively put on mass-shell; in practice

$$\text{two-particle states} \longrightarrow \text{one-particle state}$$
$$\text{three-particle states} \longrightarrow \text{two-particle states}$$

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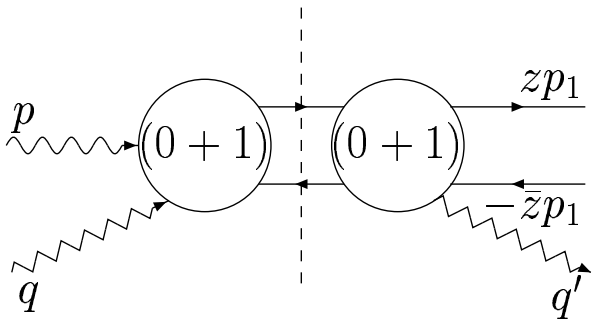
Reminder

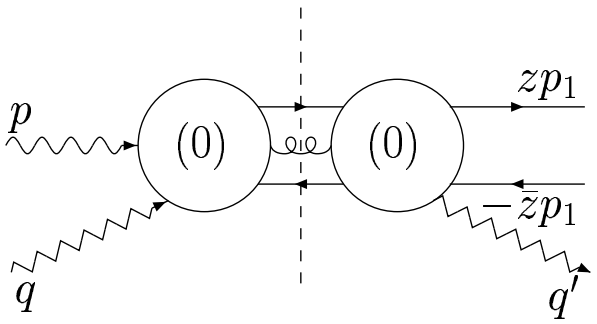
$$\Phi_{A\to A'}(\vec{q}_1,\vec{\Delta},s_0)=\frac{\delta^{cc'}}{\sqrt{N_c^2-1}}\left[\left(\frac{s_0}{\vec{q}_1^{\,2}}\right)^{\frac{1}{2}\omega(-\vec{q}_1^{\,2})}\left(\frac{s_0}{(\vec{q}_1-\vec{\Delta})^2}\right)^{\frac{1}{2}\omega(-(\vec{q}_1-\vec{\Delta})^2)}\right.$$

$$\left.\times\sum_{\{f\}}\int\frac{d\kappa}{2\pi}\theta(s_\Lambda-\kappa)d\rho_f\Gamma^c_{A\{f\}}\left(\Gamma^{c'}_{A'\{f\}}\right)^*\right]$$

$$-\frac{1}{2}\int\frac{d^{D-2}k}{\vec{k}^2(\vec{k}-\vec{\Delta})^2}\Phi^{Born}_{A\to A'}(\vec{k},\vec{\Delta},s_0)\mathcal{K}^{Born}_r(\vec{k},\vec{q}_1,\vec{\Delta})\ln\left(\frac{s_\Lambda^2}{s_0(\vec{k}-\vec{q}_1)^2}\right)$$



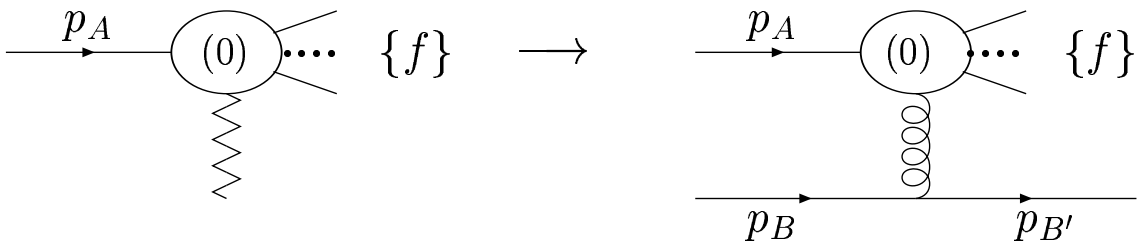
$q\bar{q}$ **intermediate state**



$q\bar{q}g$ **intermediate state**

$$T_H^{(1)}=T^{(q\bar{q})}+T^{(q\bar{q}g)}$$

How to calculate an effective vertex?



$$p_A = p_1 + \frac{m_A^2}{s} p_2 \,, \qquad p_B = p_2 \,, \qquad g^{\mu\nu} = \frac{2p_2^\mu p_1^\nu}{s} + \frac{2p_1^\mu p_2^\nu}{s} + g_\perp^{\mu\nu} \to \frac{2p_2^\mu p_1^\nu}{s}$$

Goal: $\Gamma^c_{A\{f\}}$

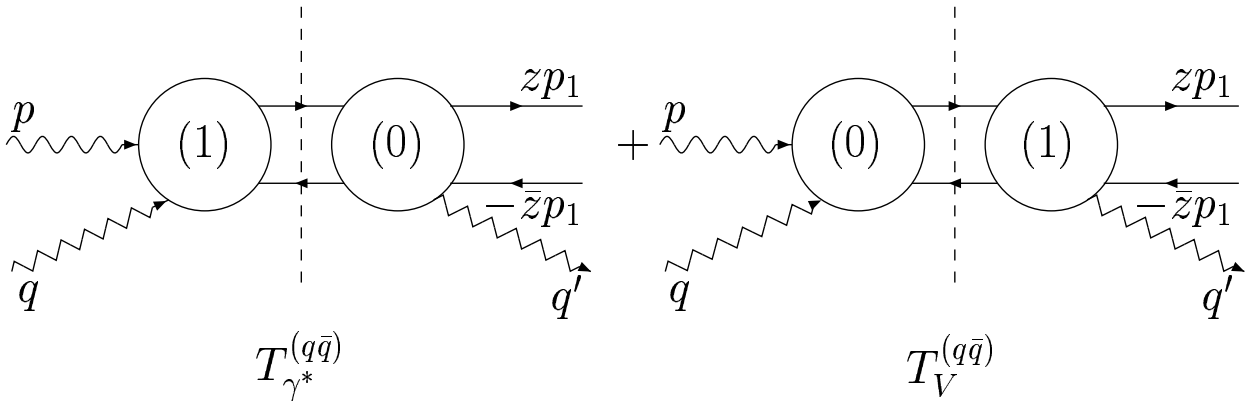
- calculate the high energy amplitude $AB \to \{f\}B'$, where B, B' are quarks, with octet color state and negative signature in the t-channel
- compare with the Regge form

$$\mathcal{A}^{(8-)}_{AB\to\{f\}B'} = \Gamma^c_{A\{f\}} \frac{s}{t} \left[\left(\frac{s}{-t} \right)^{\omega(t)} + \left(\frac{-s}{-t} \right)^{\omega(t)} \right] \Gamma^c_{BB'}$$

Born: $$\mathcal{A}^{(0)}_{AB\to\{f\}B'} = \Gamma^{c(0)}_{A\{f\}} \frac{2s}{t} \Gamma^{c(0)}_{BB'} \,, \qquad \Gamma^{c(0)}_{BB'} = g t^c_{BB'} \bar{u}_{B'} \frac{\not{p}_1}{s} u_B$$

1-loop: $$\mathcal{A}^{(1)}_{AB\to\{f\}B'} = \Gamma^{c(0)}_{A\{f\}} \frac{2s}{t} \Gamma^{c(0)}_{BB'} + \Gamma^{c(0)}_{A\{f\}} \frac{s}{t} \omega^{(1)}(t) \left[\ln\left(\frac{s}{-t} \right) + \ln\left(\frac{-s}{-t} \right) \right] \Gamma^{c(0)}_{BB'}$$
$$+ \Gamma^{c(0)}_{A\{f\}} \frac{2s}{t} \Gamma^{c(1)}_{BB'} + \Gamma^{c(1)}_{A\{f\}} \frac{2s}{t} \Gamma^{c(0)}_{BB'}$$

Quark-antiquark intermediate state



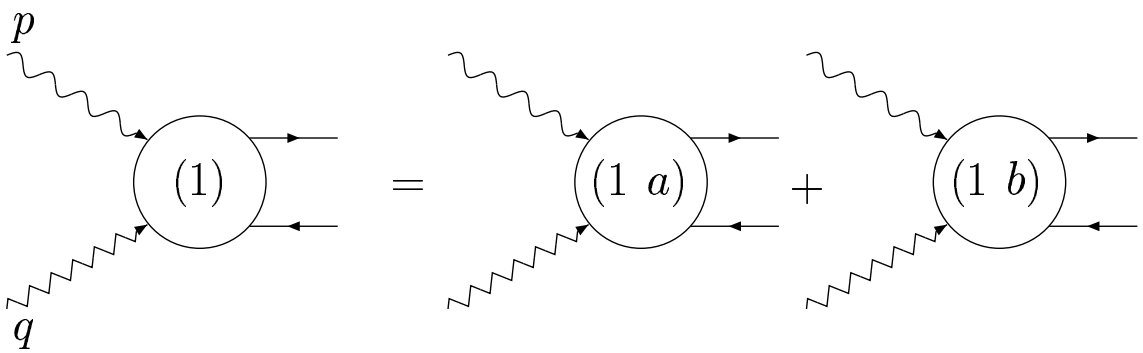
We need four “effective vertices”

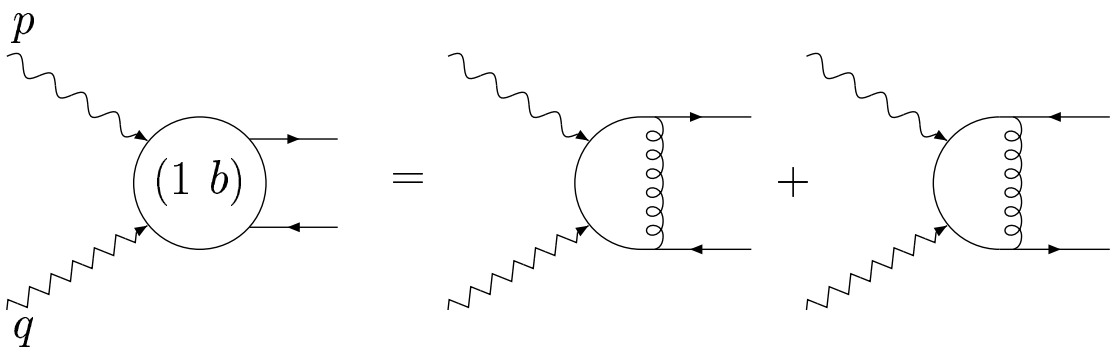
- vertex 1: photon to quark-antiquark (NLO)
- vertex 2: quark-antiquark to quark-antiquark (LO)
- vertex 3: photon to quark-antiquark (LO)
- vertex 4: quark-antiquark to quark-antiquark (NLO)

to be convoluted,

- vertex 1 $\otimes$ vertex 2
- vertex 3 $\otimes$ vertex 4

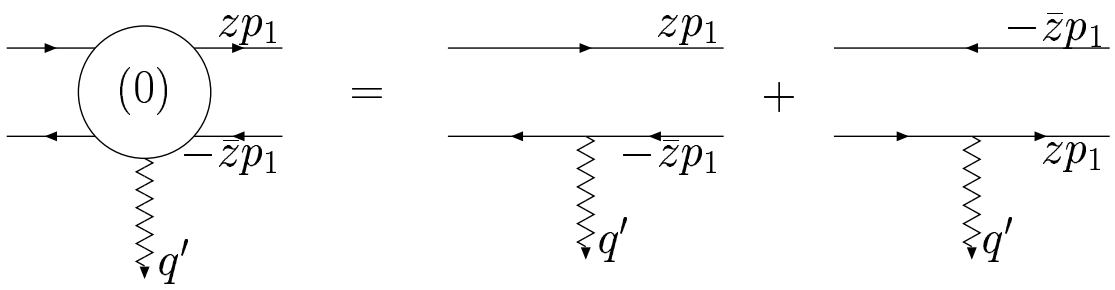
vertex 1 - photon to quark-antiquark (NLO)





[V.S. Fadin, D.Yu. Ivanov, M.I. Kotsky (2002)]

vertex 2 - quark-antiquark to quark-antiquark (LO)



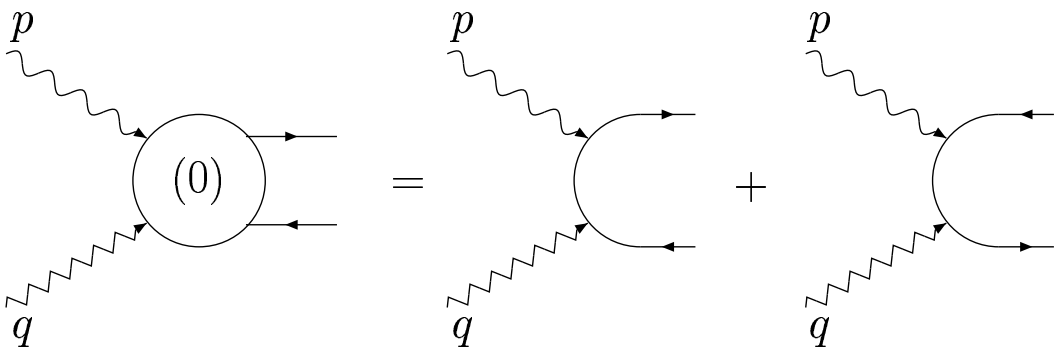
$$\left(\Gamma^{(0)}_{V_L q\bar{q}}\right)^* = \frac{g f_V}{4N_c}\,(t^{c'})_{ji}\,\phi_{\parallel}(z)\,dz\left[\bar{v}^j_\beta(q'+p_1\bar{z})\left(\frac{\not{p}_2}{s}\;\not{p}_1\right)_{\beta\alpha} + \left(\not{p}_1\frac{\not{p}_2}{s}\right)_{\beta\alpha} u^i_\alpha(p_1 z + q')\right]$$

vertex 1 $\otimes$ vertex 2

$$T^{(q\bar{q})}_{\gamma^*} = T^{(1a)}_{\gamma^*} + T^{(1b)}_{\gamma^*}$$

$T^{(1b)}_{\gamma^*}$ = “box diagrams” $\qquad$ $T^{(1a)}_{\gamma^*}$ = “other contributions”

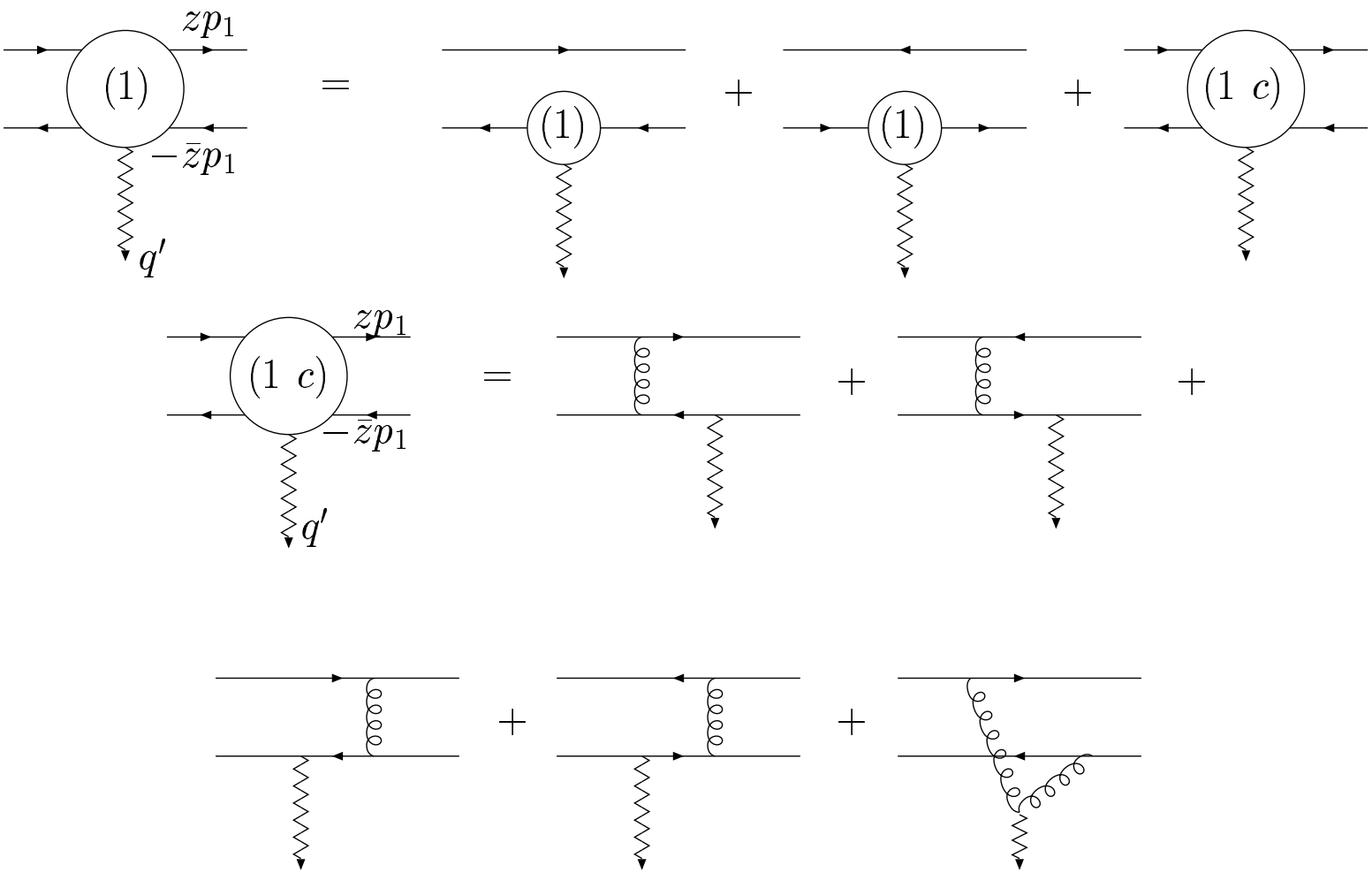
vertex 3 - photon to quark-antiquark (LO)



$$\Gamma^{(0)}_{\gamma_L^* q\bar{q}} = 2e_q g Q (t^c)_{ij} z\bar{z} \left(\frac{1}{\vec{q}_1^{\,2} + Q^2 z\bar{z}} - \frac{1}{\vec{q}_2^{\,2} + Q^2 z\bar{z}} \right) \bar{u}^i_\alpha(q_1) \left(\frac{\not{p}_2}{s} \right)_{\alpha\beta} v^j_\beta(q_2)$$

[I.F. Ginzburg, D.Yu. Ivanov (1996)]
[V.S. Fadin, D.Yu. Ivanov, M.I. Kotsky (2002)]

vertex 4 - quark-antiquark to quark-antiquark (NLO)



vertex 3 $\otimes$ vertex 4

$$T_V^{(q\bar{q})} = T_V^{(1\bar{q})} + T_V^{(1q)} + T_V^{(1c)}$$

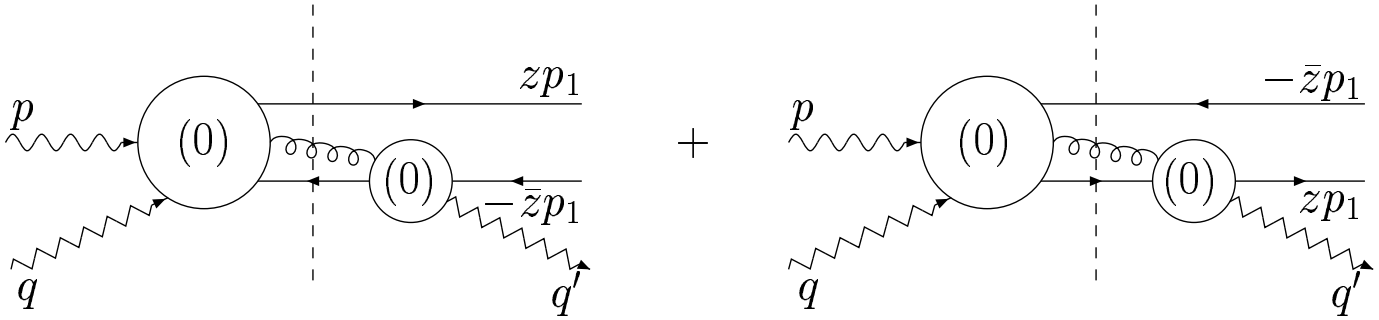
Putting together all the contributions from the quark-antiquark intermediate state

$$T^{(q\bar{q})} = T_1 + T_2 + T_3$$

$$T_1 = T_{\gamma^*}^{(1a)} \qquad T_2 = T_V^{(1c)} + T_{\gamma^*}^{(1b)} \qquad T_3 = T_V^{(1\bar{q})} + T_V^{(1q)}$$

The imaginary parts in $T_V^{(1c)}$ and $T_{\gamma^*}^{(1b)}$ cancel after the sum.

Quark-antiquark-gluon intermediate state



$$\Gamma_{V_L q\bar{q}g} = \Gamma^{q}_{V_L q\bar{q}g} + \Gamma^{\bar{q}}_{V_L q\bar{q}g}$$

We need three “effective vertices”

- vertex 1: photon to quark-antiquark-gluon (LO)
- vertex 2: antiquark-gluon to antiquark (LO)
- vertex 3: quark-gluon to quark (LO)

to be convoluted,

- vertex 1 $\otimes$ vertex 2
- vertex 1 $\otimes$ vertex 3

vertex 1 - photon to quark-antiquark-gluon (LO)

$$\Gamma_{\gamma_L^* q\bar{q}g} = -2\, e_q\, g^2\, Q\, \bar{u}^i_\alpha(zp_1) \left(\left[2\left(\vec{e}^{\,a} (\vec{T}_q \bar{z} - \vec{T}_{\bar{q}} z) \right) + z_3 \left(\not{T}_q + \not{T}_{\bar{q}} \right) \not{e}^a_\perp \right] \frac{\not{p}_2}{s} \right)_{\alpha\beta} v^j_\beta(q_2)\,,$$

$$\vec{T}_q = \left(\frac{\bar{z}\vec{q}_2 - z_2\vec{q}}{(\bar{z}\vec{q}_2 - z_2\vec{q})^2} \left(\frac{1}{Q^2} - \frac{1}{Q^2 + \frac{\vec{q}^{\,2}}{z} + \frac{\vec{q}_2^{\,2}}{z_2} + \frac{\vec{k}^2}{z_3}} \right) - \frac{\vec{q}_2}{Q^2 + \frac{\vec{q}_2^{\,2}\bar{z}}{z_2 z_3}} \frac{1}{Q^2 z_2 z_3} \right) (t^c t^a)_{ij}$$

$$+ \frac{1}{Q^2 z_2 z_3} \left(\frac{\vec{k}}{Q^2 + \frac{\vec{k}^2\bar{z}}{z_2 z_3}} + \frac{\vec{q}_2}{Q^2 + \frac{\vec{q}_2^{\,2}\bar{z}}{z_2 z_3}} \right) (t^a t^c)_{ij}\,,$$

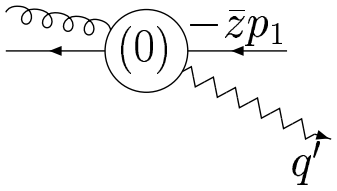
$$\vec{T}_{\bar{q}} = (t^c t^a)_{ij} \left(\frac{z\vec{q}_2 - \bar{z}_2\vec{q}}{(z\vec{q}_2 - \bar{z}_2\vec{q})^2} \left(\frac{1}{Q^2 + \frac{\vec{q}_2^{\,2}}{z_2\bar{z}_2}} - \frac{1}{Q^2 + \frac{\vec{q}^{\,2}}{z} + \frac{\vec{q}_2^{\,2}}{z_2} + \frac{\vec{k}^2}{z_3}} \right) + \frac{\vec{q}_2}{Q^2 + \frac{\vec{q}_2^{\,2}}{z_2\bar{z}_2}} \frac{1}{Q^2 z z_2 \bar{z}_2} \right.$$

$$\left. - \frac{\vec{q}_2}{Q^2 + \frac{\vec{q}_2^{\,2}\bar{z}}{z_2 z_3}} \frac{\bar{z}}{Q^2 z z_2 z_3} \right) + \frac{(t^a t^c)_{ij}}{z} \left(\frac{\vec{k}}{\vec{k}^2} \left(\frac{1}{Q^2 + \frac{\vec{q}_2^{\,2}}{z_2\bar{z}_2}} - \frac{1}{Q^2} \right) \right.$$

$$\left. + \frac{\bar{z}}{Q^2 z_2 z_3} \left(\frac{\vec{k}}{Q^2 + \frac{\vec{k}^2\bar{z}}{z_2 z_3}} + \frac{\vec{q}_2}{Q^2 + \frac{\vec{q}_2^{\,2}\bar{z}}{z_2 z_3}} \right) - \frac{\vec{q}_2}{Q^2 + \frac{\vec{q}_2^{\,2}}{z_2\bar{z}_2}} \frac{1}{Q^2 z_2 \bar{z}_2} \right)\,,$$

[V.S. Fadin, D.Yu. Ivanov, M.I. Kotsky (2002)]

vertex 2 - antiquark-gluon to antiquark (LO)



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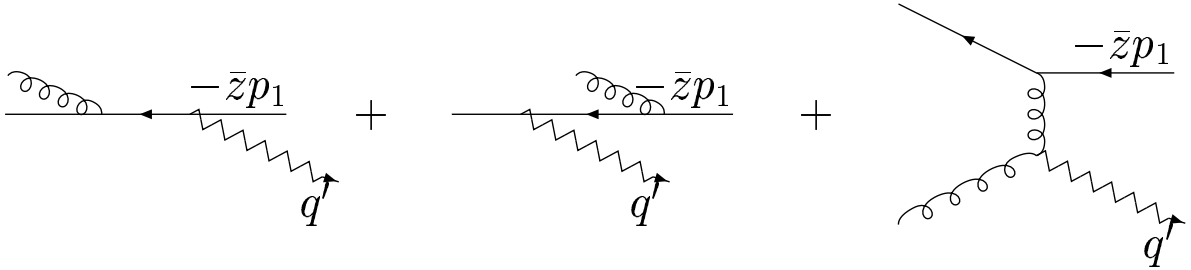

$$\left(\Gamma^{q}_{V_L q\bar{q}g}\right)^{*} = -\frac{g^2 f_V}{4N_c}\bar{v}^{j}_{\beta}(q_2)\left(\left[2\bar{z}\left(\vec{T}\vec{e}^{\,a}\right) + z_3\,\not{e}^{a}_{\perp}\not{T}\right]\frac{\not{p}_2}{s}\,\not{p}_1\right)_{\beta\alpha}\phi_{\parallel}(z)\,dz$$

$$q_2 = z_2 p_1 + \frac{\vec{q}_2^{\,2}}{z_2}\frac{p_2}{s} + q_{2\perp}$$ **momentum of cut antiquark**

$$k = z_3 p_1 + \frac{\vec{k}^2}{z_3}\frac{p_2}{s} + k_{\perp}$$ **momentum of cut gluon**

$$e^a = \frac{2(\vec{e}^{\,a}\vec{k})}{z_3\,s}p_2 + e^a_{\perp}\,, \quad e^a k = 0$$ **polarization of cut gluon**

$$\vec{T} = \left(\frac{\bar{z}\vec{q}_2 - z_2\vec{q}}{(\bar{z}\vec{q}_2 - z_2\vec{q}\,)^2} - \frac{\vec{q}_2}{\bar{z}\vec{q}_2^{\,2}}\right)(t^a t^{c'})_{ji} + \frac{1}{\bar{z}}\left(\frac{\vec{k}}{\vec{k}^2} + \frac{\vec{q}_2}{\vec{q}_2^{\,2}}\right)(t^{c'}t^a)_{ji}$$

After convolution and summation over Dirac and color indices of the cut antiquark and gluon, we get

$$-\frac{e_q g^4 f_V Q}{N_c (2\pi)^{3+2\varepsilon}} \int_0^1 \phi_{\parallel}(z)\, dz \int_0^{\bar{z}} \frac{dz_3}{z_3} d^{2+2\varepsilon}\vec{q}_2\, \theta(s_\Lambda - \kappa)$$

$$\left[(\vec{T}_q \vec{T})(2\bar{z} z_2 + (1+\varepsilon) z_3^2) + (\vec{T}_{\bar{q}} \vec{T})(z_3(2z-1) - 2z\bar{z} + (1+\varepsilon) z_3^2)\right]$$

The divergence for $z_3 \longrightarrow 0$ is cured by the $\theta(s_\Lambda - \kappa)$:

$$\kappa = \frac{\vec{k}^2}{z_3} + \frac{\vec{q}_2{}^2}{z_2} - \vec{q}\,^2 \qquad \longrightarrow \qquad z_3^{min} = \frac{\vec{k}^2}{s_\Lambda}$$

The contribution to the impact factor from $q\bar{q}g$ intermediate state can be separated in two terms:

$$T^{(q\bar{q}g)} = T_4 + T_5$$

T_4 = “divergent part for $z_3 \longrightarrow 0$ + s_Λ-counterterm”

T_5 = “remaining part (here z_3^{min} can be put equal to 0)”

Outline

1. The $\gamma^*\gamma^* \to V\,V$ forward amplitude
   - Introduction and Motivations
   - Kinematics and BFKL amplitude
2. Part 1 - The $\gamma^* \to V$ impact factor
   - Definition and general structure
   - Impact factor in the LLA
   - Impact factor in the NLA
   - Final result for the NLA impact factor
   - Cancellation of the leading energy scale uncertainty
3. Part 2 - Building up the amplitude
   - The $\gamma^* \to V$ impact factor - Reminder
   - The Green's function
   - Series representation of the amplitude
   - Numerical analysis - The "pure" BFKL regime
   - "Collinear factorization" regime
   - Analysis of systematic effects
   - Collinear improvement of the BFKL kernel
4. Conclusions

Final result for the NLA impact factor

$$\Phi_{\gamma_L^* \to V_L}(\alpha, s_0) = -\frac{4\pi e_q f_V \delta^{cc'}}{N_c Q} \int_0^1 dz\, T_H(z, \alpha, s_0, \mu_F, \mu_R)\, \phi_{||}(z, \mu_F)$$

$$T_H^{(1)} = \frac{g^2 \Gamma[1-\varepsilon]\left(Q^2/\mu^2\right)^{\varepsilon}}{(4\pi)^{2+\varepsilon}} T_H^{(0)} \left[\tilde{\tau}(z) + \tilde{\tau}(\bar{z})\right]$$

$$\tilde{\tau}(z) = \frac{C_F}{\varepsilon}\left[\frac{3}{2} + \frac{(1-2z)(\alpha - \bar{z}z)}{4c\bar{z}z} \ln\left(\frac{1+2c-2z}{2c-1+2z}\right) - \frac{(\alpha+\bar{z}z)}{4c\bar{z}z} \ln\left(\frac{2c+1}{2c-1}\right) - \frac{(\alpha-\bar{z}z)}{2\bar{z}z} \ln\left(\frac{\alpha+\bar{z}z}{\alpha}\right)\right] - \frac{\beta_0}{2\varepsilon} + \dots, \qquad c = \sqrt{\alpha + 1/4}$$

Double poles in ε which appeared in intermediate steps canceled in the sum.

Renormalization:

$$\Delta^{\alpha_s} T_H(z) = \frac{\alpha_s(\mu_R)}{4\pi} \beta_0 \left[\frac{1}{\hat{\varepsilon}} + \ln\left(\frac{\mu_R^2}{\mu^2}\right)\right] T_H^{(0)}(z), \quad \frac{1}{\hat{\varepsilon}} = \frac{1}{\varepsilon} + \gamma_E - \ln(4\pi)$$

$$\Delta^{coll} T_H(z) = -\frac{\alpha_s(\mu_F)}{2\pi}\left[\frac{1}{\hat{\varepsilon}} + \ln\left(\frac{\mu_F^2}{\mu^2}\right)\right] \int_0^1 du\, T_H^{(0)}(u)\, V^{(1)}(u, z)$$

The effect of renormalization is given by the replacements

$$\frac{\beta_0}{\varepsilon} \to \beta_0 \ln\left(\frac{Q^2}{\mu_R^2}\right), \qquad \frac{C_F}{\varepsilon} \to C_F \ln\left(\frac{Q^2}{\mu_F^2}\right).$$

$$T_H(z,\alpha,s_0,\mu_F,\mu_R)|_{\alpha\to 0} = \alpha_S(\mu_R)\frac{\alpha}{\alpha+z\bar{z}}\left\{1+\frac{\alpha_S(\mu_R)}{4\pi}\left[\tau(z)+\tau(\bar{z})\right]\right\}$$

$$\begin{aligned}
\tau(z) = {} & C_F \ln\left(\frac{Q^2}{\mu_F^2}\right)\left[\frac{3}{2}+\frac{(1-2z)(\alpha-\bar{z}z)}{4c\bar{z}z}\ln\left(\frac{1+2c-2z}{2c-1+2z}\right)-\frac{(\alpha+\bar{z}z)}{4c\bar{z}z}\ln\left(\frac{2c+1}{2c-1}\right)\right. \\
& \left.-\frac{(\alpha-\bar{z}z)}{2\bar{z}z}\ln\left(\frac{\alpha+\bar{z}z}{\alpha}\right)\right]-\frac{\beta_0}{2}\ln\left(\frac{Q^2}{\mu_R^2}\right)+n_f\left[-\frac{5}{9}+\frac{\ln\alpha}{3}\right]+C_F\left[-\frac{\ln^2\alpha}{2}-3\ln\left(\frac{\alpha+\bar{z}z}{z}\right)\right. \\
& +4\ln^2\left(c-z+\frac{1}{2}\right)-2\ln\left(c-\frac{1}{2}\right)\ln\left(c+\frac{1}{2}\right)+\frac{(\alpha+\bar{z}z)}{2\alpha\bar{z}z}\left(\ln\alpha+2c\ln\left(\frac{2c+1}{2c-1}\right)\right) \\
& -\frac{(1-2z)(\alpha+\bar{z}z)}{2c\bar{z}z}\ln(1+2c-2z)-\frac{(\alpha+\bar{z}z)}{2\bar{z}z}\ln\left(\frac{\alpha+\bar{z}z}{\alpha}\right)+\frac{(\alpha+\bar{z}z)}{\bar{z}(\alpha+z)}\ln\left(\frac{\alpha+\bar{z}z}{\alpha z}\right) \\
& +\frac{(\alpha+\bar{z}z)}{2c\bar{z}z}\left(\frac{\pi^2}{6}+\ln 4\ln\alpha+\frac{1}{2}\ln\left(\frac{1+2c}{2c-1}\right)+2\ln c\ln\left(\frac{2c-1}{4}\right)-\ln^2(2c+1)\right. \\
& \left.-\ln^2 c+\frac{1}{4}\ln^2\left(\frac{2c+1}{2c-1}\right)-2\mathrm{Li}_2\left(\frac{2c-1}{4c}\right)-\mathrm{Li}_2\left(\frac{2}{1+2c}\right)\right)+\frac{(\alpha+\bar{z}z)}{2\bar{z}z}\left(\frac{\ln^2\alpha}{2}\right. \\
& -2\ln\left(c-\frac{1}{2}\right)\ln\left(c+\frac{1}{2}\right)+\ln^2\left(\frac{z}{\alpha}\right)-\ln^2\left(\frac{\alpha+\bar{z}z}{z}\right)+2\mathrm{Li}_2\left(\frac{2z}{1-2c}\right) \\
& \left.+2\mathrm{Li}_2\left(\frac{2z}{1+2c}\right)\right)-\frac{(1-2z)(\alpha-\bar{z}z)}{2c\bar{z}z}\left(\ln\left(\frac{z}{4\alpha+1}\right)\ln\left(\frac{2\alpha+(1-2c)z}{2\alpha+(1+2c)z}\right)\right. \\
& -\ln(4\alpha+1)\ln\left(\frac{2c+1}{2c-1}\right)+2\mathrm{Li}_2\left(\frac{1-2c-2z}{1+2c-2z}\right)-\mathrm{Li}_2\left(\frac{2z}{1-2c}\right) \\
& \left.\left.+\mathrm{Li}_2\left(\frac{2z}{1+2c}\right)\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&+N_c\Bigg[\ln\left(s_0/Q^2\right)\ \ln\left(\frac{(\alpha+\bar{z}z)^2}{z^2\alpha}\right)+\frac{20}{9}-\frac{\ln\alpha}{3}+\frac{1}{2}\ln^2\left(\frac{\bar{z}}{z}\right)\\
&+\frac{1}{2}\ln^2\left(\frac{\alpha+\bar{z}z}{\alpha}\right)-2\ln\left(\frac{\alpha+z}{z}\right)\ln\left(\frac{\alpha+\bar{z}z}{\alpha\,z}\right)-\ln^2\left(\frac{\alpha+\bar{z}z}{z}\right)+3\ \ln z\ \ln\left(\frac{\alpha+\bar{z}z}{\alpha\,z}\right)\\
&+2\ \ln^2 z-\frac{(\alpha+\bar{z}z)}{2\,\alpha\,\bar{z}\,z}\left(z\ln\left(\frac{\alpha}{z}\right)+\sqrt{z\,(4\,\alpha+z)}\ln\left(\frac{z+\sqrt{z\ (4\,\alpha+z)}}{-z+\sqrt{z\ (4\,\alpha+z)}}\right)\right)\\
&+\ \mathrm{Li}_2\left(\frac{2\,z}{1-2\,c}\right)+\mathrm{Li}_2\left(\frac{2\,z}{1+2\,c}\right)+2\mathrm{Li}_2\left(-\frac{z^2}{\alpha+\bar{z}z}\right)-2\mathrm{Li}_2\left(-\frac{z}{\alpha}\right)+3\,\mathrm{Li}_2\left(-\frac{\bar{z}z}{\alpha}\right)\Bigg]\\
&+\frac{1}{N_c}\Bigg[\frac{5}{2}+\left(\frac{\alpha+\bar{z}z}{\bar{z}\,z}-\frac{3}{2}\right)\ln\alpha+\frac{1}{2}\ln\left(\frac{4\,\alpha}{(2\,c-1)^2}\right)\ln\left(\frac{2\,c+1}{2\,c-1}\right)+\frac{1}{2}\ln^2\left(\frac{1+2\,c-2\,z}{2\,c-1+2\,z}\right)\\
&+\frac{c\,(1-2\,z)\,(\alpha+\bar{z}z)}{\bar{z}^2\,z^2}\ln(1+2\,c-2\,z)+\frac{(1-2\,z)\,(\alpha+\bar{z}z)}{\bar{z}^2 z}\ln\left(\frac{z}{\alpha+\bar{z}z}\right)+\mathrm{Li}_2\left(\frac{2\,z}{1+2\,c}\right)\\
&+\ln z\ln\left(\frac{\alpha+\bar{z}z}{\alpha}\right)+\frac{(\alpha+\bar{z}z)}{4\,\bar{z}^2 z^2}\left(2\,c\ \ln\left(\frac{2\,c+1}{2\,c-1}\right)-\ln\left(\frac{\alpha+\bar{z}z}{\alpha}\right)\right)+\mathrm{Li}_2\left(\frac{2\,z}{1-2\,c}\right)\\
&-\frac{\left(\alpha^2+\alpha\,z-\bar{z}\,z^3\right)}{2\,\bar{z}\,z^3}\left(2\ln\left(c-z+\frac{1}{2}\right)\ln\left(c+z-\frac{1}{2}\right)-2\ln\left(c-\frac{1}{2}\right)\ln\left(c+\frac{1}{2}\right)\right.\\
&+\ln\left(\frac{2\,c+1}{2\,c-1}\right)\ln\left(\frac{2\,\alpha+(1+2\,c)\ z}{2\,\alpha+(1-2\,c)\ z}\right)-\ln\left(\frac{\alpha\,\bar{z}^2\,z^2}{(\alpha+\bar{z}z)^2}\right)\ln\left(\frac{\alpha+\bar{z}z}{\alpha}\right)\\
&+2\ln\left(\frac{\alpha+z}{\alpha}\right)\ln\left(\frac{\alpha\,z}{\alpha+\bar{z}z}\right)-2\mathrm{Li}_2\left(-\frac{z}{\alpha}\right)+4\mathrm{Li}_2\left(\frac{2\,z}{1-2\,c}\right)+4\mathrm{Li}_2\left(\frac{2\,z}{1+2\,c}\right)\\
&\left.-\ 2\mathrm{Li}_2\left(-\frac{\bar{z}z}{\alpha}\right)+2\mathrm{Li}_2\left(-\frac{z^2}{\alpha+\bar{z}z}\right)\right)\Bigg]
\end{aligned}
$$

$$c=\sqrt{\alpha+1/4}$$

Limits of small and large virtualities of the Reggeized gluon

$$T_H(z,\alpha,s_0,\mu_F,\mu_R)|_{\alpha\to 0} = \alpha_S(\mu_R)\frac{\alpha}{z\bar{z}}\left[1+\frac{\alpha_S(\mu_R)}{4\pi}\left(n_f\left[-\frac{10}{9}+\frac{2\ln\alpha}{3}\right]\right.\right.$$
$$-\beta_0\ln\left(\frac{Q^2}{\mu_R^2}\right)+N_c\left[\frac{67}{9}+\frac{\ln\alpha}{3}-\ln^2\alpha-3\ln(z\bar{z})+2\ln\left(\frac{s_0}{Q^2}\right)\ln\left(\frac{z\bar{z}}{\alpha}\right)\right.$$
$$\left.-\frac{5\pi^2}{6}+\ln^2 z+\ln^2\bar{z}+\ln z\ln\bar{z}\right]+C_F\left[2z\ln\bar{z}+2\bar{z}\ln z-\frac{\pi^2}{3}\right.$$
$$+z\ln^2\bar{z}+\bar{z}\ln^2 z-8-2\ln\alpha+2\,z\,\mathrm{Li}_2(z)+2\,\bar{z}\,\mathrm{Li}_2(\bar{z})$$
$$\left.\left.\left.+(3+2z\ln\bar{z}+2\bar{z}\ln z)\ln\left(\frac{Q^2}{\mu_F^2}\right)\right]\right)\right]+\mathcal{O}\left(\alpha_S^2(\mu_R)\,\alpha^2\right)$$

$$T_H(z,\alpha,s_0,\mu_F,\mu_R)|_{\alpha\to\infty} = \alpha_S(\mu_R)\left[1+\frac{\alpha_S(\mu_R)}{4\pi}\left(n_f\left[-\frac{10}{9}+\frac{2\ln\alpha}{3}\right]\right.\right.$$
$$-\beta_0\ln\left(\frac{Q^2}{\mu_R^2}\right)+N_c\left[\frac{22}{9}-\frac{20\ln\alpha}{3}-2\ln^2\alpha-\frac{\ln\bar{z}}{2\,z}-\frac{\ln z}{2\,\bar{z}}+3\ln(z\bar{z})\right.$$
$$\left.\left.\left.+2\ln\left(\frac{s_0}{Q^2}\right)\ln\left(\frac{\alpha}{z\bar{z}}\right)+4\ln\alpha\ln(z\bar{z})-2\ln z\ln\bar{z}-3\ln^2 z-3\ln^2\bar{z}\right]\right)\right]$$
$$+\mathcal{O}\left(\frac{\alpha_S^2(\mu_R)}{\alpha}\right)$$

Outline

1. The $\gamma^*\gamma^* \to V\,V$ forward amplitude
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Cancellation of the leading energy scale uncertainty

LLA: summation of terms $\sim \alpha_S^n \ln^n(s/s_0)$

Leading energy scale uncertainty given by terms $\sim \alpha_S^n \ln^{n-1} s \ln s_0$, which are NLA corrections. They must cancel in the full expression for an NLA amplitude.

Forward amplitude of the process $\gamma_1^*(Q_1^2)\gamma_2^*(Q_2^2) \to \rho_1\rho_2$

$$\mathcal{I}m_s\left(\mathcal{A}\right) = \frac{s}{(2\pi)^2}\int \frac{d^2\vec{q}_1}{\vec{q}_1^{\,2}}\Phi_1(\vec{q}_1, s_0)\int \frac{d^2\vec{q}_2}{\vec{q}_2^{\,2}}\Phi_2(-\vec{q}_2, s_0)\int\limits_{\delta-i\infty}^{\delta+i\infty}\frac{d\omega}{2\pi i}\left(\frac{s}{s_0}\right)^{\omega} G_\omega(\vec{q}_1, \vec{q}_2)$$

Transverse momentum notation: $\hat{\vec{q}}\,|\vec{q}_i\rangle = \vec{q}_i|\vec{q}_i\rangle$

$$\langle\vec{q}_1|\vec{q}_2\rangle = \delta^{(2)}(\vec{q}_1 - \vec{q}_2)\ , \qquad \langle A|B\rangle = \langle A|\vec{k}\rangle\langle\vec{k}|B\rangle = \int d^2k A(\vec{k})B(\vec{k})$$

$$\mathcal{I}m_s\left(\mathcal{A}\right) = \frac{s}{(2\pi)^2}\int\limits_{\delta-i\infty}^{\delta+i\infty}\frac{d\omega}{2\pi i}\left(\frac{s}{s_0}\right)^{\omega}\langle\Phi_1|\frac{1}{\hat{\vec{q}}_1^{\,2}}\hat{G}_\omega\frac{1}{\hat{\vec{q}}_2^{\,2}}|\Phi_2\rangle$$

$$\langle\vec{q}_1|\hat{G}_\omega|\vec{q}_2\rangle = G_\omega(\vec{q}_1, \vec{q}_2)$$

BFKL equation

$$\hat{1} = (\omega - \hat{K})\hat{G}_\omega \qquad \longrightarrow \qquad \hat{G}_\omega = (\omega - \hat{K})^{-1}$$

With LLA accuracy: $\hat{G}_\omega = (\omega - \bar{\alpha}_s \hat{K}^0)^{-1}, \qquad \hat{K} = \bar{\alpha}_s \hat{K}^0 \ , \qquad \bar{\alpha}_s = \frac{\alpha_s N_c}{\pi}$

Basis of eigenfunctions of the LLA kernel: $\{|\nu\rangle\}$

$$\hat{K}^0|\nu\rangle = \chi(\nu)|\nu\rangle \ , \qquad \chi(\nu) = 2\psi(1) - \psi\left(\frac{1}{2} + i\nu\right) - \psi\left(\frac{1}{2} - i\nu\right)$$

$$\langle\vec{q}|\nu\rangle = \frac{1}{\pi\sqrt{2}}\left(\vec{q}^{\,2}\right)^{i\nu - \frac{1}{2}} \ , \qquad \langle\nu'|\nu\rangle = \int \frac{d^2\vec{q}}{2\pi^2}\left(\vec{q}^{\,2}\right)^{i\nu - i\nu' - 1} = \delta(\nu - \nu')$$

$$G_\omega(\vec{q}_1, \vec{q}_2) = \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi^2} \frac{(\vec{q}_1^{\,2})^{\frac{1}{2} + i\nu}(\vec{q}_2^{\,2})^{\frac{1}{2} - i\nu}}{\omega - \bar{\alpha}_S \chi(\nu)}$$

$$\mathcal{I}m_s(\mathcal{A})^{\rho_1\rho_2}_{\gamma_1^*\gamma_2^*} = \frac{s}{(2\pi)^2}\int\limits_{-\infty}^{+\infty} d\nu \left(\frac{s}{s_0}\right)^{\bar{\alpha}_S\chi(\nu)} I_1(\nu,s_0)I_2(\nu,s_0)$$

$$\begin{aligned} I_1(\nu,s_0) &= \int d^2q_1 \frac{(\vec{q}_1{}^2)^{i\nu-\frac{3}{2}}}{\sqrt{2}\pi}\Phi_{\gamma_1^*\to\rho_1}(\vec{q}_1{}^2,s_0) \\ I_2(\nu,s_0) &= \int d^2q_2 \frac{(\vec{q}_2{}^2)^{-i\nu-\frac{3}{2}}}{\sqrt{2}\pi}\Phi_{\gamma_2^*\to\rho_2}(\vec{q}_2{}^2,s_0) \end{aligned}$$

$$\Phi_{\gamma_L^*\to\rho_L}\left(\alpha = \frac{\vec{q}^{\,2}}{Q^2}, s_0\right)$$

$$= -\frac{4\pi e f_\rho \delta^{cc'}}{\sqrt{2}N_c Q}\int\limits_0^1 dz\,\phi_{\|}(z)\,\alpha_S\frac{\alpha}{\alpha+z\bar{z}}\left[1+\frac{\bar{\alpha}_S}{4}\ln s_0 \ln\left(\frac{(\alpha+z\bar{z})^4}{\alpha^2 z^2\bar{z}^2}\right)+\ldots\right]$$

$$I_{1,2}(\nu,s_0) = (Q_{1,2}^2)^{\pm i\nu-1}J(\pm\nu)\frac{\pi}{\cosh(\pi\nu)}\left[1+\frac{\bar{\alpha}_\chi(\nu)}{2}\ln s_0+\ldots\right]$$

$$J(\nu) = \int\limits_0^1 dz\,\phi_{\|}(z)\,(z\bar{z})^{i\nu-\frac{1}{2}}$$

Finally,

$$\mathcal{I}m_s(\mathcal{A})^{\rho_1\rho_2}_{\gamma_1^*\gamma_2^*} \sim \frac{sf_\rho^2}{Q_1^2Q_2^2}\int\limits_{-\infty}^{+\infty} d\nu J(\nu)J(-\nu)\left(\frac{Q_1^2}{Q_2^2}\right)^{i\nu}\frac{\pi^2}{\cosh^2(\pi\nu)}$$
$$\times\left(\frac{s}{s_0}\right)^{\bar{\alpha}_S\chi(\nu)}\left[1+\frac{\bar{\alpha}_S\chi(\nu)}{2}\ln s_0+\ldots\right]^2$$

where the dependence of the amplitude on s_0 indeed cancels out.

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The $\gamma^* \longrightarrow V$ impact factor - Reminder

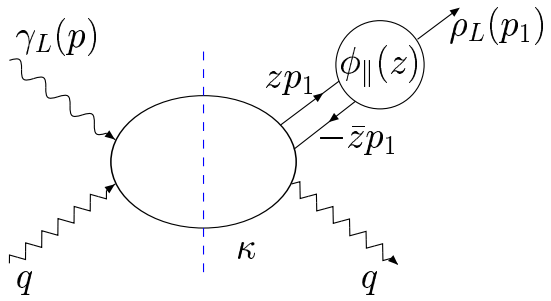
$$\mathcal{I}m_s\left(\mathcal{A}\right) = \frac{s}{(2\pi)^2}\int \frac{d^2\vec{q}_1}{\vec{q}_1^{\,2}}\Phi_1(\vec{q}_1, s_0)\int \frac{d^2\vec{q}_2}{\vec{q}_2^{\,2}}\Phi_2(-\vec{q}_2, s_0)\int_{\delta-i\infty}^{\delta+i\infty}\frac{d\omega}{2\pi i}\left(\frac{s}{s_0}\right)^{\omega} G_\omega(\vec{q}_1, \vec{q}_2)$$

$$\Phi_{1,2}(\vec{q}) = \alpha_s\, D_{1,2}\left[C^{(0)}_{1,2}(\vec{q}^{\,2}) + \bar{\alpha}_s C^{(1)}_{1,2}(\vec{q}^{\,2})\right]$$

$$D_{1,2} = -\frac{4\pi e_q f_V}{N_c Q_{1,2}}\sqrt{N_c^2 - 1}$$

$$e_q \longrightarrow \frac{e}{\sqrt{2}}\,, \qquad \frac{e}{3\sqrt{2}}\,, \qquad -\frac{e}{3}$$

$$\rho^0\,, \qquad \omega\,, \qquad \phi$$



- Leading order (photon virtuality Q^2): $C^{(0)}(\vec{q}^{\,2}) = \int_0^1 dz\, \frac{\vec{q}^{\,2}}{\vec{q}^{\,2} + z\bar{z}Q^2}\phi_\parallel(z, \mu_F)$
- Next-to-leading order: $C^{(1)}(\vec{q}^{\,2}) = \frac{1}{4N_c}\int_0^1 dz\, \frac{\vec{q}^{\,2}}{\vec{q}^{\,2} + z\bar{z}Q^2}[\tau(z) + \tau(1-z)]\phi_\parallel(z, \mu_F)$

$\tau(z)$ – the 2-page-long expression ...

$\phi_\parallel(z, \mu_F)$ is the twist-2 meson distribution amplitude $\longrightarrow \phi_\parallel^{as}(z) = 6z(1-z)$

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The Green's function

$$\mathcal{I}m_s\left(\mathcal{A}\right) = \frac{s}{(2\pi)^2}\int \frac{d^2\vec{q}_1}{\vec{q}_1^{\,2}}\Phi_1(\vec{q}_1,s_0)\int \frac{d^2\vec{q}_2}{\vec{q}_2^{\,2}}\Phi_2(-\vec{q}_2,s_0)\int\limits_{\delta-i\infty}^{\delta+i\infty}\frac{d\omega}{2\pi i}\left(\frac{s}{s_0}\right)^{\omega} G_\omega(\vec{q}_1,\vec{q}_2)$$

BFKL equation: $\delta^2(\vec{q}_1-\vec{q}_2) = \omega\, G_\omega(\vec{q}_1,\vec{q}_2) - \int d^2\vec{q}\, K(\vec{q}_1,\vec{q})\, G_\omega(\vec{q},\vec{q}_2)$

Transverse momentum notation: $\hat{\vec{q}}\,|\vec{q}_i\rangle = \vec{q}_i|\vec{q}_i\rangle$

$$\langle\vec{q}_1|\vec{q}_2\rangle = \delta^{(2)}(\vec{q}_1-\vec{q}_2) \qquad \langle A|B\rangle = \langle A|\vec{k}\rangle\langle\vec{k}|B\rangle = \int d^2k A(\vec{k})B(\vec{k})$$

$$\hat{1} = (\omega-\hat{K})\hat{G}_\omega \qquad \longrightarrow \qquad \hat{G}_\omega = (\omega-\hat{K})^{-1}$$

$$\hat{K} = \bar{\alpha}_s\hat{K}^0 + \bar{\alpha}_s^2\hat{K}^1\,, \qquad \bar{\alpha}_s = \frac{\alpha_s N_c}{\pi}$$

With NLA accuracy

$$\hat{G}_\omega = (\omega-\bar{\alpha}_s\hat{K}^0)^{-1} + (\omega-\bar{\alpha}_s\hat{K}^0)^{-1}\left(\bar{\alpha}_s^2\hat{K}^1\right)(\omega-\bar{\alpha}_s\hat{K}^0)^{-1} + \mathcal{O}\left[\left(\bar{\alpha}_s^2\hat{K}^1\right)^2\right]$$

Basis of eigenfunctions of the LLA kernel: $\{|\nu\rangle\}$

$$\hat{K}^0|\nu\rangle = \chi(\nu)|\nu\rangle \ , \qquad \chi(\nu) = 2\psi(1) - \psi\left(\frac{1}{2} + i\nu\right) - \psi\left(\frac{1}{2} - i\nu\right)$$

$$\langle\vec{q}|\nu\rangle = \frac{1}{\pi\sqrt{2}}\left(\vec{q}^{\,2}\right)^{i\nu-\frac{1}{2}} \ , \qquad \langle\nu'|\nu\rangle = \int \frac{d^2\vec{q}}{2\pi^2}\left(\vec{q}^{\,2}\right)^{i\nu-i\nu'-1} = \delta(\nu-\nu')$$

Action of the full NLA kernel on the LLA eigenfunctions:

$$\begin{aligned}\hat{K}|\nu\rangle &= \bar{\alpha}_s(\mu_R)\chi(\nu)|\nu\rangle + \bar{\alpha}_s^2(\mu_R)\left(\chi^{(1)}(\nu) + \frac{\beta_0}{4N_c}\chi(\nu)\ln(\mu_R^2)\right)|\nu\rangle \\ &+ \bar{\alpha}_s^2(\mu_R)\frac{\beta_0}{4N_c}\chi(\nu)\left(i\frac{\partial}{\partial\nu}\right)|\nu\rangle\end{aligned}$$

$$\beta_0 = \frac{11\,N_c}{3} - \frac{2n_f}{3}$$

$$\chi^{(1)}(\nu) = -\frac{\beta_0}{8\,N_c}\left(\chi^2(\nu) - \frac{10}{3}\chi(\nu) - i\chi'(\nu)\right) + \bar{\chi}(\nu)$$

$$\begin{aligned}\bar{\chi}(\nu) \quad &= \quad -\frac{1}{4}\left[\frac{\pi^2-4}{3}\chi(\nu) - 6\zeta(3) - \chi''(\nu) - \frac{\pi^3}{\cosh(\pi\nu)}\right.\\ &+ \quad \left.\frac{\pi^2\sinh(\pi\nu)}{2\nu\cosh^2(\pi\nu)}\left(3 + \left(1+\frac{n_f}{N_c^3}\right)\frac{11+12\nu^2}{16(1+\nu^2)}\right) + 4\phi(\nu)\right]\end{aligned}$$

$$\phi(\nu) \;=\; 2\int_0^1 dx\,\frac{\cos(\nu\ln(x))}{(1+x)\sqrt{x}}\left[\frac{\pi^2}{6} - \mathrm{Li}_2(x)\right]$$

$$\begin{aligned}\frac{\mathcal{I}m_s\,(\mathcal{A})}{D_1D_2} &= \frac{s}{(2\pi)^2}\int_{-\infty}^{+\infty} d\nu\left(\frac{s}{s_0}\right)^{\bar{\alpha}_s(\mu_R)\chi(\nu)}\alpha_s^2(\mu_R)c_1(\nu)c_2(\nu)\\ &\times\left[1 + \bar{\alpha}_s(\mu_R)\left(\frac{c_1^{(1)}(\nu)}{c_1(\nu)} + \frac{c_2^{(1)}(\nu)}{c_2(\nu)}\right)\right.\\ &\left.+\bar{\alpha}_s^2(\mu_R)\ln\left(\frac{s}{s_0}\right)\left(\bar{\chi}(\nu) + \frac{\beta_0}{8N_c}\chi(\nu)\left[-\chi(\nu) + \frac{10}{3} + i\frac{d\ln(\frac{c_1(\nu)}{c_2(\nu)})}{d\nu} + 2\ln(\mu_R^2)\right]\right)\right]\end{aligned}$$

$|\nu\rangle$ representation for impact factors:

$$\frac{C_1^{(0)}(\vec{q}^{\,2})}{\vec{q}^{\,2}} = \int_{-\infty}^{+\infty} d\,\nu'\, c_1(\nu')\langle\nu'|\vec{q}\rangle \qquad \frac{C_2^{(0)}(\vec{q}^{\,2})}{\vec{q}^{\,2}} = \int_{-\infty}^{+\infty} d\,\nu\, c_2(\nu)\,\langle\vec{q}|\nu\rangle$$

$$c_1(\nu) = \int d^2\vec{q}\; C_1^{(0)}(\vec{q}^{\,2})\frac{(\vec{q}^{\,2})^{i\nu-\frac{3}{2}}}{\pi\sqrt{2}} \qquad c_2(\nu) = \int d^2\vec{q}\; C_2^{(0)}(\vec{q}^{\,2})\frac{(\vec{q}^{\,2})^{-i\nu-\frac{3}{2}}}{\pi\sqrt{2}}$$

(analogous definitions for $c_1^{(1)}(\nu)$ and $c_2^{(1)}(\nu)$)

Leading-order:

$$c_{1,2}(\nu) = \frac{\left(Q_{1,2}^2\right)^{\pm i\nu-\frac{1}{2}}}{\sqrt{2}}\,\frac{\Gamma^2[\frac{3}{2}\pm i\nu]}{\Gamma[3\pm 2i\nu]}\,\frac{6\pi}{\cosh(\pi\nu)}$$

Next-to-leading-order: $\longrightarrow$ numerical calculation

The amplitude above contains also terms beyond the NLA. If only the “allowed” terms in the NLA are kept, $(\alpha_s \ln(s))^n)$ and $\alpha_s(\alpha_s \ln(s))^n$, the dependence on s_0 and μ_R disappears.

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Series representation of the amplitude

$$\frac{Q_1 Q_2}{D_1 D_2}\frac{\mathcal{I}m_s \mathcal{A}}{s} = \frac{1}{(2\pi)^2}\alpha_s(\mu_R)^2\left[b_0 + \sum_{n=1}^{\infty}\bar{\alpha}_s(\mu_R)^n\, b_n\left(\ln\left(\frac{s}{s_0}\right)^n + d_n(s_0,\mu_R)\ln\left(\frac{s}{s_0}\right)^{n-1}\right)\right]$$

LLA

$$\frac{b_n}{Q_1 Q_2} = \int_{-\infty}^{+\infty} d\nu\, c_1(\nu) c_2(\nu)\frac{\chi^n(\nu)}{n!}$$

$b_0 = \frac{9\pi}{4}\left(7\zeta(3) - 6\right)$, Born (2-gluon limit) [B. Pire, L. Szymanowsky, S. Wallon (2004)]

NLA

$$d_n = n\ln\left(\frac{s_0}{Q_1 Q_2}\right) + \frac{\beta_0}{4N_c}\left((n+1)\frac{b_{n-1}}{b_n}\ln\left(\frac{\mu_R^2}{Q_1 Q_2}\right) - \frac{n(n-1)}{2}\right.$$

$$\left. + \frac{Q_1 Q_2}{b_n}\int_{-\infty}^{+\infty} d\nu\,(n+1) f(\nu) c_1(\nu) c_2(\nu)\frac{\chi^{n-1}(\nu)}{(n-1)!}\right)$$

$$+ \frac{Q_1 Q_2}{b_n}\left(\int_{-\infty}^{+\infty} d\nu\, c_1(\nu) c_2(\nu)\frac{\chi^{n-1}(\nu)}{(n-1)!}\left[\frac{\bar{c}_1^{(1)}(\nu)}{c_1(\nu)} + \frac{\bar{c}_2^{(1)}(\nu)}{c_2(\nu)} + (n-1)\frac{\bar{\chi}(\nu)}{\chi(\nu)}\right]\right)$$

$$f(\nu) = \frac{5}{3} + \psi(3+2i\nu) + \psi(3-2i\nu) - \psi\left(\frac{3}{2}+i\nu\right) - \psi\left(\frac{3}{2}-i\nu\right)$$

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Numerical analysis - The “pure” BFKL regime

$$Q_1 = Q_2 \equiv Q \qquad \text{“pure” BFKL regime}$$

LLA: b_n coefficients (Q-independent)

$$b_0 = 17.0664 \quad b_1 = 34.5920 \quad b_2 = 40.7609 \quad b_3 = 33.0618 \quad b_4 = 20.7467$$
$$b_5 = 10.5698 \quad b_6 = 4.54792 \quad b_7 = 1.69128 \quad b_8 = 0.554475$$

NLA: $d_n(s_0, \mu_R)$ coefficients ($s_0 = Q^2 = \mu_R^2$, $n_f = 5$)

$$d_1 = -3.71087 \quad d_2 = -11.3057 \quad d_3 = -23.3879 \quad d_4 = -39.1123$$
$$d_5 = -59.207 \quad d_6 = -83.0365 \quad d_7 = -111.151 \quad d_8 = -143.06$$

NLA: $d_n^{\rm imp}(s_0, \mu_R)$ coefficients ($s_0 = Q^2 = \mu_R^2$, impact factor contribution)

$$d_1^{\rm imp} = -3.71087 \quad d_2^{\rm imp} = -8.4361 \quad d_3^{\rm imp} = -13.1984 \quad d_4^{\rm imp} = -18.0971$$
$$d_5^{\rm imp} = -23.0235 \quad d_6^{\rm imp} = -27.9877 \quad d_7^{\rm imp} = -32.9676 \quad d_8^{\rm imp} = -37.9618$$

Large NLA corrections!

d_n coefficients negative and increasingly large in absolute value.
The contribution from the kernel dominates only for $n \geq 4$.

Optimization of the perturbative expansion needed!
Principle of minimal sensitivity (PMS) [P.M. Stevenson (1981)]: require the minimal sensitivity to the change of both s_0 and μ_R.

Test of the PMS method

Total cross section for the production of e^+e^- in the collision of two heavy ions at high energies [G. Racah (1937)]

$$\sigma = \frac{28\alpha_{EM}^4 Z_1^2 Z_2^2}{27\pi m_e^2}\left(l^3 + Al^2 + Bl + C\right) + \mathcal{O}\left(\frac{1}{(p_1 p_2)}\right)$$

$Z_{1,2}$ are the ions charges, m_e the electron mass, $p_{1,2}$ the ions' four-momenta, $m_{1,2}$ the masses of the ions and $l = \ln \frac{2(p_1 p_2)}{m_1 m_2}$

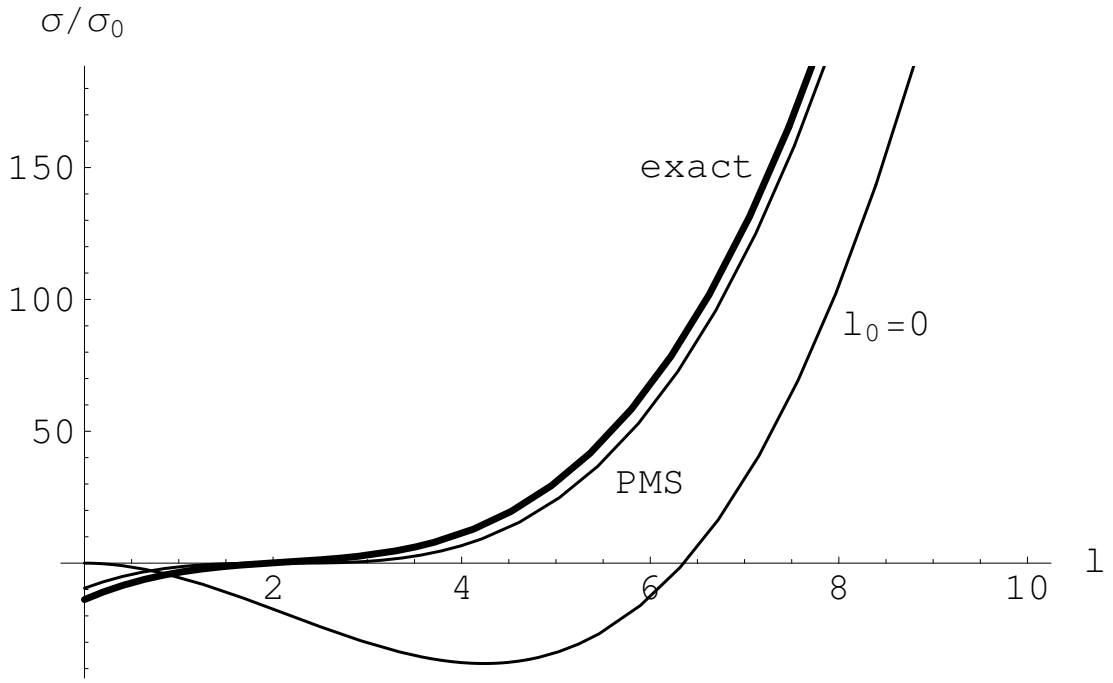
$$\begin{aligned} A &= -178/28 = -6.35714 \\ B &= \frac{1}{28}(7\pi^2 + 370) = 15.6817 \\ C &= -\frac{1}{28}\left(348 + \frac{13}{2}\pi^2 - 21\zeta(3)\right) = -13.8182 \end{aligned}$$

Let's imagine that only the coefficient A in front of the first subleading logarithm is known and construct the following approximation (in analogy with NLA BFKL):

$$\sigma^{app} = \sigma_0 \left((l - l_0)^3 + (A + 3l_0)(l - l_0)^2 \right) , \quad \sigma_0 = \frac{28\alpha_{EM}^4 Z_1^2 Z_2^2}{27\pi m_e^2}$$

where l_0 is an arbitrary parameter, which acts as energy shift; the dependence of the cross section on l_0 is subleading.

Minimal sensitivity to the change of $l_0 \longrightarrow l_0 = -A/3 = 2.11905$



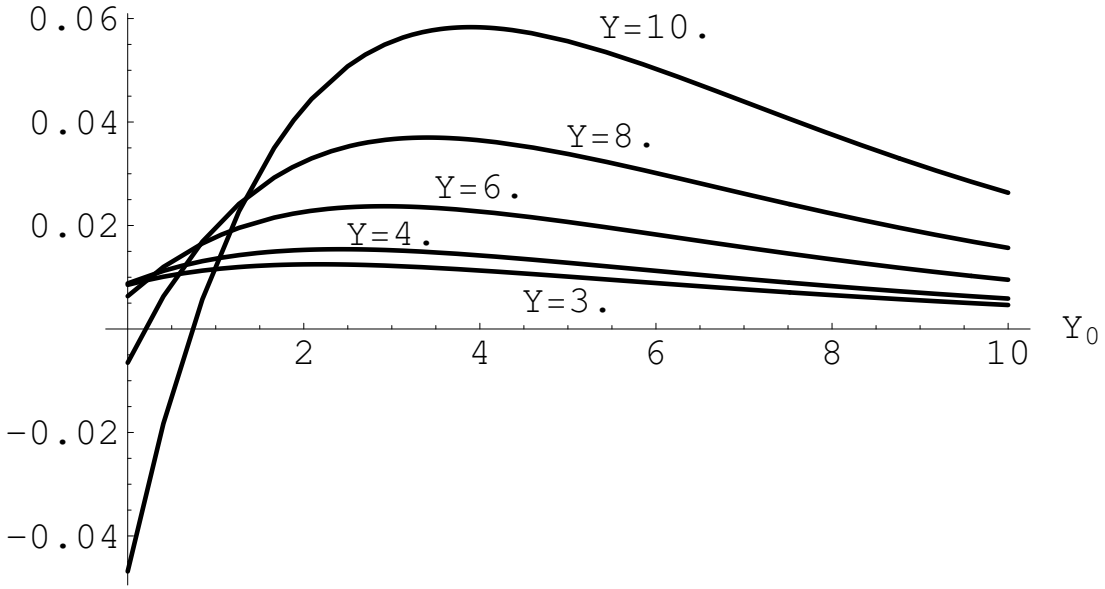
$$\frac{Q^2}{D_1 D_2}\frac{\mathcal{I}m_s\mathcal{A}}{s} = \frac{1}{(2\pi)^2}\alpha_s(\mu_R)^2\left[b_0 + \sum_{n=1}^{\infty}\bar{\alpha}_s(\mu_R)^n\, b_n\left((Y-Y_0)^n + d_n(s_0,\mu_R)(Y-Y_0)^{n-1}\right)\right]$$

$$Y \equiv \ln\left(\frac{s}{Q^2}\right), \qquad Y_0 \equiv \ln\left(\frac{s_0}{Q^2}\right)$$

Strategy: for each fixed Y calculate the amplitude for varying Y_0 at fixed μ_R and vice versa, up to finding the optimal values for which the amplitude is least sensitive to variations of them.

In practice, there are wide regions in Y_0 and μ_R where the amplitude is very weekly dependent on Y_0 and μ_R.

Example: Q^2=24 GeV^2, $n_f = 5$



$\frac{Q^2}{D_1 D_2} \frac{\mathcal{I}m_s \mathcal{A}}{s}$ *vs* Y_0

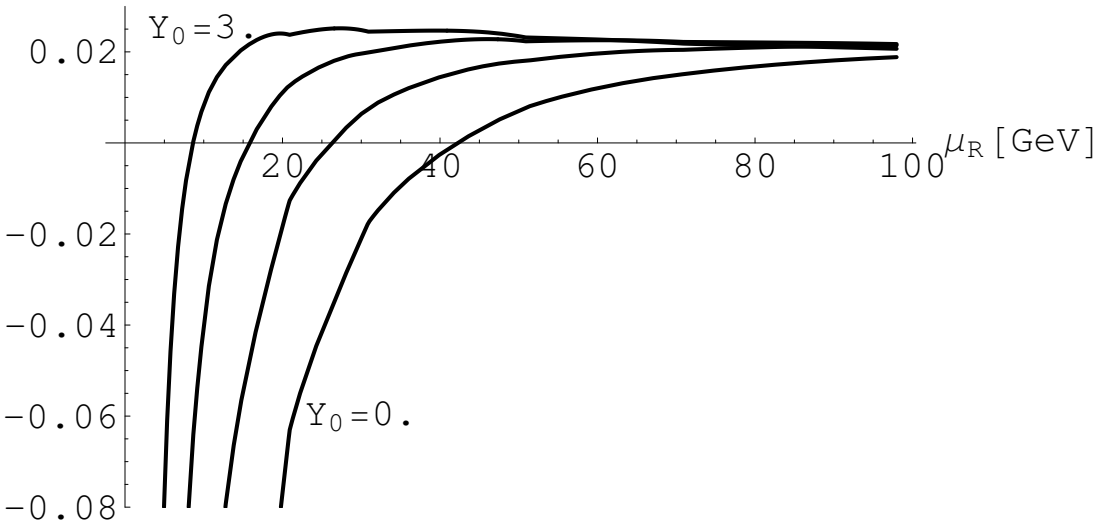
Y=10, 8, 6, 4, 3

$\mu_R = 10\,Q$

$\frac{Q^2}{D_1 D_2} \frac{\mathcal{I}m_s \mathcal{A}}{s}$ *vs* μ_R

Y_0=3, 2, 1, 0

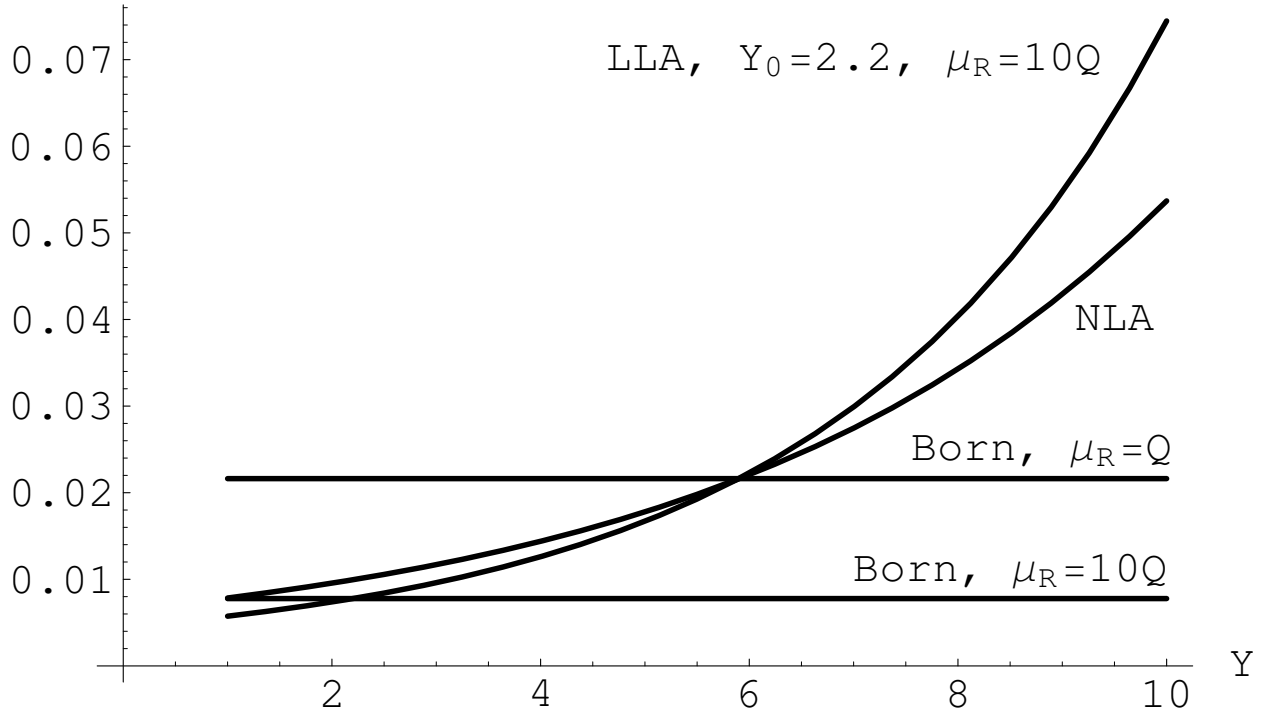
Y=6



Finally,

$\frac{Q^2}{D_1 D_2} \frac{\mathcal{I}m_s \mathcal{A}}{s}$ *vs* Y

Q^2=24 GeV^2, $n_f = 5$



Lessons

- The Born approximation does not give necessarily the estimate from below.
- The optimal values for μ_R are "unnaturally" larger than Q (new scale or nature of the BFKL series?).

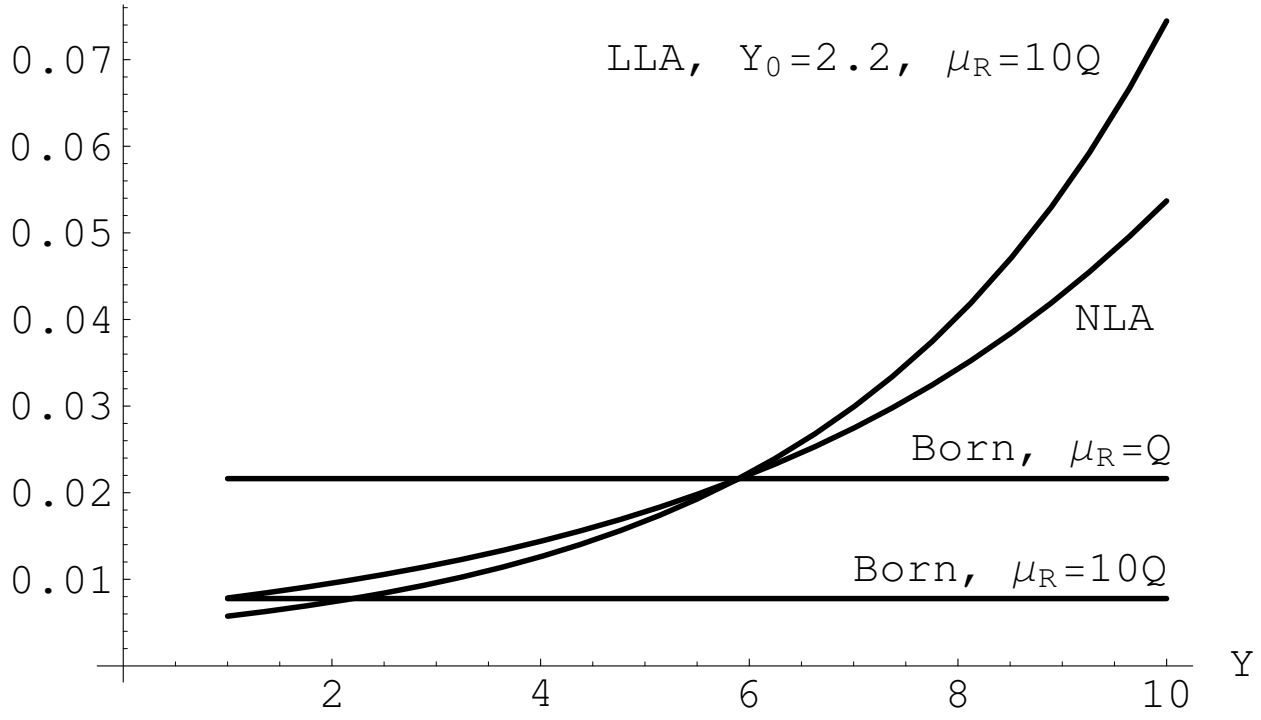
$\longrightarrow$ Since NLA corrections are large and since the exact amplitude should be renorm- and energy scale invariant, the NNLA terms should be large and of the opposite sign with respect to the NLA.

If the NNLA corrections were known and we would apply PMS to the NNLA amplitude, we would obtain more natural values of μ_R.

Finally,

$\frac{Q^2}{D_1 D_2} \frac{\mathcal{I}m_s \mathcal{A}}{s}$ *vs* Y

Q^2=24 GeV^2, $n_f = 5$



Lessons

- The Born approximation does not give necessarily the estimate from below.
- The optimal values for μ_R are "unnaturally" larger than Q (new scale or nature of the BFKL series?).

$\longrightarrow$ Since NLA corrections are large and since the exact amplitude should be renorm- and energy scale invariant, the NNLA terms should be large and of the opposite sign with respect to the NLA.

If the NNLA corrections were known and we would apply PMS to the NNLA amplitude, we would obtain more natural values of μ_R.

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“Collinear factorization” regime

$\gamma^*(Q_1)\gamma^*(Q_2) \to V\,V$ for strongly ordered photon virtualities, $Q_1^2 \gg Q_2^2$

$$\frac{Q_1 Q_2}{D_1 D_2}\frac{\mathcal{I}m_s\mathcal{A}}{s} = \frac{1}{(2\pi)^2}\alpha_s(\mu_R)^2\left[b_0 + \sum_{n=1}^{\infty}\bar{\alpha}_s(\mu_R)^n\, b_n\left(\ln\left(\frac{s}{s_0}\right)^n + d_n(s_0,\mu_R)\ln\left(\frac{s}{s_0}\right)^{n-1}\right)\right]$$

$$\frac{b_n}{Q_1 Q_2} = \int_{-\infty}^{+\infty} d\nu\, c_1(\nu)c_2(\nu)\frac{\chi^n(\nu)}{n!}$$

$$c_1(\nu)c_2(\nu) = \frac{1}{Q_1 Q_2}\left(\frac{Q_1^2}{Q_2^2}\right)^{i\nu}\frac{9\,\pi^3(1+4\nu^2)\sinh(\pi\nu)}{32\,\nu\,(1+\nu^2)\cosh^3(\pi\nu)}$$

$$\chi(\nu) = 2\psi(1) - \psi\left(\frac{1}{2}+i\nu\right) - \psi\left(\frac{1}{2}-i\nu\right)$$

$d_n(s_0,\mu_R)$ is the (much) more complicated expression involving the NLA impact factors and the NLA BFKL eigenvalues in the ν-representation

- Leading-twist approximation: close the ν-integration contour in the upper plane and take only the residue at the pole $\nu = i/2$.

- LLA: near the pole, $\nu = \frac{i}{2} + \Delta$

$$c_1(\Delta)c_2(\Delta) = \left(\frac{Q_1^2}{Q_2^2}\right)^{i\Delta} \frac{1}{Q_1^2}\left(\frac{-3}{\Delta^2} + \frac{i}{\Delta} + \frac{34}{3} + O(\Delta)\right)$$

$$\chi(\Delta) = -\frac{i}{\Delta} + O(\Delta^2)$$

$$\text{Residue}_{\Delta=0}\left(\frac{(Q_1^2/Q_2^2)^{i\Delta}}{\Delta^n}\right) = i^{n-1}\frac{\log^{n-1}(Q_1^2/Q_2^2)}{(n-1)!}$$

$$b_n = \frac{2\pi}{3}\frac{Q_2}{Q_1}L^{n-1}\frac{9L^2 - 3L(n+1) + 34n(n+1)}{n!(n+1)!} + O(L^{n-2})$$

$$L \equiv \log(Q_1^2/Q_2^2)$$

In the NLA we have contributions from the kernel and from the impact factors.

- From the kernel, among others

$$\bar{\chi}(\Delta) = -\frac{i}{2\Delta^3} + \frac{1}{6}\left(\frac{11}{2} + \frac{n_f}{N^3}\right)\frac{1}{\Delta^2} + \frac{i}{36}\left(55 + \frac{13 n_f}{N^3}\right)\frac{1}{\Delta} + O(\Delta^0)$$

- In the NLA impact factors are very complicated, however

$$c_1^{(1)}(\nu) = \int d^2\vec{q}\; C_1^{(1)}(\vec{q}^{\,2})\frac{(\vec{q}^{\,2})^{i\nu-\frac{3}{2}}}{\pi\sqrt{2}} = \int d^2\vec{q}\; C_1^{(1)}(\vec{q}^{\,2})\frac{(\vec{q}^{\,2})^{-2+i\Delta}}{\pi\sqrt{2}}$$

near the pole ($\nu = \frac{i}{2} + \Delta$); only the small-$\vec{q}^{\,2}$ limit of $C_1^{(1)}(\vec{q}^{\,2})$ needed

Analogously,

$$c_2^{(1)}(\nu) = \int d^2\vec{q}\; C_2^{(1)}(\vec{q}^{\,2})\frac{(\vec{q}^{\,2})^{-i\nu-\frac{3}{2}}}{\pi\sqrt{2}} = \int d^2\vec{q}\; C_2^{(1)}(\vec{q}^{\,2})\frac{(\vec{q}^{\,2})^{-1-i\Delta}}{\pi\sqrt{2}}$$

near the pole ($\nu = \frac{i}{2} + \Delta$); only the large-$\vec{q}^{\,2}$ limit for $C_2^{(1)}(\vec{q}^{\,2})$ needed

$$b_n d_n = 2\pi \frac{Q_2}{Q_1} \frac{1}{(n!)^2} \left\{ L^{n+1} \left[\frac{3\beta_0}{8N} \frac{n(n^2+n+2)}{n+1} - \frac{n}{n+1} \left(\frac{3}{4N^2} + \frac{17}{4} + \frac{11n}{4} \right) - \frac{n_f}{N^3} \frac{n(n-1)}{2(n+1)} \right.\right.$$

$$\left. + \frac{3n}{(n+1)} \log \frac{s_0}{Q_1^2} \right] + L^n \left[-n \log \frac{s_0}{Q_1^2} - \frac{11n(n-1)}{12} \frac{n_f}{N^3} + \frac{\beta_0}{4N} \left(-3n(n+1) \log \frac{Q_1^2}{\mu_R^2} \right.\right.$$

$$\left.\left.\left. -n \left(\frac{n(n-21)}{2} - 10 \right) \right) - n \left(\frac{19}{6} + \frac{11n}{3} + \frac{\pi^2}{2} - \frac{51}{24N^2} \right) \right] \right\}$$

$$L \equiv \log(Q_1^2/Q_2^2)$$

- Terms with $\alpha_s(\mu_R)^{n+2} L^{n+2}$ canceled out, as it should be
- The same procedure can be applied to the case of the $\gamma^*\gamma^*$ total cross section

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Analysis of systematic effects

There are two main sources of systematic effects in the numerical determinations performed so far:

- change of representation of the NLA amplitude
- change of the method of optimization of the perturbative series

So far, we have considered series representation and PMS optimization method, at $Q_1^2 = Q_2^2 = Q^2$=24 GeV^2, $n_f = 5$.

In the following we take into account also

- "exponentiated" representation of the NLA amplitude
- other optimization methods: FAC, BLM

“Exponentiated” representation of the NLA amplitude

$$\frac{\mathcal{I}m_s(\mathcal{A})}{D_1 D_2} = \frac{s}{(2\pi)^2} \int_{-\infty}^{+\infty} d\nu \left(\frac{s}{s_0}\right)^{\bar{\alpha}_s(\mu_R)\chi(\nu)+\bar{\alpha}_s^2(\mu_R)\left(\bar{\chi}(\nu)+\frac{\beta_0}{8N_c}\chi(\nu)\left[-\chi(\nu)+\frac{10}{3}\right]\right)}$$

$$\times \alpha_s^2(\mu_R) c_1(\nu) c_2(\nu) \left[1 + \bar{\alpha}_s(\mu_R)\left(\frac{c_1^{(1)}(\nu)}{c_1(\nu)} + \frac{c_2^{(1)}(\nu)}{c_2(\nu)}\right)\right.$$

$$\left. + \bar{\alpha}_s^2(\mu_R) \ln\left(\frac{s}{s_0}\right) \frac{\beta_0}{8N_c}\chi(\nu)\left(i\frac{d\ln(\frac{c_1(\nu)}{c_2(\nu)})}{d\nu} + 2\ln(\mu_R^2)\right)\right]$$

By “exponentiation” we mean “transfer of an effect from the kernel to the Green’s function” or, equivalently, account of that effect “to all orders".

The last term in the above equation originates from the scale non-invariant part of the NLA kernel, i.e. from the running of the coupling, and is not exponentiated.

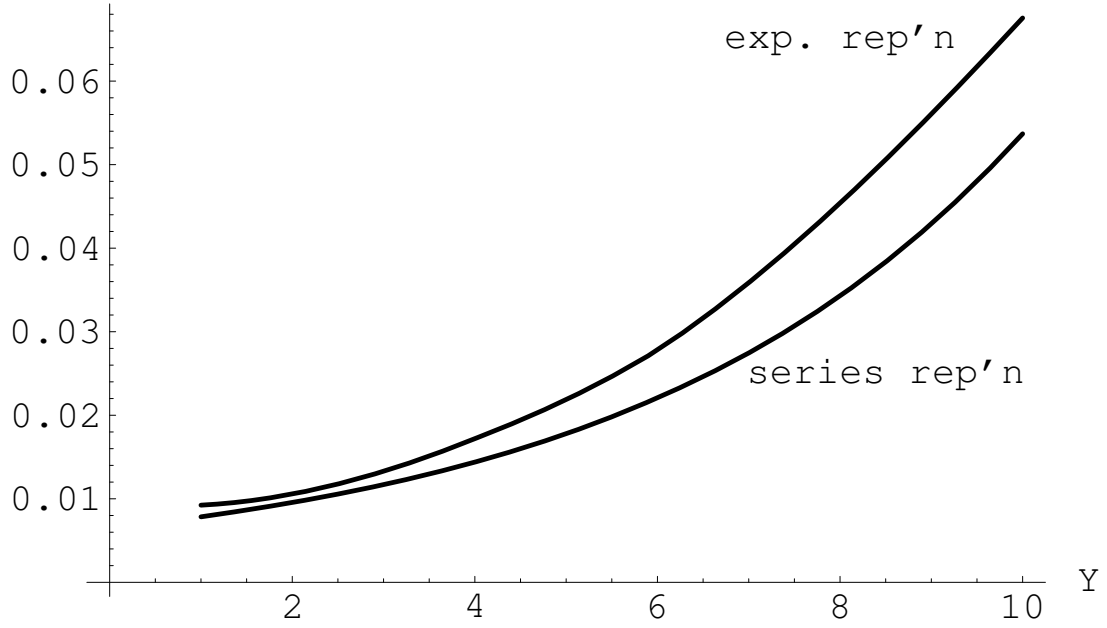
An exact account of the derivative term in the BFKL equation leads to a radical change of both the spectrum and the eigenfunctions of the NLA BFKL kernel.

[N. Armesto, J. Bartels and M. A. Braun (1998)]

PMS method - Exponentiated representation

$\frac{Q^2}{D_1 D_2} \frac{\mathcal{I}m_s \mathcal{A}}{s}$ *vs* Y

Q^2=24 GeV^2, $n_f = 5$



- Rough agreement with the PMS method applied to the series representation.
- The optimal value for μ_R is slightly smaller than in the previous case.

FAC method

- Fast apparent convergence (FAC) [G. Grunberg (1980)]:
  require that the NLO corrections identically vanish.

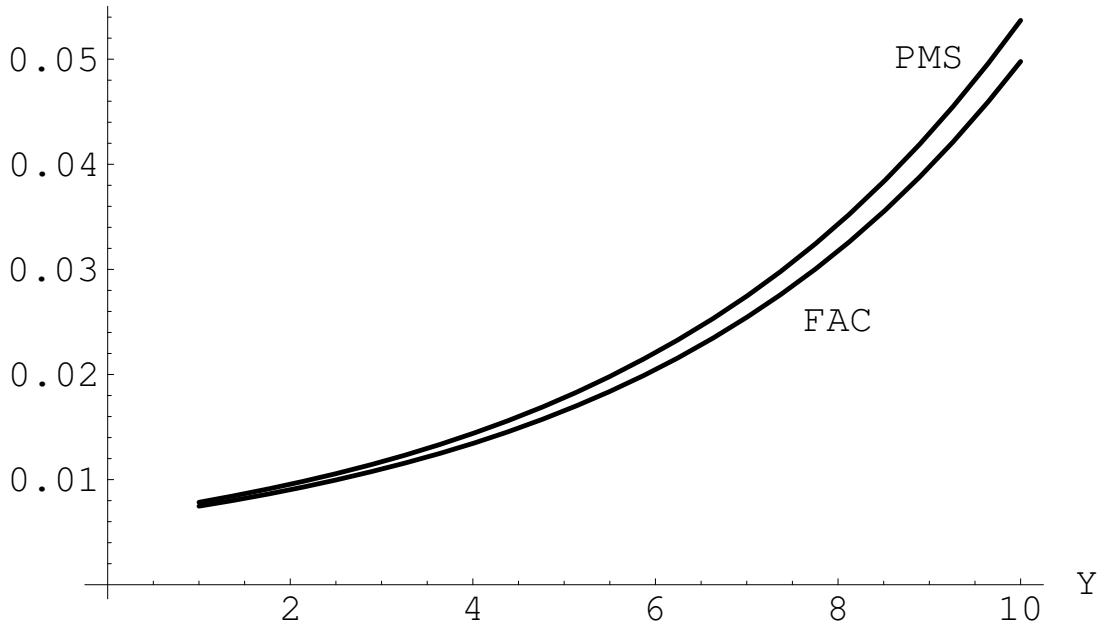
- Strategy: for each fixed Y calculate determine the line of Y_0 and μ_R values for which the NLO corrections cancel; then, the optimal values of Y_0 and μ_R along this line are chosen according to “minimum sensitivity”.

FAC method - Series representation

$$\frac{Q^2}{D_1 D_2}\frac{\mathcal{I}m_s\mathcal{A}}{s} = \frac{1}{(2\pi)^2}\alpha_s(\mu_R)^2\left[b_0+\sum_{n=1}^{\infty}\bar{\alpha}_s(\mu_R)^n\, b_n\left((Y-Y_0)^n+d_n(s_0,\mu_R)(Y-Y_0)^{n-1}\right)\right]$$

$\frac{Q^2}{D_1 D_2}\frac{\mathcal{I}m_s\mathcal{A}}{s}$ *vs* Y

Q^2=24 GeV^2, $n_f = 5$



- Despite the very different strategy, FAC and PMS give quite consistent results

FAC method - Exponentiated representation

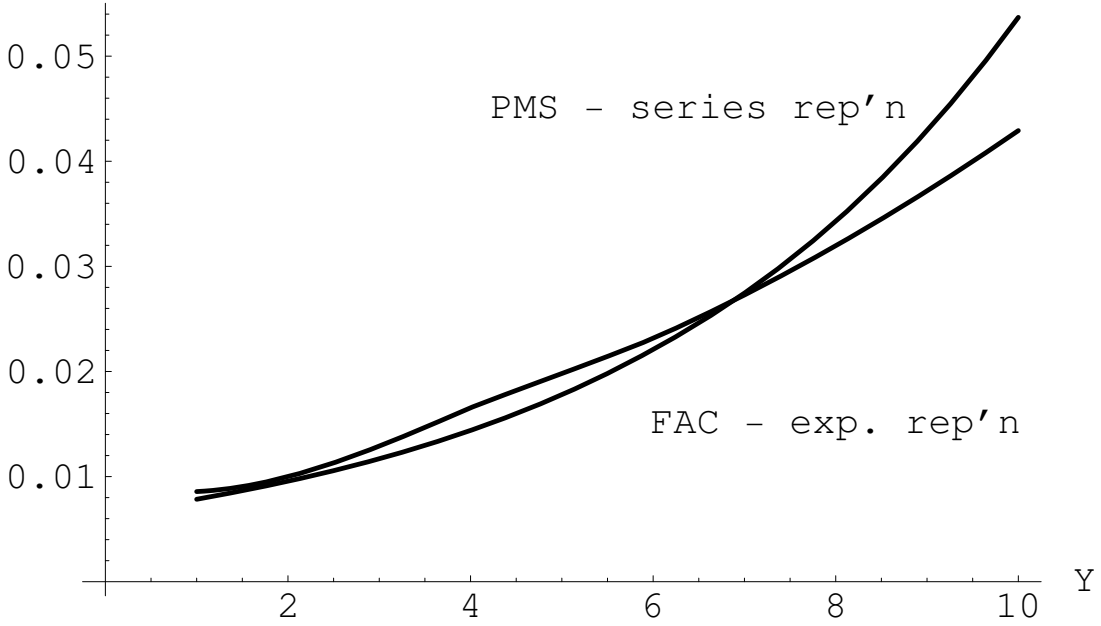
$$\frac{\mathcal{I}m_s\left(\mathcal{A}_{NLA}\right)}{D_1 D_2} = \frac{s}{(2\pi)^2}\int\limits_{-\infty}^{+\infty} d\nu \left(\frac{s}{s_0}\right)^{\bar{\alpha}_s(\mu_R)\chi(\nu)+\bar{\alpha}_s^2(\mu_R)\left(\bar{\chi}(\nu)+\frac{\beta_0}{8N_c}\chi(\nu)\left[-\chi(\nu)+\frac{10}{3}\right]\right)}$$

$$\times\alpha_s^2(\mu_R)c_1(\nu)c_2(\nu)\left[1+\bar{\alpha}_s(\mu_R)\left(\frac{c_1^{(1)}(\nu)}{c_1(\nu)}+\frac{c_2^{(1)}(\nu)}{c_2(\nu)}\right)\right.$$

$$\left.+\bar{\alpha}_s^2(\mu_R)\ln\left(\frac{s}{s_0}\right)\frac{\beta_0}{8N_c}\chi(\nu)\left(i\frac{d\ln(\frac{c_1(\nu)}{c_2(\nu)})}{d\nu}+2\ln(\mu_R^2)\right)\right]$$

$$\frac{\mathcal{I}m_s\left(\mathcal{A}_{LLA}\right)}{D_1 D_2} = \frac{s}{(2\pi)^2}\int\limits_{-\infty}^{+\infty} d\nu \left(\frac{s}{s_0}\right)^{\bar{\alpha}_s(\mu_R)\chi(\nu)}\alpha_s^2(\mu_R)c_1(\nu)c_2(\nu)$$

$$\mathcal{I}m_s\left(\mathcal{A}_{NLA}\right) = \mathcal{I}m_s\left(\mathcal{A}_{LLA}\right) + \left[\mathcal{I}m_s\left(\mathcal{A}_{NLA}\right) - \mathcal{I}m_s\left(\mathcal{A}_{LLA}\right)\right]$$



$\frac{Q^2}{D_1 D_2}\frac{\mathcal{I}m_s\mathcal{A}}{s}$ vs Y

Q^2=24 GeV^2, $n_f = 5$

BLM method

- [S.J. Brodsky, G.P. Lepage, P.B. Mackenzie (1983)] optimization method: perform a finite renormalization to a physical scheme and then choose the renormalization scale in order to remove the β_0-dependent part.

Strategy (applied to the amplitude in the series representation only):

- finite renormalization to the MOM-scheme (ξ=0)

$$\alpha_S \to \alpha_S \left[1 + T_{MOM}(\xi = 0)\frac{\alpha_S}{\pi}\right]$$

$$T_{MOM}(\xi = 0) = T^{conf}_{MOM} + T^{\beta}_{MOM}$$

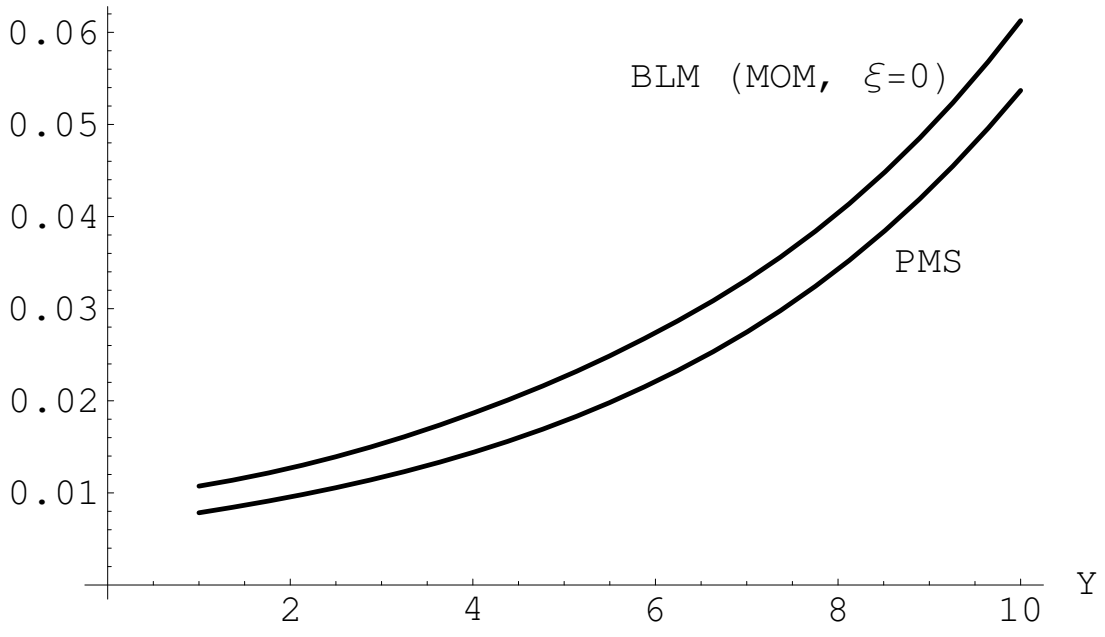
$$T^{conf}_{MOM} = \frac{N_C}{8}\frac{17}{2}I \qquad T^{\beta}_{MOM} = -\frac{\beta_0}{2}\left[1 + \frac{2}{3}I\right] \qquad I \simeq 2.3439$$

- Y_0 and μ_R chosen in order to make the term proportional to β_0 in the resulting amplitude vanish (the β_0-dependence in the series representation of the amplitude is hidden into the d_n coefficients)
- optimal values for Y_0 and μ_R determined according to “minimum sensitivity”

BLM method - Series representation

$\frac{Q^2}{D_1 D_2} \frac{\mathcal{I}m_s \mathcal{A}}{s}$ *vs* Y

Q^2=24 GeV^2, $n_f = 5$



- Drawback: for each given Y, Y_0 "wants" to be as large as Y.

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Collinear improvement of the BFKL kernel

Main idea: compare the NLA BFKL amplitude for deep inelastic scattering with the DGLAP formula and deduce the subleading terms to be added to the NLA BFKL Green's function. [G.P. Salam (1998)]

If these terms catch the essential subleading dynamics, their inclusion must stabilize the perturbative series.

The $\gamma^* \gamma^* \to V\ V$ forward amplitude can be used as a test-field for this method.

Collinear improvement in the LLA

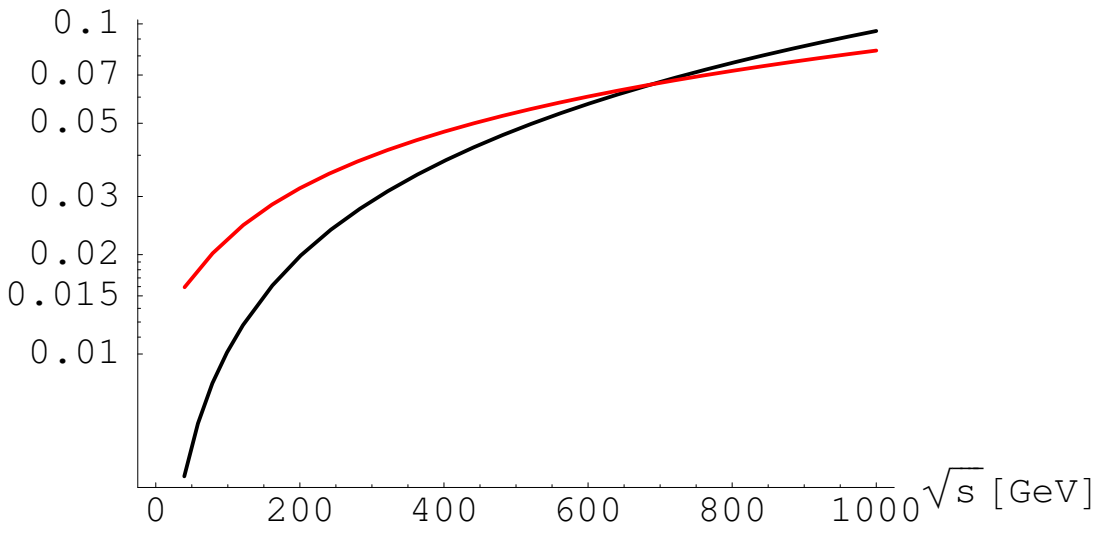
[R. Enberg, B. Pire, L. Szymanowsky, S. Wallon (2005)] made an estimate of NLA effects for the same amplitude, using the following ingredients:

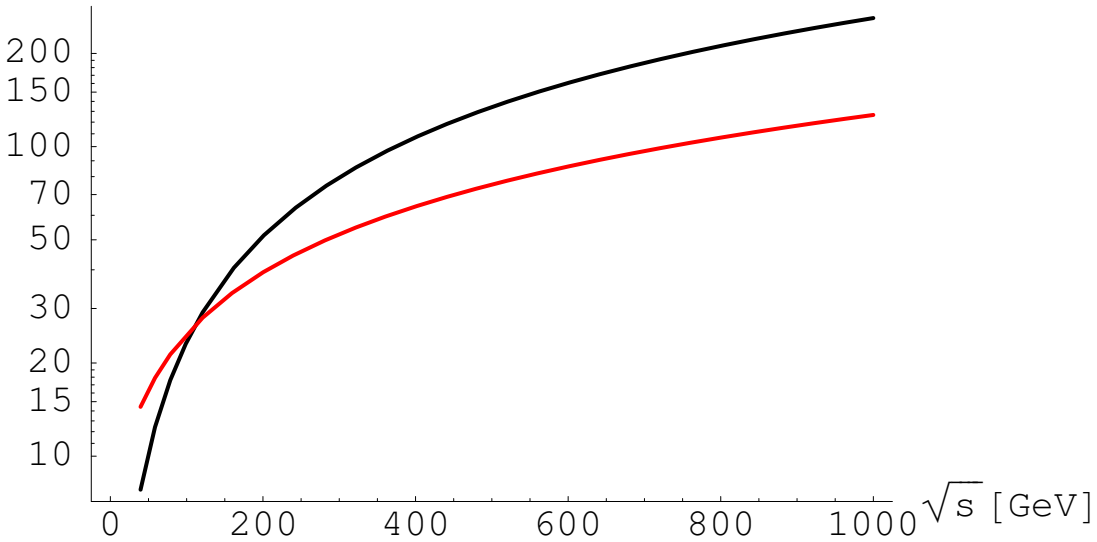
- LO impact factors for the $\gamma^* \to V$ transition;
- BLM scale fixing for the running of the coupling in the prefactor of the amplitude; the BLM scale is found using the NLO $\gamma^* \to V$ impact factors;
- RG-resummed BFKL kernel; the resummation is performed on the LLA BFKL kernel at fixed coupling

[V.A. Khoze, A.D. Martin, M.G. Ryskin, W.J. Stirling (2004)]

$\left.\frac{d\sigma}{dt}\right|_{t_{min}}$ in pb $\cdot$ GeV^{-2}

Q^2=16 GeV^2 $\longrightarrow$, $n_f = 4$





$\longleftarrow$ Q^2=4 GeV^2, $n_f = 3$

Courtesy R. Enberg for the curves in red.

Collinear improvement in the NLA $\longrightarrow$ F. Caporale's talk

- General tendency of the optimal values towards "naturalness"
- Improved stability under change of the representation of the amplitude
- Reasonable agreement with unimproved determinations
- Numerical determinations feasible even for strongly ordered photon virtualities

Conclusions

- Closed analytical expression found for the $\gamma^*\gamma^* \to V\,V$ forward amplitude in the Regge limit of QCD with next-to-leading order accuracy
- For equal photons' virtualities, i.e. in the BFKL regime
  - the next-to-leading order corrections are large and of opposite sign with respect to the leading order contribution
  - the PMS optimization method allows to get stable results for the amplitude; the optimal value of the renormalization scale μ_R turns out to be much larger than the kinematical scale of the problem; this could be a manifestation of the nature of the BFKL series
  - the energy dependence of the amplitude is reasonably stable under change of representation (exponentiated vs series) or of optimization method (FAC, BLM)
- For strongly ordered photons' virtualities,
  - the structure of the amplitude is compatible with the leading-twist collinear factorization
  - numerical determinations are feasible if collinearly improved kernels are used