Jets and Hadronic Final States

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Lectures III, IV

OUTLINE, L. III, IV

Lecture III

A. Reprise: how we get away with pQCD

B. Vector bosons: Q_T, its factorization & resummation

C. Threshold resummation

Lecture IV

A. Resummed jet shapes and power corrections

B. Evolution with Color Exchange

C. Generalizations and limitations

- Resummation: Organization of soft and collinear radiation to all orders in PT

- Why resum?

  - Tests of perturbative stability for inclusive cross sections.
  - The only way to calculate certain critical distributions for W, Z transverse momentum & jet event shapes.
  - As such, tests of QCD to *all* orders: LO, NLO . . .
  - A window to the perturbative/nonperturbative transition.
  - An analytic complement to, stimulus for & test of parton shower techniques and tools.
  - Nice formulas (a matter of taste).

- Depends on very some general concepts too.

IIIA. Reprise: how we get away with perturbative QCD

The sorrows of QCD perturbation theory:

1. Confinement

$$\int \mathrm{e}^{-iq\cdot x}\langle 0|\, T[\phi_a(x)\ldots]\,|0\rangle$$

has no $q^2 = m^2$ pole for any field (particle) ϕ_a that transforms nontrivially under color (confinement)

2. The pole at $p^2 = m_\pi^2$

$$\int \mathrm{e}^{-iq\cdot x}\langle 0|\, T[\pi(x)\ldots]\,|0\rangle$$

is not accessible to perturbation theory (χSB etc., etc.)

- **And yet we use infrared safety & asymptotic freedom:**

$$Q^2\ \hat{\sigma}_{\rm SD}(Q^2,\mu^2,\alpha_s(\mu)) = \sum_n c_n(Q^2/\mu^2)\ {\alpha_s}^n(\mu) + \mathcal{O}\left(1/Q^p\right)$$
$$= \sum_n c_n(1)\ {\alpha_s}^n(Q) + \mathcal{O}\left(1/Q^p\right)$$

- **What are we really calculating? PT for color singlet operators**

  – $\int \mathrm{e}^{-iq\cdot x}\langle 0|\, T[J(x)J(0)\ldots]\,|0\rangle$ **for color singlet currents**

  $\mathrm{e}^+\mathrm{e}^-$ **total, sum rules etc. "no scale"**

– Another class of color singlet matrix elements:

$$\lim_{R\to\infty}\int dx_0\int d\hat{n}\, f(\hat{n})\, \mathrm{e}^{-iq\cdot y}\langle 0|\, J(0)T[\hat{n}_i\Theta_{0i}(x_0,R\hat{n})J(y)]\,|0\rangle$$

With Θ_{0i} the energy momentum tensor

– These are what we really calculate: jet cross sections, etc.

If the "weight" $f(\hat{n})$ introduces no new dimensional scale,

and all $d^k f/d\hat{n}^k$ bounded, then

individual final states have IR divergences, but these cancel in sum over collinear splitting/merging & soft parton emission because they respect energy flow.

We regularize these divergences dimensionally (typically) and “pretend” to calculate the long-distance enhancements only to cancel them in infrared safe quantities

It is this intermediate step that makes the calcualtions tough, and is part [not all] of why higher-order calculations are hard!

The goals of experiment are remarkably similar – to control late stage interactions in calorimeters.

Resummation organizes large, or potentially large, terms from high orders in α_s at the short-distance scale.

- **Onward to factorization**

$$Q^2\sigma_{\rm phys}(Q,m) = \omega_{\rm SD}(Q/\mu, \alpha_s(\mu)) \otimes f_{\rm LD}(\mu, m) + \mathcal{O}\left(1/Q^p\right)$$

  - μ **= factorization scale;** m**= IR scale (**m may **be perturbative)**
  - **New physics in** $\omega_{\rm SD}$**;** $f_{\rm LD}$ **"universal"**
  - **ep DIS inclusive,** $\rm pp \to$ **jets,** $Q\bar{Q}$**,** $\pi(p_T)$ **. . .**
  - **Exclusive decays:** $B \to \pi\pi$
  - **Exclusive limits:** $\rm e^+e^- \to JJ$ **as** $m_J \to 0$

- **Whenever there is factorization, there is evolution**

$$0 = \mu \frac{d}{d\mu} \ln \sigma_{\rm phys}(Q, m)$$

$$\mu \frac{d \ln f}{d\mu} = -P(\alpha_s(\mu)) = -\mu \frac{d \ln \omega}{d\mu}$$

PDF f **<u>or</u> Fragmentation** D

- **Wherever there is evolution there is resummation**

$$\ln \sigma_{\rm phys}(Q, m) = \exp \left\{ \int_q^Q \frac{d\mu'}{\mu'} P\left(\alpha_s(\mu')\right) \right\}$$

- **Infrared safety & factorization proofs:**
  - **(1) ω_{SD} incoherent with long-distance dynamics**
  - **(2) Mutual incoherence when $v_{\mathrm{rel}} = c$:**
    Jet-jet factorization Ward identities.
  - **(3) Wide-angle soft radiation sees only total color flow:**
    jet-soft factorization Ward identities.
  - **(4) Dimensionless coupling and renormalizability**
    $\Leftrightarrow$ no worse that logarithmic divergence in the IR:
    fractional power suppression $\Rightarrow$ finiteness

IIIB. Vector bosons: Q_T, its factorization

Every final state from a hard scattering carries the imprint of QCD dynamics from all distance scales.

– **Look at transverse momentum distribution at order α_s.**

$$q(p_1) + \bar{q}(p_2) \longrightarrow \gamma^*(Q) + g(k)$$

– **Treat this $2 \to 2$ process at lowest order (α_s) "LO" in factorized cross section, so that $\mathbf{k} = -\mathbf{Q}_T$.**

– **Factorized cross section at fixed $\mathbf{Q}_T$:**

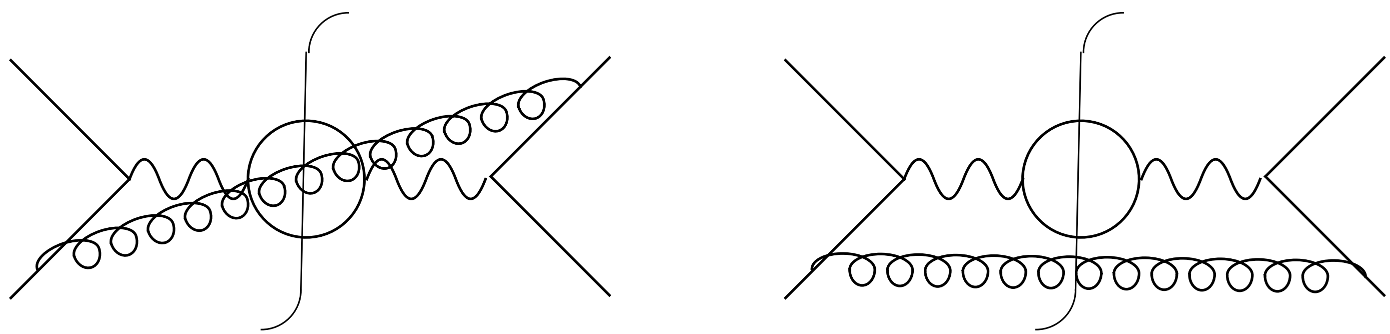
$$\frac{d\sigma_{NN\to\mu^+\mu^-+X}(Q,p_1,p_2)}{dQ^2 d^2\mathbf{Q}_T} = \int_{\xi_1,\xi_2} \sum_{a=q\bar{q}} \frac{d\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu,\xi_1 p_1,\xi_2 p_2,\mathbf{Q}_T)}{dQ^2 d^2\mathbf{Q}_T} \times f_{a/N}(\xi_1,\mu)\, f_{\bar{a}/N}(\xi_2,\mu)$$

– **Recall:** μ **is the factorization scale that separates IR (f) from UV ($d\hat{\sigma}$) in quantum corrections.**

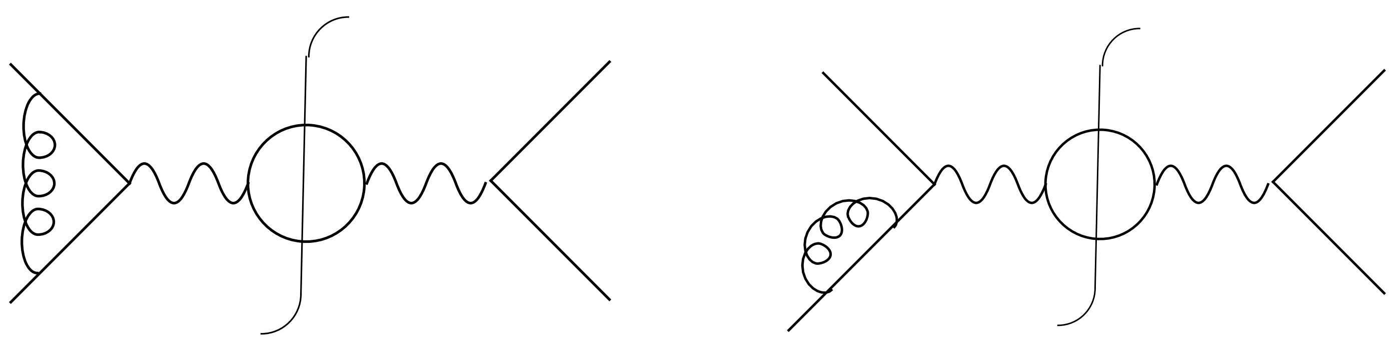
– μ **appears in $\hat{\sigma}$ through $\alpha_s(\mu)$ and $\ln(\mu/Q)$, so choosing $\mu \sim Q$ can improve perturbative predictions.**

– **Evolution:** $\mu df(x,\mu)/d\mu = \int_x^1 P(x/\xi)\, f(\xi,\mu)$ **makes energy extrapolations possible.**

– **The diagrams at order α_s.**
Gluon emission contributes at $Q_T \neq 0$.



Virtual corrections contribute only at $Q_T = 0$



– **The result is finite for $\mathbf{Q}_T \neq 0$ . . .**

$$\frac{d\hat{\sigma}^{(1)}_{q\bar{q}\to\gamma^* g}}{dQ^2\, d^2\mathbf{Q}_T} = \sigma_0 \frac{\alpha_s C_F}{\pi^2} \left(1 - \frac{4\mathbf{Q}_T^2}{(1-z)^2 \xi_1 \xi_2 S}\right)^{-1/2}$$

$$\times \left[\frac{1}{\mathbf{Q}_T^2}\frac{1+z^2}{1-z} - \frac{2z}{(1-z)Q^2}\right]$$

as long as $\mathbf{Q}_T \neq 0$, $z = Q^2/\xi_1\xi_2 S \neq 1$.

Q_T **integral** $\to \frac{\ln(1-z)}{1-z}$; z **integral** $\to \frac{\ln \mathbf{Q}_T^2}{\mathbf{Q}_T^2}$.

Both singularities cancel in the inclusive cross section.
Both inspire resummation of higher order corrections.

The leading singularity in $\mathbf{Q}_T$

– **As we'll see later:** $1 - z \sim 2k_0/Q \geq 2|\mathbf{k}_T|/Q$

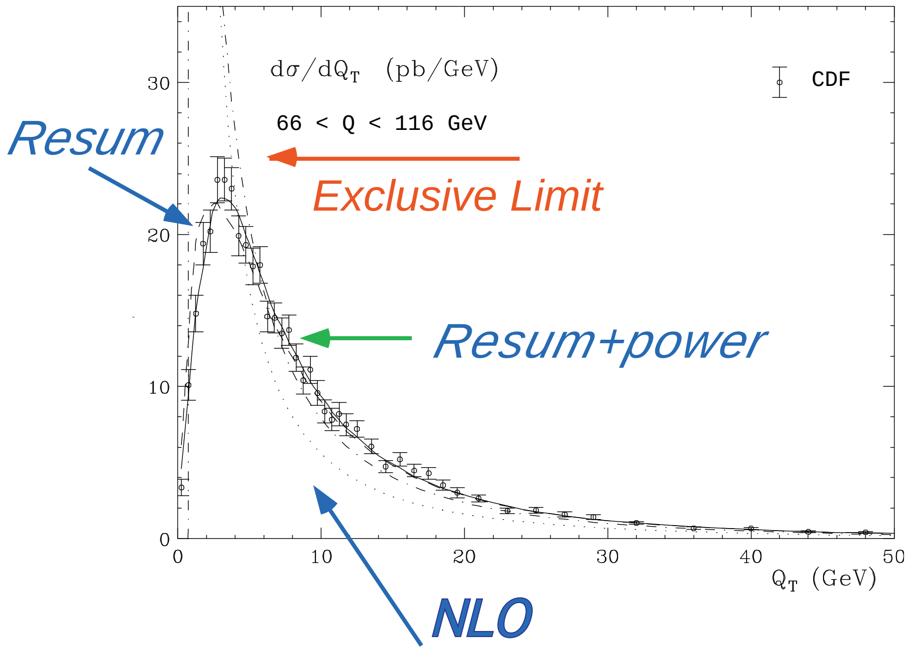
– z **integral: If Q^2/S not too big, PDFs nearly constant:**

$$\frac{1}{\mathbf{Q}_T^2} \int_{1-Q^2/S}^{1-\mathbf{Q}_T^2/Q^2} \frac{dz}{1-z} = \frac{1}{\mathbf{Q}_T^2} \ln\left[\frac{Q^2}{\mathbf{Q}_T^2}\right]$$

$\Rightarrow$ **Prediction for Q_T dependence:**

$$\frac{d\sigma_{NN\to\mu^+\mu^-+X}(Q,\mathbf{Q}_T)}{dQ^2 d^2\mathbf{Q}_T} = \frac{\alpha_s C_F}{\pi} \frac{1}{\mathbf{Q}_T^2} \ln\left[\frac{Q^2}{\mathbf{Q}_T^2}\right]$$
$$\times \sum_{a=q\bar{q}} \int_{\xi_1 \xi_2} \frac{\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu)}{dQ^2} f_{a/N}(\xi_1,\mu)\, f_{\bar{a}/N}(\xi_2,\mu)$$

– Compare to: Z p_T from Run I



(from Kulesza, G.S., Vogelsang (2002))

– $\ln Q_T/Q_T$ works pretty well for large Q_T

– At smaller Q_T reach a maximum, then a decrease near "exclusive" limit (parton model kinematics)

– Most events are at "low" $Q_T \ll Q = m_Z$.

Getting to $Q_T \ll Q$: Transverse momentum resummation

(Logs of Q_T)/Q_T to all orders

How? Variant factorization and separation of variables

q and $\bar{q}$ "arrive" at point of annihilation with transverse momentum of radiated gluons in initial state.

q and $\bar{q}$ radiate independently (fields don't overlap!).

Final-state QCD radiation too late to affect cross section

$$\frac{d\sigma_{NN\to\mu^+\mu^-+X}(Q,\mathbf{Q}_T)}{dQ^2 d^2\mathbf{Q}_T}$$

Summarized by: Q_T-factorization:

$$\frac{d\sigma_{NN\to QX}}{dQd^2Q_T} = \int d\xi_1 d\xi_2\ d^2\mathbf{k}_{1T} d^2\mathbf{k}_{2T} d^2\mathbf{k}_{sT}\,\delta\left(Q_T - k_{1T} - k_{2T} - k_{sT}\right)$$
$$\times\ H(\xi_1 p_1, \xi_2 p_2, Q, n)_{a\bar{a}\to Q+X}$$
$$\times \mathcal{P}_{a/N}(\xi_1, p_1\cdot n, k_{1T})\,\mathcal{P}_{\bar{a}/N}(\xi_2, p_2\cdot n, k_{2T})\ U_{a\bar{a}}(k_{sT}, n)$$

The $\mathcal{P}'s$: new Transverse momentum-dependent PDFs

Also need U: "soft function" for wide-angle radiation.

– **Caution: Extensions to less inclusive cross sections are highly nontrivial.** **(viz: Collins and Qiu (2007))**

Symbolically:

$$\frac{d\sigma_{NN\to QX}}{dQd^2Q_T} \qquad H\times\mathcal{P}_{a/N}(\xi_1,p_1\cdot n,k_{1T})\,\mathcal{P}_{\bar{a}/N}(\xi_2,p_2\cdot n,k_{2T})$$
$$\otimes_{\xi_i,k_{iT}}\,U_{a\bar{a}}(k_{sT},n)$$

We will solve for the k_T dependence of the $\mathcal{P}$'s.

New factorization variables: n^μ apportions gluons k:

$$p_i\cdot k < n\cdot k \;\Rightarrow\; k\;\in\mathcal{P}_i$$
$$p_a\cdot k, p_{\bar{a}}\cdot k \;>\; n\cdot k \;\Rightarrow\; k\in U$$

Convolution in $k_{i,T}$s $\Rightarrow$ Fourier $\mathrm{e}^{i\vec{Q}_T\cdot\vec{b}}$

The factorized cross section in "impact parameter space":

$$\frac{d\sigma_{NN\to QX}(Q,b)}{dQ} = \int d\xi_1 d\xi_2$$
$$\times\ H(\xi_1 p_1, \xi_2 p_2, Q, n)_{a\bar{a}\to Q+X}$$
$$\times \mathcal{P}_{a/N}(\xi_1, p_1\cdot n, b)\, \mathcal{P}_{\bar{a}/N}(\xi_2, p_2\cdot n, b)\ U_{a\bar{a}}(b,n)$$

Now we can resum by separating variables!

the LHS independent of $\mu_{\rm ren}$, n $\Rightarrow$ two equations

$$\mu_{\rm ren}\frac{d\sigma}{d\mu_{\rm ren}} = 0 \qquad n^\alpha \frac{d\sigma}{dn^\alpha} = 0$$

Method of Collins and Soper, and Sen (1981)

Change in $\mathcal{P}$ must cancel change in (UV) H and (IR) U:

$$p\cdot n\,\frac{\partial}{\partial p\cdot n}\,\ln\,\mathcal{P}(p\cdot n/\mu, b\mu) = G(p\cdot n/\mu) + K(b\mu)$$

G matches H, K matches U. Renormalization indep. of n^{μ}:

$$\mu\,\frac{\partial}{\partial\mu}\left[\,G(p\cdot n/\mu) + K(b\mu)\,\right] = 0$$

$$\mu\,\frac{\partial}{\partial\mu}\,G(p\cdot n/\mu) \;=\; A(\alpha_s(\mu)) \;=\; -\,\mu\,\frac{\partial}{\partial\mu}\,K(b\mu)$$

Solve this one first.

$$G(p\cdot n/\mu) + K(b\mu) = G(p\cdot n/\mu) + K(\mu/p\cdot n)$$
$$- \int_{1/b}^{p\cdot n} \frac{d\mu'}{\mu'} A_a(\alpha_s(\mu'))$$

Notice the scale in the coupling is now a variable.

The consistency equation for the jet becomes

$$p\cdot n \frac{\partial}{\partial p\cdot n} \ln \mathcal{P}(p\cdot n/\mu, b\mu) = G(p\cdot n/\mu) + K(\mu/p\cdot n)$$
$$- \int_{1/b}^{p\cdot n} \frac{d\mu'}{\mu'} A(\alpha_s(\mu'))$$

Integrate $p\cdot n$ **and get double logs in** $b \to \alpha_s^n \frac{\ln^{2n-1}(Q/Q_T)}{Q_T}$**.**

Transformed solution back to Q_T: all the (Logs of Q_T)/Q_T:

$$\frac{d\sigma_{NN\mathrm{res}}}{dQ^2\, d^2\vec{Q}_T} = \sum_a H_{a\bar{a}}(\alpha_s(Q^2)) \int \frac{d^2 b}{(2\pi)^2}\, e^{i\vec{Q}_T\cdot\vec{b}} \exp\left[\, E^{\mathrm{PT}}_{a\bar{a}}(b,Q,\mu)\,\right]$$

$$\times \sum_{a=q\bar{q}} \int_{\xi_1\xi_2} \frac{d\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu)}{dQ^2}\, f_{a/N}(\xi_1,1/b)\, f_{\bar{a}/N}(\xi_2,1/b)$$

"Sudakov" exponent suppresses large $b \leftrightarrow$ small Q_T:

$$E^{\mathrm{PT}}_{a\bar{a}} = -\int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2} \left[2A_q(\alpha_s(k_T))\, \ln\left(\frac{Q^2}{k_T^2}\right) + 2B_q(\alpha_s(k_T))\right]$$

With $B = 2(K+G)_{\mu=p\cdot n}$, and lower limit: $1/b$ (NLL)

$*$ **Leading log: fixed $\alpha_s(Q)$, $A = (\alpha_s/\pi) \times A^{(1)}$ only**

$$\frac{d\sigma_{NN\mathrm{res}}}{dQ^2\, d^2\vec{Q}_T} = \sum_a H_{a\bar{a}}(\alpha_s(Q^2)) \int \frac{d^2b}{(2\pi)^2}\, e^{i\vec{Q}_T\cdot\vec{b}} \exp\left[-\, A^{(1)}(\alpha_s(Q)/\pi)\, \ln^2(bQ)\right]$$

$$\times \sum_{a=q\bar{q}} \int_{\xi_1\xi_2} \frac{d\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu)}{dQ^2}\; f_{a/N}(\xi_1, 1/b)\, f_{\bar{a}/N}(\xi_2, 1/b)$$

$*$ **If ignore evolution of the f's, get an overall factor**

$$\frac{d\sigma_{NN\to\mu^+\mu^-+X}(Q,\mathbf{Q}_T)}{dQ^2 d^2\mathbf{Q}_T} = \frac{\partial}{\partial Q_T^2}\, e^{-\left[\, A^{(1)}(\alpha_s(Q)/\pi)\, \ln^2(Q^2/Q_T^2)\right]}$$

$$\times \sum_{a=q\bar{q}} \int_{\xi_1\xi_2} \frac{\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu)}{dQ^2}\; f_{a/N}(\xi_1, \mu)\, f_{\bar{a}/N}(\xi_2, \mu)$$

$*$ **Comments:**

The functions $A_i(\alpha_s)$ **and** $B_i(\alpha_s)$ **are anomalous dimensions.**

They can be calculated by comparison to low orders.

In particular, $A_i(\alpha_s)$ **is the numerator of the** $1/(1-x)$ **term in splitting function** $P_{ii}(x)$

because it's the infrared divergent $(x \to 1)$ **coefficient of the collinear** $(b \to \infty)$ **singularity.**

$*$ $A_q(\alpha_s) = \frac{\alpha_s}{\pi} C_q \left(1 + \frac{\alpha_s}{\pi} K + \ldots\right)$, $K = C_A \left(\frac{67}{18} - \frac{\pi^2}{6}\right) - \frac{5 n_F}{9}$

$*$ **Logs from LO, NLO in** $A_q = A_q^{(1)}(\alpha_s/\pi) + \ldots$**, LO in** B_q

$$E_{q\bar{q}} = -\,2\int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2}\ \left[A_q(\alpha_s(k_T))\ \ln\left(\frac{Q^2}{k_T^2}\right)\ +\ B(\alpha_s(k_T))\right]$$

$$\sim 2C_i \int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2}\ \left[\left\{\frac{\alpha_s(k_T)}{\pi} + K\,\frac{\alpha_s(k_T)}{\pi}\right\}\ln\left(\frac{Q^2}{k_T^2}\right)\ + 2\frac{\alpha_s(k_T)}{\pi}\right]$$

$$\sim 2C_i\,\frac{\alpha_s(Q)}{\pi}\ \int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2}\left[\ \left\{1 + \left(\frac{\alpha_s(Q)}{\pi}\right)\ (K -\ \beta_0)\right\}\ln\left(\frac{Q^2}{k_T^2}\right)\right.$$

$$\left. +2\,\frac{\alpha_s(Q)}{\pi}\right]$$

* **The pattern:**

$$
\begin{aligned}
2C_i\,\frac{\alpha_s(Q)}{\pi}\int_{1/b^2}^{Q^2}\frac{dk_T^2}{k_T^2}\Bigg[\ &\left\{1+\left(\frac{\alpha_s(Q)}{\pi}\right)\left(K-\frac{\beta_0}{4\pi}\right)\right\}\ln\left(\frac{Q^2}{k_T^2}\right)\\
&+2\,\frac{\alpha_s(Q)}{\pi}\Bigg]\\
&\sim \alpha_s \ln^2(bQ)(1+\alpha_s\,\ln(bQ)+\dots)\\
&\quad+\alpha_s\ln(bQ)(1+\alpha_s\,\ln(bQ)+\dots)\\
&\quad+\dots
\end{aligned}
$$

* **These are LL($A^{(1)}$), NLL ($B^{(1)}$, $A^{(2)}$), and so on**

* **NLL is good so long as $\alpha_s(Q)\ln bQ \le 1$.**

* **Evaluating a resummed cross sections: re-enter NPQCD.**

We start with:

$$E^{\rm PT} = - \int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2} \left[2A_q(\alpha_s(k_T)) \ln\left(\frac{Q^2}{k_T^2}\right) + B_q(\alpha_s(k_T)) \right]$$

With running coupling:

$$\alpha_s(k_T) = \frac{\alpha_s(Q)}{1 + \frac{\alpha_s(Q)}{4\pi}\beta_0 \ln\left(\frac{k_T^2}{Q^2}\right)} = \frac{4\pi}{\beta_0 \ln\left(\frac{k_T^2}{\Lambda^2_{\rm QCD}}\right)}$$

Singularity in integral at $b^2 = Q^2 \ \exp[-4\pi/\beta_0\alpha_s(Q)] \sim \frac{1}{\Lambda^2}$**.**

* **Problem:** **how to do the inverse transform with the running coupling when $k_T^{\min} \sim 1/b$ gets small?**

* **At least four approaches:**

1) Work in Q_T-space directly to some approximation
(The originals: Dokshitzer, Diakanov & Troyan ($\sim$ 1979).
Revived by Ellis & Veseli Kulesza & Stirling
who re-derived this from b-space. ($\sim$ 2000))

2) Insert a "soft landing" on the k_T integral by replacing

$$1/b \rightarrow \sqrt{1/b^2 + 1/b_*^2}$$

for some fixed b_*. (Collins, Soper "b_*" prescription (1982), ResBos)

3) **Extrapolation of $E^{\rm PT}$ into NP region** (Qiu, Zhang (2002)).

4) **Minimal: avoid the singularity at $1/b = \Lambda_{\rm QCD}$ by monkeying with the b-space contour integral.**
(This technique introduced in threshold resummation;
then adapted by Laenen, GS and Vogelsang,
and Bozzi, Catani, de Florian and Grazzini. (2000-2003))

Any of these "define" PT. All will fit the data qualitatively, and with a little work quantitatively.

But all require new parameters for quantitative fit.
This is not all bad . . . let's see why.

A bit more consideration generalizes (for the A-term) for small k_T to some upper limit μ_I:

$$
\begin{aligned}
E^{\rm soft} = &\ \frac{1}{2\pi} \int_0^{\mu_I^2} \frac{d^2 k_T}{k_T^2}\, A_q(\alpha_s(k_T))\, \ln\left(\frac{Q^2}{k_T^2}\right)\, \left(e^{i\mathbf{b}\cdot\mathbf{k}_T} - 1\right) \\
\sim - &\ \int_0^{\mu_I^2} \frac{dk_T^2}{k_T^2}\, (\mathbf{b}\cdot\mathbf{k}_T)^2 A_q(\alpha_s(k_T))\, \ln\left(\frac{Q^2}{k_T^2}\right) + \cdots \\
\sim - &\ b^2 \int dk_T^2\, A_q(\alpha_s(k_T))\, \ln\left(\frac{Q^2}{k_T^2}\right)
\end{aligned}
$$

$\theta(k_T - 1/b) \Rightarrow (e^{i\mathbf{b}\cdot\mathbf{k}_T} - 1)$**; in fact, correct to all orders,**

Note the expansion is for b " small enough" only.

What is $-\, b^2 \int dk_T^2\, A_q(\alpha_s(k_T))\, \ln\left(\frac{Q^2}{k_T^2}\right)$ **?**

Don't really know, but it suggests a nonperturbative correction of the form (exhibiting the μ_I is unconventional)

$$E^{\rm NP} = -\, b^2 \mu_I^2 \left(g_1 \ln\left(\frac{Q}{\mu_I}\right) + g_2 \right)$$

Since this is an exponent, whatever the definition of the pertrubative resummed cross section, it is smeared with a Gaussian whose width in b (k_T) space decreases (increases) with $\ln Q$.

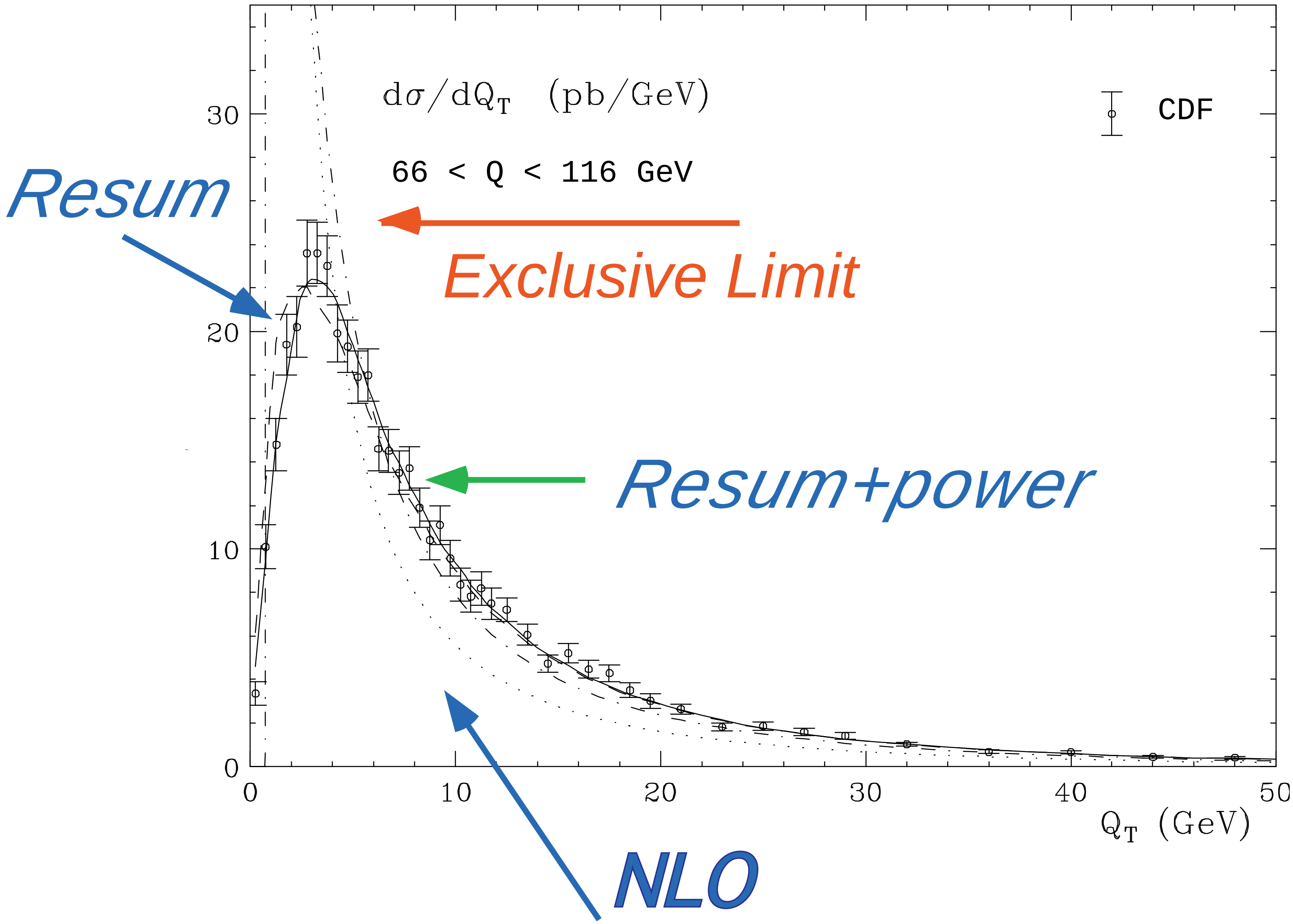
In summary

$$\frac{d\sigma(Q_T)}{dQ^2\, d^2\vec{Q}_T} = \sum_a H_{a\bar{a}}(\alpha_s(Q^2)) \int \frac{d^2b}{(2\pi)^2}\, e^{i\vec{Q}_T\cdot\vec{b}}\, e^{E_{a\bar{a}}^{\mathrm{PT}}(b,Q,\mu)}\; e^{\,-\;\mu_I^2 b^2 (g_1\;\ln\left(\frac{Q}{\mu_I}\right)+g_2)}$$

$$\times \sum_{a=q\bar{q}} \int_{\xi_1\xi_2} \frac{d\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu)}{dQ^2}\; f_{a/N}(\xi_1, 1/b)\; f_{\bar{a}/N}(\xi_2, 1/b)$$

$$= \;\pi \int d^2\mathbf{k}_T\; \frac{e^{-k_T^2/4[\mu_I^2(g_2\ln(Q/k_T)+g_2)]}}{\mu_I^2(g_2\ln(Q/k_T)+g_2)}\; \frac{d\sigma_{NN}(\mathbf{Q_T}-\mathbf{k_T})}{dQ^2\, d^2\vec{Q}_T}$$

Which gives curves like the one we saw before.



Successful phenomenology for W and Z.
In principle, can also fit to fixed-target Drell-Yan with the same set of NP parameters.

Qiu and Zhang show that NP corrections are dominant for fixed-target Q^2.

What about those $1/(1-z)$ (soft gluon energy) singularities?

$*$ **This is threshold resummation . . .**

IIIC. Threshold Resummation

Integrate over Q_T: the NLO total DY cross section

Integrate over Q_T at fixed $z = \frac{Q^2}{\xi_1 \xi_2 S}$. $Q_T \to 0$ is singular.

Add diagrams with virtual gluons: $\textit{their}\ k_T$ **integrals are singular.**

Factorize low $\mathrm{k}_T = -\mathrm{Q}_T < \mu$ gluons just as in DIS.

The remainder is now finite at fixed Q_T, $z \neq 1$.

The left-over NLO $\hat{\sigma}$ is not a normal function of z!

Because $d\sigma/dQ^2$ begins at ${\alpha_s}^0$, this is next-to-leading order (NLO) here.

$\hat{\sigma}_{\bar{q}q}$ for Drell-Yan at NLO

$$\frac{d^2\hat{\sigma}^{(1)}_{q\bar{q}\to\gamma^* g}(z,Q^2,\mu^2)}{dQ^2}$$

$$= \sigma_0(Q^2)\ \left(\frac{\alpha_s(\mu)}{\pi}\right)\ \left\{2(1+z^2)\ \left[\frac{\ln(1-z)}{1-z}\right]_+ - \frac{(1+z^2)\ln\,z}{(1-z)} + \left(\frac{\pi^2}{3}-4\right)\ \delta(1-z)\right\}$$

$$+\ \sigma_0(Q^2)\ C_F\ \frac{\alpha_s}{\pi}\ \left[\frac{1+z^2}{1-z}\right]_+\ \ln\left(\frac{Q^2}{\mu^2}\right)$$

- **Plus distributions: "generalized functions" (c.f. delta function). μ-dependence: evolution.**

- **What they are, how they work**

$$\int_0^1 dx\, \frac{f(x)}{(1-x)_+} \equiv \int_0^1 dx\, \frac{f(x)-f(1)}{(1-x)}$$

$$\int_0^1 dx\ f(x) \left(\frac{\ln(1-x)}{1-x} \right)_+ \equiv \int_0^1 dx\ (\,f(x)-f(1)\,)\ \frac{\ln(1-x)}{(1-x)}$$

and so on . . . where $f(x)$ **will be parton distributions**

- $f(x)$ **term: real gluon, with momentum fraction** $1-x$

- $f(1)$ **term: virtual, with elastic kinematics**

- **If** $f(x)$ **is changing rapidly, find a large correction.**

- *A Special Distribution*

- *DGLAP "evolution kernel" = "splitting function"*

$$P_{qq}(x) = C_F \, \frac{\alpha_s}{\pi} \, \left[\frac{1+x^2}{1-x} \right]_+$$

- **Nonsinglet, leading order**

- **A neat bit of soft-gluon kinematics:** $p_q + p_{\bar{q}} = q + k \Rightarrow$**:**

$$z = \frac{Q^2}{\xi_1 \xi_2 S} = \frac{(p_q + p_{\bar{q}} - k)^2}{(p_q + p_{\bar{q}})^2}$$
$$\sim 1 - \frac{2Q \cdot k}{Q^2}$$

And in the $\vec{Q} = 0$ (c.m.) frame,

$$1 - z = \frac{2k_0}{Q}$$

So one singularity is from $\mathrm{k}_T = 0$, one from $k_0 = 0$, for any number of soft partons in the final state.

$z \to 1$ **is called "partonic threshold".**

- **Back to the one-loop DY hard-scattering**

$$\frac{d\hat{\sigma}^{(1)}_{q\bar{q}\to\gamma^* g}}{dQ^2\, d^2\mathbf{Q}_T} = \sigma_0 \frac{\alpha_s C_F}{\pi^2} \left(1 - \frac{4\mathbf{Q}_T^2}{(1-z)^2 \xi_1 \xi_2 S}\right)^{-1/2} \times \left[\frac{1}{\mathbf{Q}_T^2}\frac{1+z^2}{1-z} - \frac{2z}{(1-z)Q^2}\right]$$

- **Factorized cross section at fixed $\mathbf{Q}_T$:**

$$\frac{d\sigma_{NN\to\mu^+\mu^-+X}(Q,p_1,p_2)}{dQ^2 d^2\mathbf{Q}_T} = \int_{\xi_1,\xi_2} \sum_{a=q\bar{q}} \frac{d\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu,\xi_1 p_1,\xi_2 p_2,\mathbf{Q}_T)}{dQ^2 d^2\mathbf{Q}_T} \times f_{a/N}(\xi_1,\mu)\, f_{\bar{a}/N}(\xi_2,\mu)$$

Integrate over Q_T: the NLO total DY cross section

Integrate over Q_T at fixed $z = \frac{Q^2}{\xi_1 \xi_2 S}$. $Q_T \to 0$ is singular.

Add diagrams with virtual gluons: $their\ k_T$ integrals are singular.

Factorize low $\mathrm{k}_T = -\mathrm{Q}_T < \mu$ gluons as in DIS.

The remainder is now finite at fixed Q_T, $z \neq 1$.

- **The Q_T-integrated NLO partonic cross section**

$$\frac{d^2\hat{\sigma}^{(1)}_{q\bar{q}\to\gamma^* g}(z,Q^2,\mu^2)}{dQ^2}$$

$$= \sigma_0(Q^2)\ \left(\frac{\alpha_s(\mu)}{\pi}\right)\ \left\{2(1+z^2)\ \left[\frac{\ln(1-z)}{1-z}\right]_+ - \frac{(1+z^2)\ln\ z}{(1-z)} + \left(\frac{\pi^2}{3}-4\right)\ \delta(1-z)\right\}$$

$$+\ \sigma_0(Q^2)\ C_F\ \frac{\alpha_s}{\pi}\ \left[\frac{1+z^2}{1-z}\right]_+\ \ln\left(\frac{Q^2}{\mu^2}\right)$$

- **Plus distributions: “generalized functions” (c.f. delta function).**

- **What they are, how they work**

$$\int_0^1 dx\, \frac{f(x)}{(1-x)_+} \quad \equiv \int_0^1 dx\, \frac{f(x)-f(1)}{(1-x)}$$

$$\int_0^1 dx\ f(x) \left(\frac{\ln(1-x)}{1-x} \right)_+ \quad \equiv \int_0^1 dx\ \left(f(x) - f(1) \right)\ \frac{\ln(1-x)}{(1-x)}$$

and so on . . . where $f(x)$ will be parton distributions

- $f(x)$ **term: real gluon, with momentum fraction $1-x$.**
- $f(1)$ **term: virtual, with elastic kinematics.**
- **If $f(x)$ is changing rapidly, find a large correction.**

- **A Special Distribution is the**

- **DGLAP "evolution kernel" = "splitting function":**

$$P_{qq}(z) = C_F \, \frac{\alpha_s}{\pi} \, \left[\frac{1+z^2}{1-z}\right]_+ \to \frac{A(\alpha_s)}{1-z} + \dots$$

- **Nonsinglet, leading order**

- **A neat bit of soft-gluon kinematics:** $p_q + p_{\bar{q}} = q + k \Rightarrow$

$$z = \frac{Q^2}{\xi_1 \xi_2 S} = \frac{(p_q + p_{\bar{q}} - k)^2}{(p_q + p_{\bar{q}})^2}$$

$$z \;\sim 1 - \frac{2Q \cdot k}{Q^2}$$

And in the $\vec{Q} = 0$ (c.m.) frame,

$$1 - z = \frac{2k_0}{Q}$$

So one singularity in $\hat{\sigma}^{(1)}$ is from $\mathrm{k}_T = 0$, one from $k_0 = 0$, for any number of soft partons in the final state.

$z \rightarrow 1$ is called "partonic threshold".

- **Threshold resummation is resummation for the plus distributions.**

- **Same method as for Q_T, but now fix $k_{\rm soft} \sim \frac{1}{2}(1-z)Q$.**

  Laplace or Mellin transform $e^{-N2k_0/Q} \sim z^N$ and $\overline{\rm MS}$ collinear subtraction gives (here NLL accuracy shown) $\exp[\, E_a^{\rm thr}(N,Q)\,]$:

$$E_a^{\rm thr}(N,Q) = \int_{Q^2/N^2}^{Q^2} \frac{du^2}{u^2}\, 2A_a\left(\alpha_s(u)\right)\, \ln \frac{Nu}{Q}$$

Threshold: small $1-z \sim 2k_0/Q$, large N: enhancement:

$$2C_i \frac{\alpha_s(Q)}{\pi} \int_{Q^2/N^2}^{Q^2} \frac{dk_T^2}{k_T^2} \left[\left\{ 1 + \left(\frac{\alpha_s(Q)}{\pi} \right) \left(K - \frac{\beta_0}{4\pi} \right) \right\} \ln\left(\frac{Q^2}{k_T^2} \right) \right.$$

$$\left. + 2 \frac{\alpha_s(Q)}{\pi} \right]$$

$$\sim \alpha_s \ln^2(N)(1 + \alpha_s \ln(N) + \dots)$$

$$+ \alpha_s \ln(N)(1 + \alpha_s \ln(N) + \dots)$$

$$+ \dots$$

- **As for Q_T, these are LL($A^{(1)}$), NLL ($B^{(1)}$, $A^{(2)}$), and so on.**

- **And again, NLL is good so long as $\alpha_s(Q) \ln N \leq 1$.**

In this case, the enhancement is entirely due to the subtraction of collinear singularities.

The $\overline{\mathrm{MS}}$ distributions decrease faster in N than the partonic cross section.

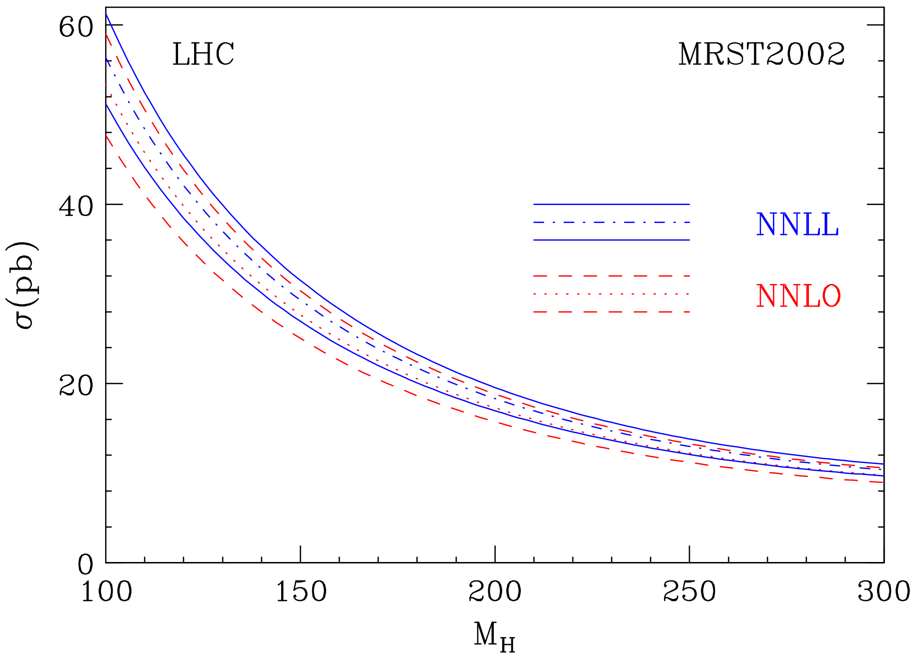
- **Inverse transform to the cross section:**

$$\frac{d\sigma^{\rm res}_{NN}}{dQ^2} = \sum_a \ \hat{\sigma}^{(0)}_a(Q,\mu) \int_{C_N} \frac{dN}{2\pi i} \left(\frac{Q^2}{S}\right)^{-N} \ \exp\left[\, E_a^{\rm thr}(N,Q,\mu) \,\right]$$
$$\times f_{a/N}(N,\mu)\, f_{\bar{a}/N}(N,\mu)$$

Formalism is similar for W, Z, H. "Electroweak annihilation"

Typical collider result . . .

• **Logs: threshold resummation vs. fixed order for H at LHC**



(from Catani, de Florian, Grazzini, Nason (2003))

• **Modest change & decrease in μ-dependence**
$\rightarrow$ **increased confidence. But see Sec. VII.)**

IVA. Jet shapes and $1/Q$ corrections

- **Angularity event shapes**

  (C.F. Berger, Kúcs, GS (2003), Berger, Magnea (2004))

$$\tau_a = \frac{1}{Q} \sum_{i \text{ in } N} E_i \, (\sin\theta_i)^a \, (1 - |\cos\theta_i|)^{1-a}$$

- θ_i **angle to thrust (**$a = 0$**) axis (**$\hat{n}$ **that gives** τ_0^{min}**).**

- **Jet "broadening":** $a = 1$**; total cross section:** $a \to -\infty$**.**

- **Cross section is a convolution in contributions of each jet and a soft radiation function**

$$\sigma\,(\tau_a, Q, a) \;=\; H_{IJ} \int dt_s \prod_{\text{jets}\;i} \int dt_i\, S_{JI}(t_s) \prod_i J_i(t_i, p_{Ji})$$
$$\times\, \delta(\sum_i t_i + t_s - \tau_a)$$

- **Thus, general resummed cross section can be written as an inverse transform**

$$\sigma\,(\tau_a, Q, a) \;=\; \int_C d\nu\, \mathrm{e}^{\nu\,\tau_a}\, H_{IJ}\, S_{JI}(\nu) \prod_i J_i(\nu, p_{Ji})$$

in terms of $f(\nu) = \int_0^\infty dt\, e^{-\nu t}\, f(t)\,.$

- **NLL resummed cross section is from an inverse transform:**

$$\sigma\,(\tau_a, Q, a) \;=\; \sigma_{\rm tot} \int_C d\nu\, {\rm e}^{\nu\,\tau_a}\, \left[\, J_i(\nu, p_{Ji})\,\right]^2$$

- **At NLL can define $S_{c\bar{c}} = 1$: indepenent jet "shower" evolution.** (Catani, Turnock, Trentadue, Webber (1990-92))

So we need the resummed jet function in transform space

$$J_i(\nu, p_{Ji}) \;=\; \int_0 d\tau_a \, \mathrm{e}^{-\nu \tau_{Ji}} \; J_i(\tau_{Ji}, p_{Ji}) = \mathrm{e}^{\frac{1}{2}E(\nu,Q,a)}$$

where the same reasoning as above gives:

$$E(\nu, Q, a) \;=\; 2 \int_0^1 \frac{du}{u} \Bigg[\int_{u^2Q^2}^{uQ^2} \frac{dp_T^2}{p_T^2} A\left(\alpha_s(p_T)\right) \left(\mathrm{e}^{-u^{1-a}\nu (p_T/Q)^a} - 1\right)$$
$$+ \frac{1}{2} B\left(\alpha_s(\sqrt{u}Q)\right) \left(\mathrm{e}^{-u(\nu/2)^{2/(2-a)}} - 1\right) \Bigg]$$

Again, nonperturbative scales are implied by resummmed PT. But now, an expansion in powers of $1/Q$ . . .

Shape function approach for $\mathrm{e}^+\mathrm{e}^-$ jets

- $p_T > \kappa$, **PT**

- $p_T < \kappa$, **expand exponentials: isolate "shape function".**

- **Low p_T ($< \kappa \leftrightarrow \mu_I$) replaced by $f_{\rm NP}$**

$$
\begin{aligned}
E(\nu, Q, a) &= E_{\rm PT}(\nu, Q, \kappa, a) \\
&\quad + \frac{2}{1-a} \sum_{n=1}^{\infty} \frac{1}{n\, n!} \left(-\frac{\nu}{Q} \right)^n \int_0^{\kappa^2} \frac{dp_T^2}{p_T^2}\, p_T^n\, A\left(\alpha_s(p_T)\right) \left[1 - \left(\frac{p_T}{Q} \right)^{n(1-a)} \right] \\
&\quad + \dots \\
&\equiv E_{\rm PT}(\nu, Q, \kappa, a) + \ln \tilde{f}_{a,{\rm NP}} \left(\frac{\nu}{Q}, \kappa \right)
\end{aligned}
$$

Shape function properties

- $f_{\rm NP}$ **factorizes under moments** $\rightarrow$ **convolution**

$$\sigma(\tau_a, Q) = \frac{1}{2\pi i}\int_C d\nu\, \tilde{f}_{a,{\rm NP}}\left(\frac{\nu}{Q}\right)\, \sigma_{\rm PT}(\nu, Q, a)$$
$$= \int d\xi\, f_{a,{\rm NP}}(\xi)\ \sigma(\tau_a - \xi, Q)$$

- $f_{\rm NP}$ **function of** ν/Q**only**

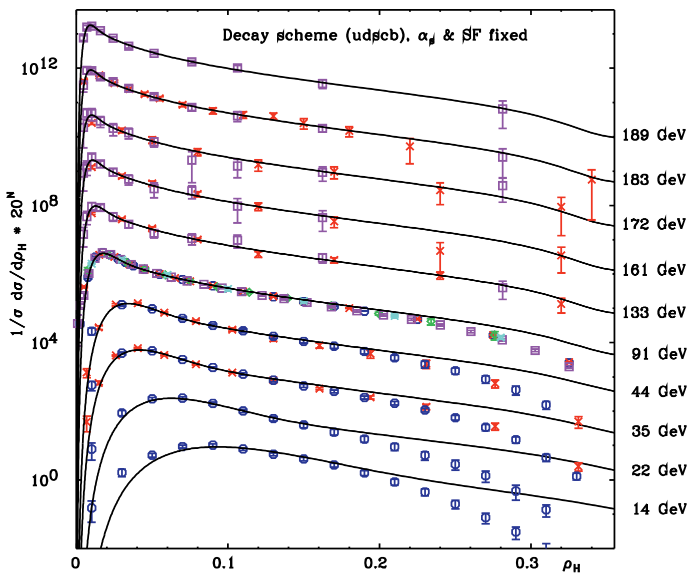
- **Linear in** ν/Q**: shift in PT distribution**

(Korchemsky & GS (1995), Dokshitzer & Webber (1997))

$$\tilde{f}_{a,{\rm NP}}\left(\frac{\nu}{Q}\right) \rightarrow {\rm e}^{\nu\left(\tau_a - \frac{1}{1-a}\frac{\lambda_1}{Q}\right)}$$

- **Shape function phenomenology for thrust**

(Korchemsky,GS, Belitsky; Gardi Rathsman,Magnea (1998 . . .))



Strategy: $f_{\rm NP}(\epsilon)$ at Z pole; predict other Q (viz. PDFs)

- **Scaling property for τ_a event shapes**

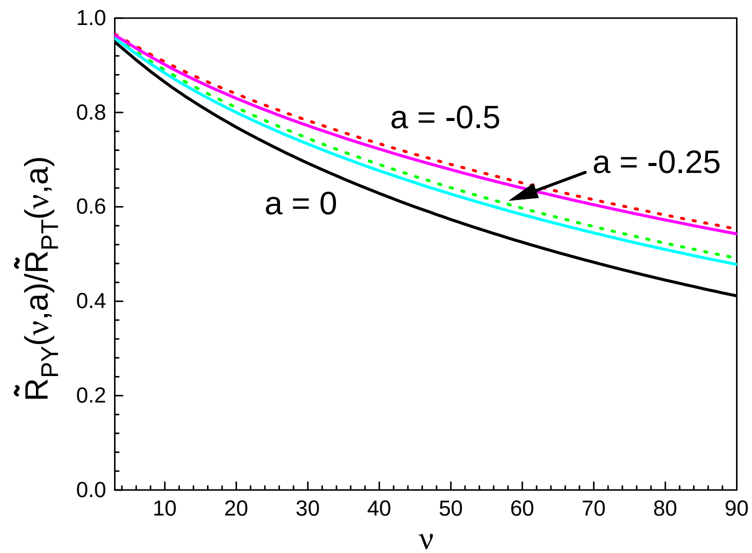
- **(Approximate rapidity-independence of NP dynamics)**

$$\ln \tilde{f}_{a,\mathrm{NP}}\left(\frac{\nu}{Q},\kappa\right) = \frac{1}{1-a}\sum_{n=1}^{\infty} \lambda_n(\kappa)\left(-\frac{\nu}{Q}\right)^n$$

$$\tilde{f}_a\left(\frac{\nu}{Q},\kappa\right) = \left[\tilde{f}_0\left(\frac{\nu}{Q},\kappa\right)\right]^{\frac{1}{1-a}}$$

- **All a-dependence is in the exponent.**

- **What PYTHIA gives**



- **Intriguing, but untested as yet.**

  (L3 data have been analyzed, with encouraging preliminary results.)

IVB. Evolution with Color Exchange

- **What distinguishes hadron colliders.**
- **Multiloop scattering amplitudes in dimensional regularization**

(Catani (1998) Tejeda-Yeomans & GS (2002) Kosower (2003)) Aybat, Dixon & GS (2006)

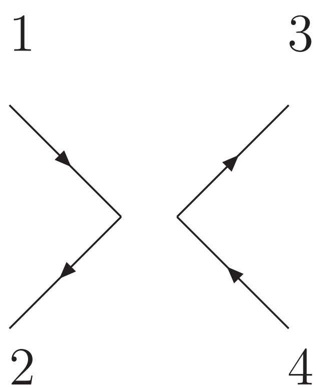
  - **Amplitude for partonic process**

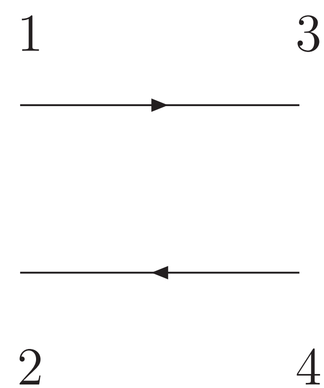
$$\mathrm{f}: \quad f_A(p_A, r_A) + f_B(p_B, r_B) \to f_1(p_1, r_1) + f_2(p_2, r_2)$$

$$\mathcal{M}^{[\mathrm{f}]}_{\{r_i\}}\left(p_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \mathcal{M}^{[\mathrm{f}]}_{L}\left(p_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) (c_L)_{\{r_i\}}$$

- **Need to control poles in ϵ for factorized calculations at fixed order and for resummation. Evol. for "soft" functions S_{IJ}.**

- **Example:** $q\bar{q}$ **tensors** $(c_L)_{\{r_i\}}$**:**

$c_1 \equiv$ 

$c_2 \equiv$ 

  – **Jet/soft factorization at amplitude level.** **(Sen (1983)):**

$$\mathcal{M}_L^{[\mathrm{f}]}\left(p_i,\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)=\prod_{i=A,B,1,2} J_i^{[\mathrm{virt}]}\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)$$

$$\times\mathbf{S}_{LI}^{[\mathrm{f}]}\left(p_i,\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)h_I^{[\mathrm{f}]}\left(\wp_i,\frac{Q^2}{\mu^2},\alpha_s(\mu^2)\right)$$

- Soft function labelled by color exchange (singlet, octet . . .)

- Factors require dimensional regularization

- Same factorization $\rightarrow$ resummation

- Poles at 2- and higher loops . . .

- Relation to supersymmetric Yang-Mills theories
  (Bern, Czakon, Dixon, Kosower & Smirnov (2006) verified structure to 4 loops.)

- **Dimensionally-regularized jets**

  (Magnea & GS (1990))

$$J_i\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right) = \exp\Bigg\{ \frac{1}{4}\int_0^{-Q^2} \frac{d\xi^2}{\xi^2}\Bigg[\mathcal{K}^{[i]}(\alpha_s(\mu^2),\epsilon)$$

$$+\mathcal{G}^{[i]}\left(-1,\bar{\alpha_s}\left(\frac{\mu^2}{\xi^2},\alpha_s(\mu^2),\epsilon,\right)\epsilon\right)$$

$$+\frac{1}{2}\int_{\xi^2}^{\mu^2}\frac{d\tilde{\mu}^2}{\tilde{\mu}^2}\gamma_K^{[i]}\left(\bar{\alpha_s}\left(\frac{\mu^2}{\tilde{\mu}^2},\alpha_s(\mu^2),\epsilon\right)\right)\Bigg]\Bigg\}.$$

$$J_i\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right) = \exp\left[\sum_{n=1}^{\infty}\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n \sum_{n=1}^{n+1}\frac{E_m^{[i]\,(n)}(\varepsilon)}{\varepsilon^n} + \text{finite}\right]$$

- γ_K, $\mathcal{K}$ **related to** A **above**

– **Dimensionally-regularized** S

$$\mathbf{S}^{[\mathrm{f}]}\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)$$

$$= \mathrm{P}\ \exp\left[-\frac{1}{2}\int_0^{-Q^2}\frac{d\tilde{\mu}^2}{\tilde{\mu}^2}\mathbf{\Gamma}^{[\mathrm{f}]}\left(\bar{\alpha}_s\left(\frac{\mu^2}{\tilde{\mu}^2},\alpha_s(\mu^2),\epsilon\right)\right)\right]$$

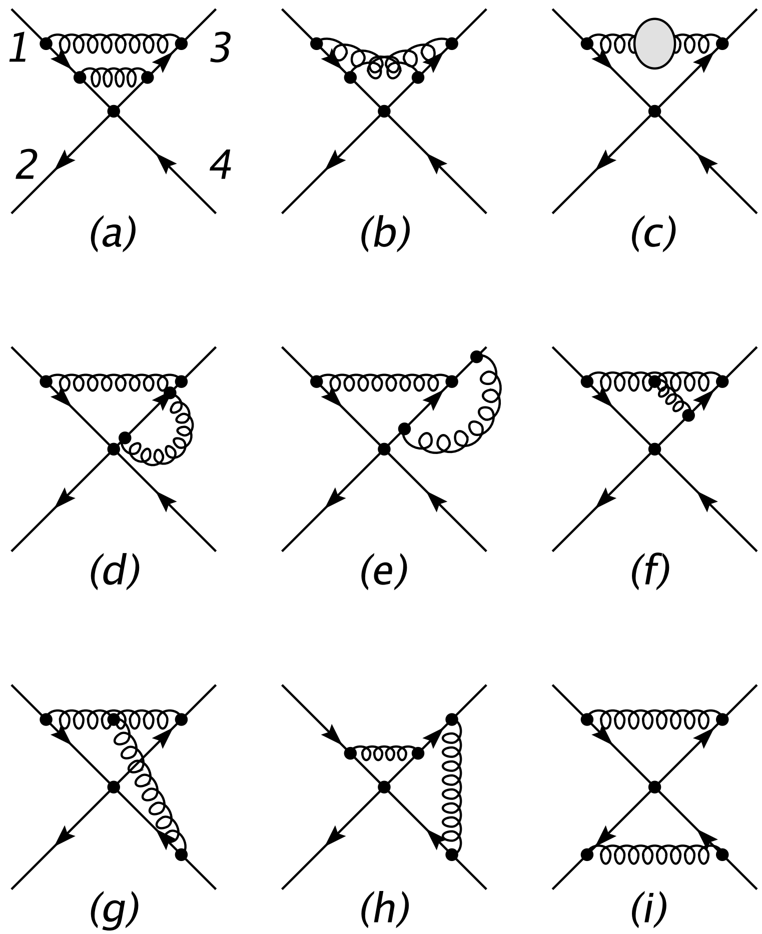
$\Gamma^{[\mathrm{f}]}$**: anomalous dimension; color mixing**

- **New result for all massless $2 \to n$ processes**
  (Aybat, Dixon, GS (2006))

$$\Gamma_S = \frac{\alpha_s}{\pi}\left(1 + \frac{\alpha_s}{\pi} K\right) \Gamma^{(1)}_{S'} \; + \; \cdots$$

$\Gamma^{(2)} = (K/2)\Gamma^{(1)}$ **with same K as in the DGLAP splitting.**

Related to the "CMW" or MC/bremsstrahlung scheme.
(Catani, Marchesini & Webber (1990))



The diagrams with 3g vertices vanish!

To NNLO, "single-web" exchange generalizes single gluon.

(C.F. Berger, 2002)

- **The full two-loop single-pole terms $\times$ LO are simply**

$$\left[\sum_{i\in \mathrm{f}} \frac{E_1^{[i]\,(2)}}{\varepsilon} + \frac{1}{4\varepsilon}\Gamma_S^{[\mathrm{f}]\,(2)}\right] \times \; \mathrm{LO}$$

- $E_1^{[i]\ \ (2)}$ **is 2 loop single pole in Sudakov form factor**
  (Ravindran, Smith, van Neerven (2005))

  Agrees with Jantzen, Kuhn, Penin, Smirnov (2005, 2006) in EW logs.

- **Hints of unexpected simplicity for IR gluons.**

- **Increasing insight into the structure of final states.**

IVC. Generalizations and limitations
1) Factorization with no hard scattering: BFKL

(Sen (1980) Balitsky (1996) Kúcs (2003))

- **Regge limit in PT for elastic scattering:** $p_A + p_B \to p'_A + p'_B$

$$-(p'_A - p'_B)^2 - t \ll s = (P_A + p_B)^2$$

- **Elastic amplitude:** $M(t,s)$.

- **Special case:** $A \to \gamma^*(q)$, **large** $S = (q + p_B)^2$, **fixed** $Q^2 = -q^2$.
This corresponds to $x = Q^2/(2p_B \cdot q) \sim Q^2/S \to 0$,
which is the small-x limit of DIS by the optical theorem:

$$\sigma^{\rm tot}_{\gamma^*(q)+p} \propto {\rm Im}\ M^{\rm elastic}_{\gamma^*(q)+p}$$

$$M(t,s) = \sum_{m,\ell} \int \left(\prod_{i=1}^{m-1} \mathrm{d}^{D-2} k_{i\perp} \right) \left(\prod_{j=1}^{\ell-1} \mathrm{d}^{D-2} p_{j\perp} \right)$$

$$\times \Gamma_A^{(m)\, a_1 \ldots\, a_m}(p_A, q, n, k_{1\perp}, \ldots, k_{m\perp})$$

$$\times S'^{\,(n,\ell)}_{a_1 \ldots\, a_n,\, b_1 \ldots\, b_ell ll}(q, n; k_{1\perp}, \ldots, k_{n\perp}; p_{1\perp}, \ldots, p_{m\perp})$$

$$\times \Gamma_B^{(\ell)\, b_1 \ldots\, b_m}(p_B, q, n; p_{1\perp}, \ldots, p_{\ell\perp})$$

- **Factorization at fixed rapidity separation:**

  Jets, $\Gamma_{A,B}$ **& soft,** S**; no** H**. Introduce vector** n^μ **as above.**

- **Evolution equations (in $\ln s \sim$ rapidity $\sim \ln(1/x)$) give**

- **generically m convolutions at $N^m LL$**

$$\left(p_A\cdot n\,\frac{\partial}{\partial\,p_A\cdot n} - 1\right)\,\Gamma_A^{(\ell)\ a_1\ldots\ a_\ell}(p_A,q,n;k_{1\perp},\ldots,k_{\ell\perp}) =$$

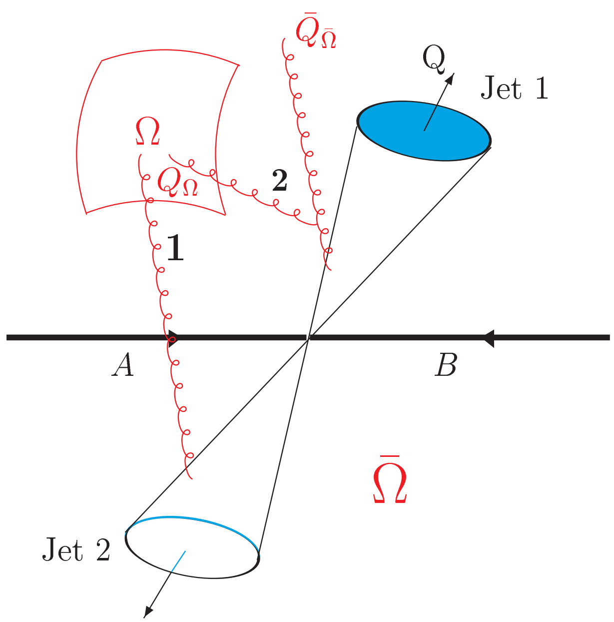
$$\sum_m \int \prod_{j=1}^m \mathrm{d}^{D-2} l_{j\perp}\,\mathcal{K}^{(\ell,m)}_{a_1\ldots\ a_n;\,b_1\ldots\ b_m}(k_{1\perp},l_{1\perp},\ldots;q,n)$$

$$\times\,\Gamma_B^{(m)\ b_1\ldots\ b_m}(p_A,q,n;l_{1\perp}\ldots)$$

- **Can project onto different color exchange:**
  octet, $m=0$ LL reggeized gluon
  singlet, $m=1$, BFKL LL pomeron ordered in rapidity, not k_T . . .

2) Non-global logs: color and energy flow

(Dasgupta & Salam (2001) . . .)



- **Simplest cases: 2 jets. Measure distribution $\Sigma_\Omega(E)$**

- **Very interesting case: energy flow between jets in WW fusion to H.**

- **Choices for Cross Section:**

- **a) Inclusive in $\bar{\Omega}$ $\rightarrow$ Number of jets not fixed!**

- **b) Correlation with event shape τ_a . . . :**
  fixes number of jets $\rightarrow$ factorization

  (Berger, Kúcs, GS (2003), Dokshitzer, Marchesini (2003), Banfi, Salam, Zanderighi (2004,5))

- **Contrast: for number of jets not fixed: nonlinear evolution! The approximate evolution equation for Σ :**
  (Banfi, Marchesini, Smye (2002)) LL in E/Q, large-N_c) **Define:** $\partial_\Delta \equiv E\,(\partial/\partial E)$

$$\partial_\Delta \Sigma_{ab}(E) = -\partial_\Delta R_{ab}\,\Sigma_{ab}(E) + \int_{k \text{ not in } \Omega} dN_{ab\to k}\,\left(\Sigma_{ak}\Sigma_{kb} - \Sigma_{ab}\right)$$

$$dN_{ab\to k} \equiv \frac{d\Omega_k}{4\pi}\,\frac{\beta_a \cdot \beta_b}{\beta_k \cdot \beta_b\,\beta_k \cdot \beta_a} \quad (\text{“}dipolesource\text{”})$$

$$R_{ab} \equiv \int_E^Q \frac{dE'}{E'}\,\int_\Omega dN_{ab\to k}\,, \quad (suppression\ due\ to\ uncancelled\ virtual\ gluons)$$

- Origin of the nonlinearity
  - ∂_E can come from unobserved "hard" gluon $G(k)$.
  - New hard gluon $G(k)$ acts as new, recoil-less source.
  - Large-N_c limit: $\bar{q}(a)G(k)q(b)$ sources $\rightarrow \bar{q}(a)q(k) \oplus \bar{q}(k)q(a)$.
  - "Global" event shapes don't allow an extra hard gluon. (observed everywhere), but fixing an event shape may limit the number of events.
  - We are far from a full understanding.

3) Large threshold effects in observed hadrons

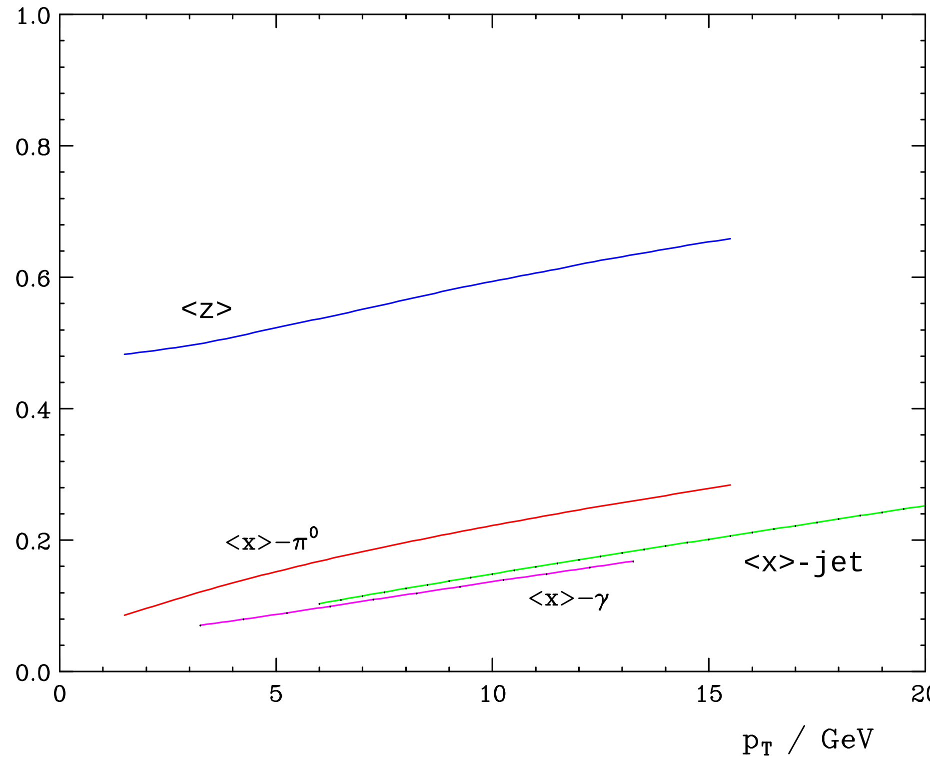
- **Pions at fixed target and RHIC** (Vogelsang and de Florian, 2004)

$$\frac{p_T^3\, d\sigma(x_T)}{dp_T} = \sum_{a,b,c} \int_0^1 dx_1\, f_{a/H_1}\left(x_1, \mu_F^2\right) \int_0^1 dx_2\, f_{b/H_2}\left(x_2, \mu_F^2\right)$$
$$\times \int_0^1 dz\, z^2\, D_{h/c}\left(z, \mu_F^2\right)$$
$$\times \int_0^1 d\hat{x}_T\, \delta\left(\hat{x}_T - \frac{x_T}{z\sqrt{x_1 x_2}}\right) \int_{\hat{\eta}_-}^{\hat{\eta}_+} d\hat{\eta}\, \frac{\hat{x}_T^4\, \hat{s}}{2}\, \frac{d\hat{\sigma}_{ab\to cX}(\hat{x}_T^2, \hat{\eta})}{d\hat{x}_T^2 d\hat{\eta}}$$

$\hat{\eta}$: **pseudorapidity at parton level**

$$\hat{\eta}_+ = -\hat{\eta}_- = \ln\left[(1 + \sqrt{1 - \hat{x}_T^2})/\hat{x}_T\right]$$

- **Averages for distribution x and fragmentation z's**



RHIC 200 GeV midrapidity average z for pions, and average x for pions, photons, jets at (NLO). Thanks to Werner Vogelsang.

- **Large z enhances threshold singularities.**

- **Singularities at one loop:**

$$\frac{\hat{s}\, d\hat{\sigma}^{(1)}_{ab\to cX}(v,w)}{dv\, dw} \approx \frac{\hat{s}\, d\hat{\hat{\sigma}}^{(0)}_{ab\to cd}(v)}{dv} \left[A'\,\delta(1-w) + B' \left(\frac{\ln(1-w)}{1-w}\right)_+ + C' \left(\frac{1}{1-w}\right)_+ \right]$$

- **For resummation, take x_T^{2N} moments $\rightarrow$ factorization:**

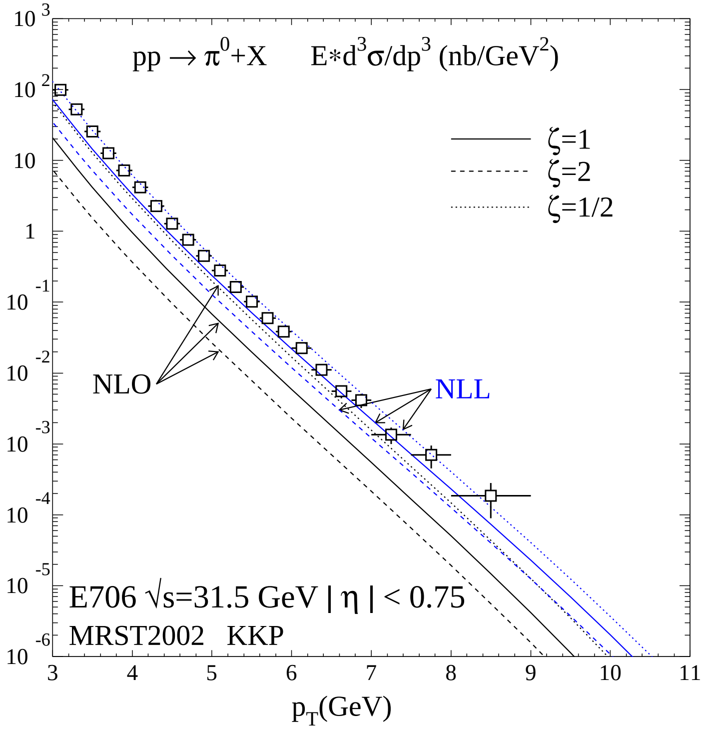
$$\hat{\sigma}^{(\mathrm{res})}_{ab\to cd}(N) = C_{ab\to cd}\, \Delta^a_N\, \Delta^b_N\, \Delta^c_N\, J^d_N \left[\sum_I G^I_{ab\to cd}\, \Delta^{(\mathrm{int})ab\to cd}_{I\,N} \right] \hat{\sigma}^{(\mathrm{Born})}_{ab\to cd}(N)$$

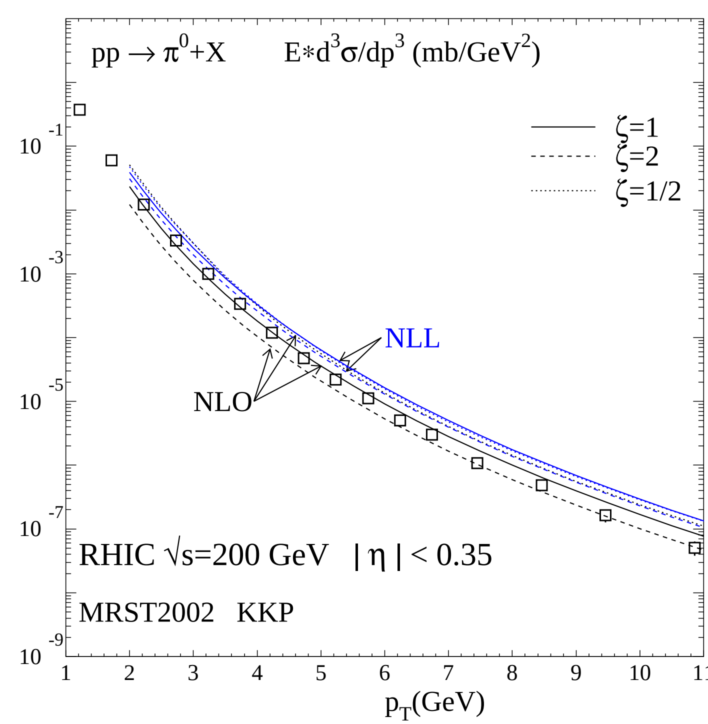
- **A typical NLL resummed factor:**

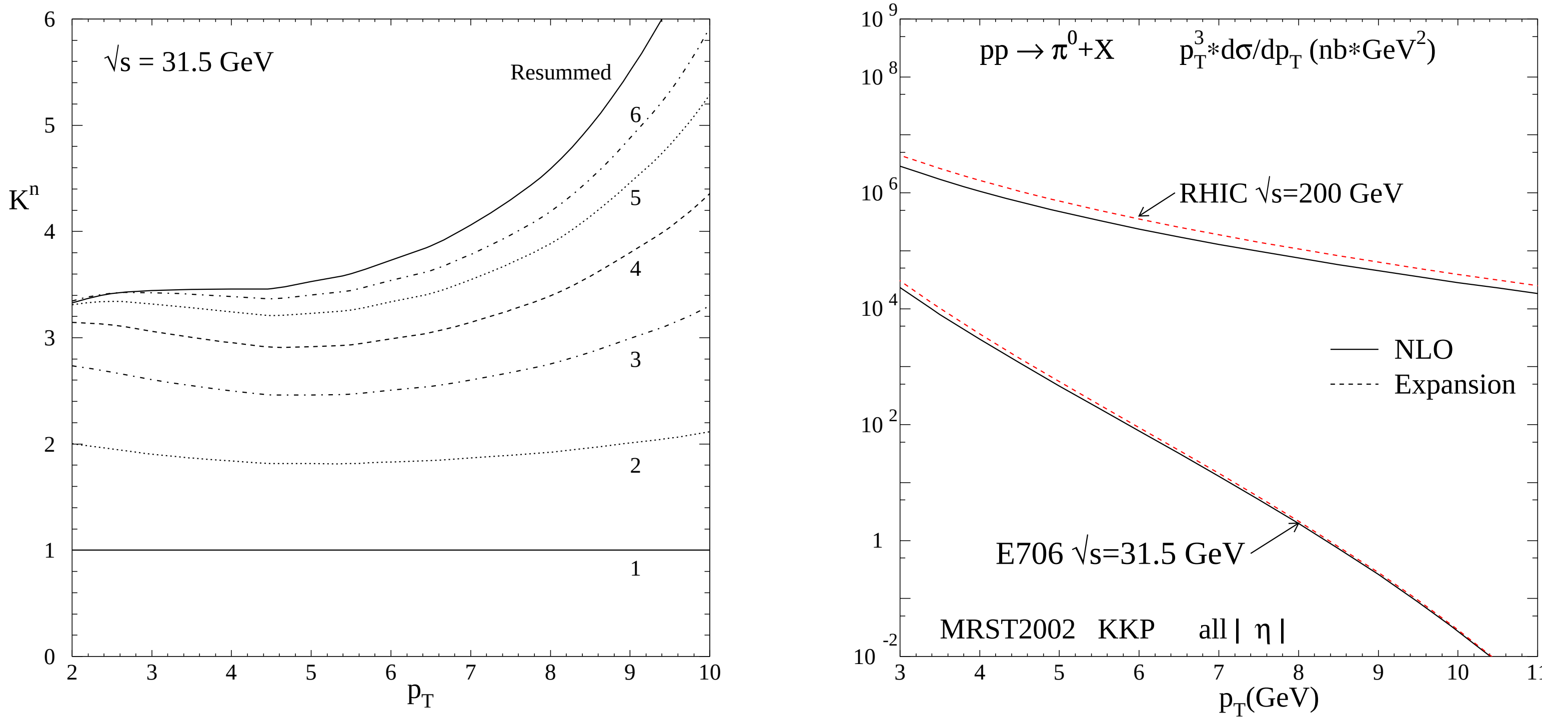
$$\Delta^a_N = \exp\left[\int_0^1 \frac{z^{N-1}-1}{1-z} \int_{\mu_{FI}^2}^{(1-z)^2 Q^2} \frac{dq^2}{q^2} A_a(\alpha_s(q^2)) \right]$$

$$A = C_F(\alpha_s/\pi)(1 + K(\alpha_s/\pi))\ + \ \ldots$$

- **Invert the moments: resolve a long-standing fixed-target vs. collider puzzle.**







- **Left: expansion of resummed cross section to fixed orders.**

- **Right: exact NLO vs. NLO expansion.**

- **Shows in π^0 1PI cross sections threshold resummation is more accurate and more important in fixed target range.**

Conclusions/Summary

- Time's up for a sample of a large subject.

- Resummation is absolutely necessary for many distributions (Q_T, event shape) just one step away from inclusive cross sections, because most events are found in regions with ordered scales ($Q_T \ll Q$, $m_{\rm jet} \ll E_{\rm jet}$).

- It is a useful tool to approach precision in certain collider cross sections (DY, $\sigma_{t\bar{t}}^{\rm tot}$).

- It can serve to suggest the form of NP effects.

- **Resummations can be derived from factorizations.**
- **Many puzzles remain, esp. connected to energy flow for non-global cross sections.**
- **Among the many topics not covered:**
  - Other approaches to resummation, based on parton showers effective theories . . . [generally similar results]
  - Threshold resummation for QCD hard scattering
  - Joint resummation of Q_T and threshold effects
  - Resummation for electroweak scattering
  - Resummation of heavy-quark logarithms
  - Much more on small-x resummations, high density QCD (HERA, RHIC).

- **Resummation just scratches the surface of QCD. But it makes a mark.**