Quantum Chromodynamics

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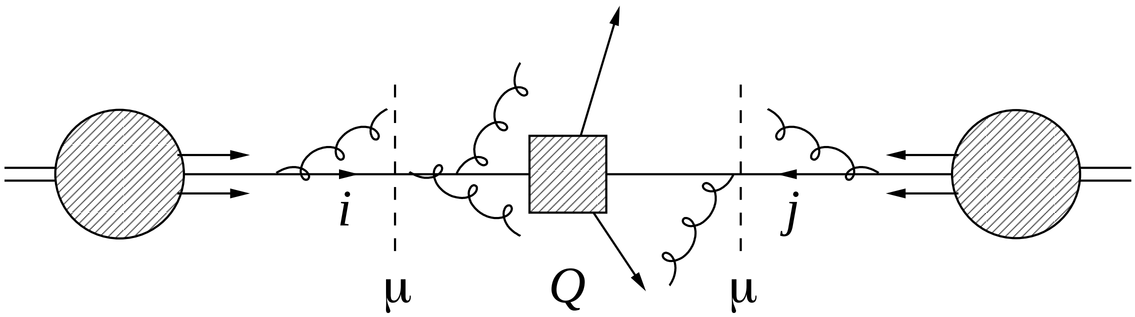
Lecture 4: Hadron-hadron interactions and spinor techniques

- Hadron-Hadron interactions
- Lepton pair production
- Spinor techniques
- W production
- $W g$ production
- W transverse momentum distribution
- W mass measurement
- W + jets

Hadron-hadron processes

- In hard hadron-hadron scattering, constituent partons from each incoming hadron interact at short distance (large momentum transfer Q^2).



- For hadron momenta P_1, P_2 ($S = 2P_1 \cdot P_2$), form of cross section is

$$\sigma(S) \quad = \quad \sum_{i,j} \int dx_1 dx_2 D_i(x_1, \mu^2) D_j(x_2, \mu^2) \hat{\sigma}_{ij}(\hat{s} = x_1 x_2 S, \alpha_S(\mu^2), Q^2/\mu^2)$$

where μ^2 is factorization scale and $\hat{\sigma}_{ij}$ is subprocess cross section for parton types i, j.

  - Notice that factorization scale is in principle arbitrary: affects only what we call part of subprocess or part of initial-state evolution (parton shower).
  - Unlike e^+e^- or ep, we may have interaction between spectator partons, leading to *soft underlying event* and/or *multiple hard scattering*.

Factorization of the cross section

- Why does the factorization property hold and when it should fail?
- For a heuristic argument Consider the simplest hard process involving two hadrons

$$H_1(P_1) + H_2(P_2) \to V + X.$$

- Do the partons in hadron H_1, through the influence of their colour fields, change the distribution of partons in hadron H_2 before the vector boson is produced? Soft gluons which are emitted long before the collision are potentially troublesome.
- A simple model from classical electrodynamics. The vector potential due to an electromagnetic current density J is given by

$$A^\mu(t,\vec{x}) = \int dt' d\vec{x}' \, \frac{J^\mu(t',\vec{x}')}{|\vec{x}-\vec{x}'|} \, \delta(t' + |\vec{x}-\vec{x}'| - t) \; ,$$

where the delta function provides the retarded behaviour required by causality.

Consider a particle with charge e travelling in the positive z direction with constant velocity β. The non-zero components of the current density are

$$J^t(t',\vec{x}') = e\delta(\vec{x}' - \vec{r}(t'))\ ,$$
$$J^z(t',\vec{x}') = e\beta\delta(\vec{x}' - \vec{r}(t')),\quad \vec{r}(t') = \beta t'\hat{z},$$

$\hat{z}$ is a unit vector in the z direction. At an observation point (the supposed position of hadron H_2) described by coordinates x, y and z, the vector potential (after performing the integrations using the current density given above) is

$$A^t(t,\vec{x}) = \frac{e\gamma}{\sqrt{[x^2+y^2+\gamma^2(\beta t - z)^2]}}$$
$$A^x(t,\vec{x}) = 0$$
$$A^y(t,\vec{x}) = 0$$
$$A^z(t,\vec{x}) = \frac{e\gamma\beta}{\sqrt{[x^2+y^2+\gamma^2(\beta t - z)^2]}},$$

where $\gamma^2 = 1/(1-\beta^2)$. Target hadron H_2 is at rest near the origin, so that $\gamma \approx s/m^2$.

- Note that for large γ and fixed non-zero $(\beta t - z)$ some components of the potential tend to a constant independent of γ, suggesting that there will be non-zero fields which are not in coincidence with the arrival of the particle, even at high energy.
- However at large γ the potential is a pure gauge piece, $A^\mu = \partial^\mu \chi$ where χ is a scalar function
- Covariant formulation using the vector potential A has large fields which have no effect.
- For example, the electric field along the z direction is

$$E^z(t, \vec{x}) = F^{tz} \equiv \frac{\partial A^z}{\partial t} + \frac{\partial A^t}{\partial z} = \frac{e\gamma(\beta t - z)}{[x^2 + y^2 + \gamma^2(\beta t - z)^2]^{\frac{3}{2}}}.$$

The leading terms in γ cancel and the field strengths are of order $1/\gamma^2$ and hence of order m^4/s^2. The model suggests the force experienced by a charge in the hadron H_2, at any fixed time before the arrival of the quark, decreases as m^4/s^2.

Parton luminosity

- parton luminosity is determined by the parton distribution functions, $f_i(x_1, \mu^2)$ and $f_j(x_2, \mu^2)$.
- the available centre-of-mass energy-squared of the parton-parton collision, $\hat{s}$, is less than the overall hadron-hadron collision energy, s, by a factor of $x_1 x_2 \equiv \tau$.
- Define differential parton luminosities

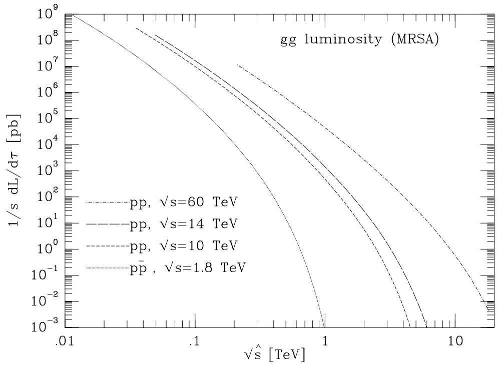
$$\tau \frac{dL_{ij}}{d\tau} = \frac{1}{1+\delta_{ij}} \int_0^1 dx_1 dx_2$$

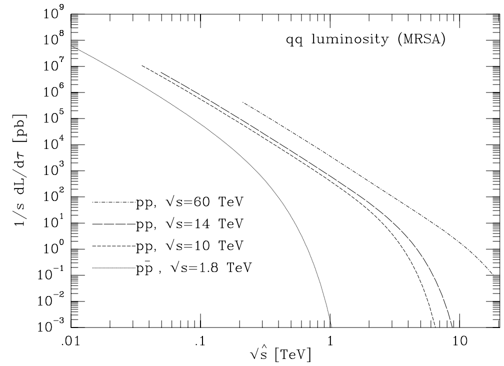
$$\times \Big[(x_1 f_i(x_1, \mu^2)\ x_2 f_j(x_2, \mu^2)) + (1 \leftrightarrow 2) \Big] \delta(\tau - x_1 x_2).$$

case of identical partons.

- We now assume that $\hat{\sigma}$ depends only on $\hat{s}$.

$$\sigma(s) = \sum_{\{ij\}} \int_{\tau_0}^1 \frac{d\tau}{\tau} \left[\frac{1}{s} \frac{dL_{ij}}{d\tau} \right] \left[\hat{s} \hat{\sigma}_{ij} \right],$$


gg luminosity (MRSA)
pp, √s=60 TeV
pp, √s=14 TeV
pp, √s=10 TeV
pp̄ , √s=1.8 TeV
1/s dL/dτ [pb]
√ŝ [TeV]


qq luminosity (MRSA)
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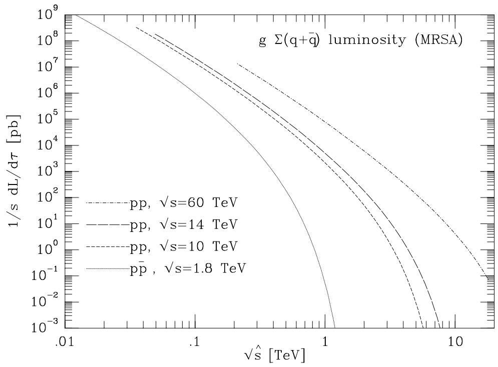
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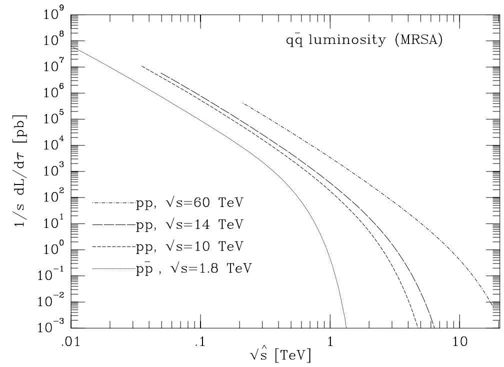
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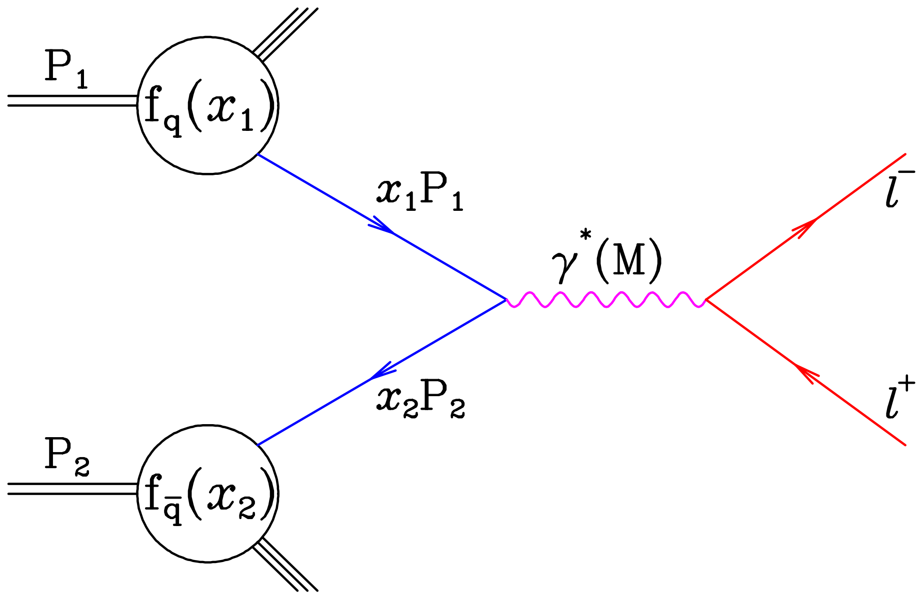
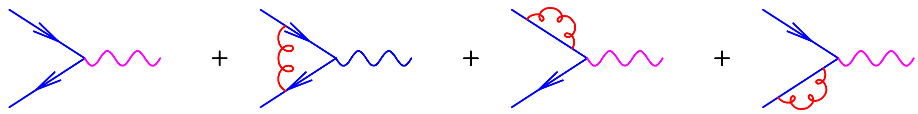
- Inverse of $e^+e^- \rightarrow q\bar{q}$ is Drell-Yan process.



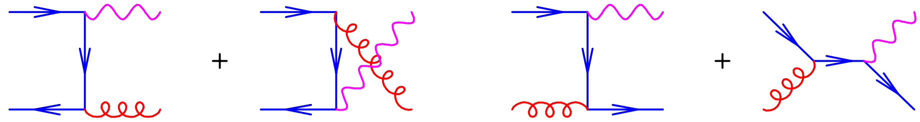
At $\mathcal{O}(\alpha_S^0)$, mass distribution of lepton pair is given by

$$\frac{d\hat{\sigma}}{dM^2}(q\bar{q} \to \gamma^* \to l^+l^-) = \frac{4\pi\alpha^2}{\hat{s}}\frac{1}{3}Q_q^2\,\delta(M^2 - \hat{s})$$

- Factor of $1/3 = 1/N$ instead of $3 = N$ because of *average* over colours of incoming q.



(a)



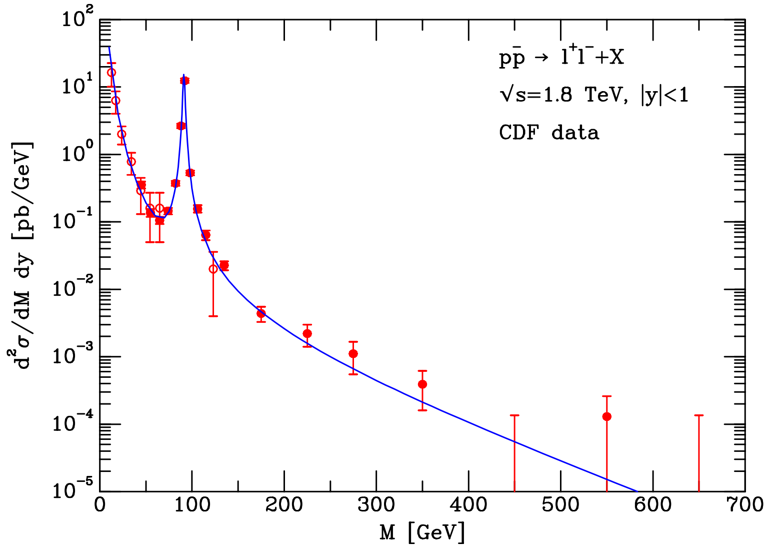
(b) (c)

- In higher orders *vertex corrections* (a) have $M^2 = \hat{s}$, *gluon emission* (b) and *QCD Compton* (c) diagrams give $M^2 < \hat{s}$.

- Rapidity of lepton pair in overall c.m. frame is

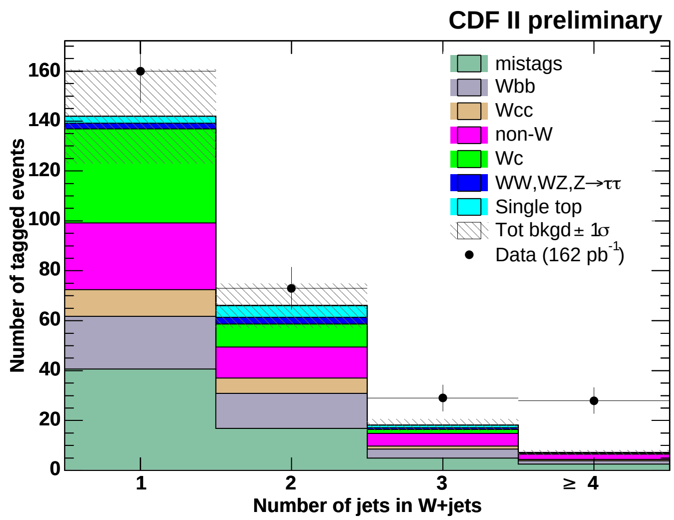
$$y \equiv \frac{1}{2} \ln \left(\frac{p^0 + p_3}{p^0 - p_3} \right) = \frac{1}{2} \ln \left(\frac{x_1}{x_2} \right)$$

where $p^\mu = p_1^\mu + p_2^\mu$.



- $W^\pm$ production is similar, sensitive to different parton distributions, e.g. $u\bar{d} \to W^+ \to l^+\nu_l$.

- We need an efficient way to calculate parton sub-processes.
- At a hadron collider many parton sub-processes can give similar signatures.
- A recent example from CDF from the top search shows events $W +$ jets events with 1 tagged jet.



Spinor techniques

- At a high energy collider we need the amplitudes for processes involving many final state particles.
- The fermions involved in high energy processes can often be taken to be massless.
- For massless fermions helicity is a good quantum number. The left and right-handed fermions are uncoupled.
- These techniques can be used for massive particles (although they are more cumbersome in that case).

  We choose an explicit representation for the gamma matrices. The Bjorken and Drell representation is,

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}, \gamma^5 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix},$$

  The Weyl representation is more suitable at high energy

$$\gamma^0 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \gamma^i = \begin{pmatrix} \mathbf{0} & -\sigma^i \\ \sigma^i & \mathbf{0} \end{pmatrix}, \gamma^5 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix},$$

$$\gamma_+ = \frac{1}{2}(1+\gamma_5) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \gamma_- = \frac{1}{2}(1-\gamma_5) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix},$$

  In the Weyl representation upper and lower components have different helicities.

- The massless spinors solns of Dirac eqn are

$$u_+(p) = \begin{bmatrix} \sqrt{p^+} \\ \sqrt{p^-} e^{i\varphi_p} \\ 0 \\ 0 \end{bmatrix}, \quad u_-(p) = \begin{bmatrix} 0 \\ 0 \\ \sqrt{p^-} e^{-i\varphi_p} \\ -\sqrt{p^+} \end{bmatrix},$$

where

$$e^{\pm i\varphi_p} \equiv \frac{p^1 \pm ip^2}{\sqrt{(p^1)^2 + (p^2)^2}} = \frac{p^1 \pm ip^2}{\sqrt{p^+ p^-}}, \qquad p^{\pm} = p^0 \pm p^3.$$

In this representation the Dirac conjugate spinors are

$$\overline{u_+}(p) \equiv u_+^\dagger(p)\gamma^0 = \left[0, 0, \sqrt{p^+}, \sqrt{p^-} e^{-i\varphi_p}\right]$$

$$\overline{u_-}(p) = \left[\sqrt{p^-} e^{i\varphi_p}, -\sqrt{p^+}, 0, 0\right]$$

- Normalization

$$u_\pm^\dagger u_\pm = 2p^0$$

Charge conjugation

In QED we have

$$\Big((+i\nabla^{\mu}+eA^{\mu})\gamma_{\mu}-m\Big)\psi=0$$

Taking the complex conjugate

$$\Big((-i\nabla^{\mu}+eA^{\mu})\gamma_{\mu}^{*}-m\Big)\psi^{*}=0$$

The equation satisfied by the charge conjugate state is

$$\Big((+i\nabla^{\mu}-eA^{\mu})\gamma_{\mu}-m\Big)\psi_{C}=0$$

- The operation of charge conjugation is therefore given by

  $$\psi_{C}=C\gamma^{0}\psi^{*}$$

  where the matrix C is determined up to a phase by the condition $(C\gamma^{0})\gamma^{\mu *}(C\gamma^{0})^{-1}=-\gamma^{\mu}$.
- Since for our representation $\gamma^{0}\gamma^{\mu\,*}\gamma^{0}=\gamma^{\mu\,T}$ the defining condition on matrix C can be written

  $$C^{-1}\gamma^{\mu}C=-\gamma^{\mu T}$$

We choose the phase such that

$$\begin{aligned} C &= -i\gamma^2\gamma^0 = \begin{pmatrix} i\sigma^2 & \mathbf{0} \\ \mathbf{0} & -i\sigma^2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \end{aligned}$$

- For free particle spinors we have that

$$v_\pm(p) = C\overline{u}_\pm^T(p),$$

- For the special case of massless spinors and our choice of phase we find that

$$v_\pm(p) = C\overline{u}_\pm^T(p) = u_\mp(p)$$

so there are only two independent massless spinors.

Introduce a bra and ket notation spinors corresponding to momenta p_i, $i = 1, 2, \ldots, n$ labelled by the index i

$$|i^{\pm}\rangle \equiv |p_i^{\pm}\rangle \equiv u_{\pm}(p_i) = v_{\mp}(p_i),$$
$$\langle i^{\pm}| \equiv \langle p_i^{\pm}| \equiv \overline{u_{\pm}}(p_i) = \overline{v_{\mp}}(p_i).$$

We define the basic spinor products by

$$\langle ij\rangle \equiv \langle i^{-}|j^{+}\rangle = \overline{u_{-}}(p_i)u_{+}(p_j), \quad [ij] \equiv \langle i^{+}|j^{-}\rangle = \overline{u_{+}}(p_i)u_{-}(p_j).$$

The helicity projection implies that products like $\langle i^{+}|j^{+}\rangle$ vanish.

$$\langle i+|j+\rangle = \langle i-|j-\rangle = \langle ii\rangle = [ii] = 0$$

$$\langle ij\rangle = -\langle ji\rangle, \quad [ij] = -[ji]$$

We get explicit formulae for the spinor products valid for the case when both energies are positive, $p_i^0 > 0, \quad p_j^0 > 0$

$$\langle i\, j\rangle = \sqrt{p_i^- p_j^+}\, e^{i\varphi_{p_i}} - \sqrt{p_i^+ p_j^-}\, e^{i\varphi_{p_j}} = \sqrt{|s_{ij}|}\, e^{i\phi_{ij}},$$

$$[i\,j] \quad = \quad \sqrt{p_i^+ p_j^-}\, e^{-i\varphi_{p_j}} - \sqrt{p_i^- p_j^+}\, e^{-i\varphi_{p_i}} \;=\; -\sqrt{|s_{ij}|}\, e^{-i\phi_{ij}}$$

where $s_{ij} \;=\; (p_i + p_j)^2 \;=\; 2p_i \cdot p_j$, and

$$\cos\phi_{ij} \;=\; \frac{p_i^1 p_j^+ - p_j^1 p_i^+}{\sqrt{|s_{ij}| p_i^+ p_j^+}}\,, \qquad \sin\phi_{ij} \;=\; \frac{p_i^2 p_j^+ - p_j^2 p_i^+}{\sqrt{|s_{ij}| p_i^+ p_j^+}}\,.$$

- The spinor products are, up to a phase, square roots of Lorentz products.
- The collinear limits of massless gauge amplitudes have square-root singularities; spinor products lead to very compact analytic representations of gauge amplitudes.
- spinor products should have simple properties under crossing symmetry, i.e. for energies negative. Define the spinor product $\langle i\,j\rangle$ by analytic continuation from the positive energy case, but with p_i replaced by $-p_i$ if $p_i^0 < 0$, and similarly for p_j; and with an extra multiplicative factor of i for each negative energy particle. We define $[i\,j]$ through the identity

$$\langle i\,j\rangle\,[j\,i] = \langle i^-|j^+\rangle\langle j^+|i^-\rangle = \mathrm{Tr}(\tfrac{1}{2}(1-\gamma_5)\not{p}_i\not{p}_j) = 2p_i \cdot p_j = s_{ij}.$$

We also have the following identities which are useful to manipulate the spinor products

$$\langle i^{\pm}|\gamma^{\mu}|i^{\pm}\rangle \;=\; 2p_i^{\mu}, \qquad\qquad |i^{\pm}\rangle\langle i^{\pm}| \;=\; \frac{1}{2}(1\pm\gamma_5)\not{p}_i$$

$$\langle ij\rangle^* = -\mathrm{sign}(i\cdot j)[ij] = \mathrm{sign}(i\cdot j)[ji]$$

$$|\langle ij\rangle|^2 = \langle ij\rangle\langle ij\rangle^* = 2|i\cdot j| \equiv |s_{ij}|$$

$$\langle ij\rangle[ji] = 2i\cdot j \equiv s_{ij}$$

We have the charge conjugation relations

$$\langle i\pm|\gamma_{\mu_1}\dots\gamma_{\mu_{2n+1}}|j\pm\rangle = \langle j\mp|\gamma_{\mu_{2n+1}}\dots\gamma_{\mu_1}|i\mp\rangle$$

$$\langle i\pm|\gamma_{\mu_1}\dots\gamma_{\mu_{2n}}|j\mp\rangle = -\langle j\pm|\gamma_{\mu_{2n}}\dots\gamma_{\mu_1}|i\mp\rangle$$

By explicit construction we can prove the identity

$$|j+\rangle\langle k-| - |k+\rangle\langle j-| = \langle kj\rangle\frac{1}{2}(1+\gamma_5)$$

which can be used to prove a relation which is extremely useful to simplify analytic results.

$$\langle ij\rangle\langle kl\rangle = \langle il\rangle\langle kj\rangle + \langle ik\rangle\langle jl\rangle$$

From the Fierz transformation ($\gamma_{\pm} = \frac{1}{2}(1 \pm \gamma_5)$).

$$(\gamma^\mu \gamma_+)_{ij} \otimes (\gamma_\mu \gamma_-)_{kl} = 2(\gamma_+)_{kj} \otimes (\gamma_-)_{il}$$

we can prove the simple relation

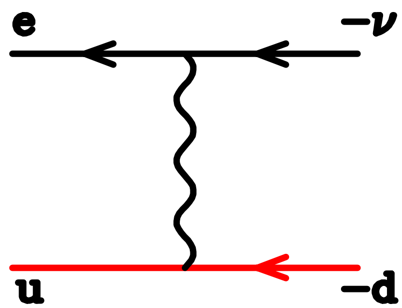
$$\langle i + |\gamma_\mu| j+\rangle \langle k - |\gamma^\mu| l-\rangle = 2\langle kj\rangle [il].$$

So that we also have the identity valid for massless spinors only

$$\begin{aligned} & \langle i \pm |\gamma^\mu| j\pm\rangle \, \gamma_\mu \\ \equiv \quad & \langle i \pm |\gamma^\mu| j\pm\rangle \, \gamma_\mu \gamma_\mp + \langle i \mp |\gamma^\mu| j\mp\rangle \, \gamma_\mu \gamma_\pm \\ = \quad & 2 \Big[\, |i\mp\rangle\langle j \mp | + |j\pm\rangle\langle i \pm | \, \Big] \end{aligned}$$

Examples from W production

As an example of the use of spinor techniques we will consider the production of a vector boson. For the process

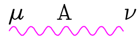


$$d(p_d) + \bar{u}(p_u) \to e^-(p_e) + \bar{\nu}(p_\nu) \; ,$$

where the momentum labels are indicated in brackets,

$$M = \frac{-ig_W^2}{2} \delta^{ij} \langle u - |\gamma_\mu| d- \rangle \langle e - |\gamma^\mu| \nu - \rangle \frac{1}{s_{e\nu} - M_W^2}$$

and $g_W^2 = 4\sqrt{2} G_F M_W^2$, $s_{e\nu} = (p_e + p_\nu)^2$,.

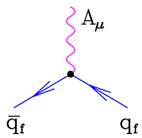


$$\left[-g^{\mu\nu}+\frac{q^{\mu}q^{\nu}}{q^{2}}\right]\frac{i}{q^{2}}$$

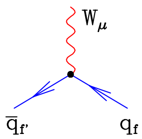
$$\left[-g^{\mu\nu}+\frac{q^{\mu}q^{\nu}}{M^{2}}\right]\frac{i}{(q^{2}-M^{2})}$$

$$\frac{i}{\not{q}-m}$$

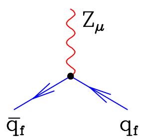
$$\frac{i}{(q^{2}-M_{H}^{2})}$$



$$-ieQ_{f}\gamma_{\mu}$$

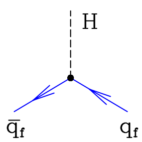


$$\frac{-ig_{w}}{2\sqrt{2}}\,\gamma^{\mu}(1-\gamma_{5})(T^{+})_{f'f} \qquad g_{w}=\frac{e}{\sin\theta_{w}}$$



$$\frac{-ig_{w}}{2\cos\theta_{W}}\,\gamma^{\mu}(V_{f}-A_{f}\gamma_{5})$$

$(V_{f}=T_{f}^{3}-2Q_{f}\sin^{2}\theta_{W},\quad A_{f}=T_{f}^{3})$



$$\frac{-ig_{w}m_{f}}{2M_{W}}$$

Using the Fierz and Charge conjugation identities

$$M = -ig_W^2 \delta^{ij} [d\nu] \langle ue \rangle \frac{1}{s_{e\nu} - M_W^2} \; .$$

Notice how all the gamma-matrix algebra has just collapsed; there is nothing left. Squaring the amplitude and averaging over initial colours and spins we obtain the simple result

$$\overline{\sum} |\mathcal{M}(d\bar{u} \to e^- \bar{\nu})|^2 \quad = \quad \frac{1}{3} g_W^4 |V_{ud}|^2 \, \frac{(u \cdot e)^2}{(s_{e\nu} - M_W^2)^2 + M_W^2 \Gamma_W^2} \; .$$

If we define θ^* to be the e^+ polar angle of emission in the W^+ rest frame, measured with respect to the direction of the incident $\bar{p}$, and if we assume that all incoming quarks (antiquarks) are constituents of the proton (antiproton), then for the above matrix elements we have

$$(u \cdot e)^2 \; = \; \frac{M_W^4}{16} \, (1 + \cos\theta^*)^2 \; .$$

There is a simple angular momentum argument for this. The W couples to negative helicity fermions and positive helicity antifermions. Angular momentum conservation therefore requires the outgoing fermion (electron) to preferentially follow the direction of the incoming fermion (quark), which is usually the direction of the incoming proton.

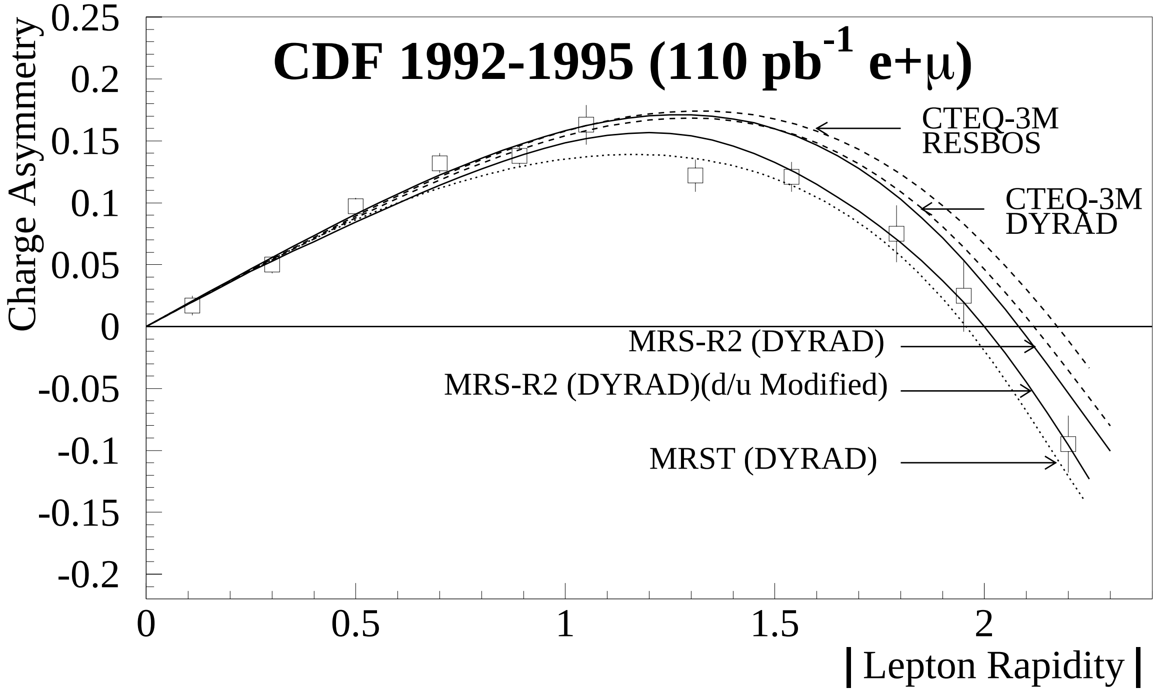


Figure 1: Lepton asymmetry as measured by the CDF collaboration

The asymmetry is controlled both by the matrix element and by the parton distributions.

Polarization vectors

We can introduce a representation of the polarization vectors in terms of the spinors. In terms of the momentum p of the gluon and an arbitrary gauge vector b such that $p \cdot b \neq 0$ we have

$$\varepsilon_\mu^\pm(p, b) = \pm \frac{\langle p \pm |\gamma_\mu| b \pm \rangle}{\sqrt{2} \langle b \mp | p \pm \rangle}$$

Hence we have that

$$\varepsilon_\mu^+(p, b) = \frac{\langle p + |\gamma_\mu| b + \rangle}{\sqrt{2} \langle b p \rangle}, \quad \not{\varepsilon}^+(p, b) = \frac{\sqrt{2}\Big[|p-\rangle \langle b - | + |b+\rangle \langle p + | \Big]}{\langle b p \rangle}$$

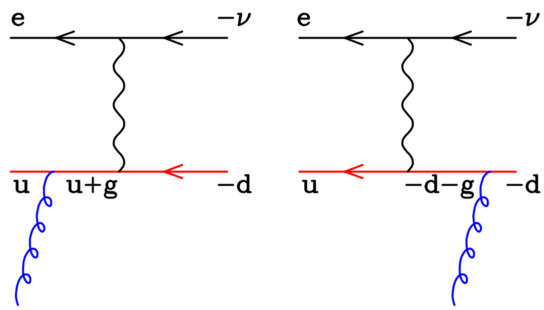
$$\varepsilon_\mu^-(p, b) = \frac{\langle p - |\gamma_\mu| b - \rangle}{\sqrt{2} [p b]}, \quad \not{\varepsilon}^-(p, b) = \frac{\sqrt{2}\Big[|p+\rangle \langle b + | + |b-\rangle \langle p - | \Big]}{[p b]}$$

The polarization vectors are orthogonal to both p and to b.

$$\varepsilon^{\pm\,\mu}(p, b) \varepsilon_\mu^{\pm\,*}(p, b) = -1, \quad \varepsilon^{\pm\,\mu}(p, b) \varepsilon_\mu^{\mp\,*}(p, b) = 0$$

$W + g$ production

We shall now calculate the matrix element with an additional gluon $d + \bar{u} \to e^- + \nu + g$



Leaving out a common overall factor of

$$\frac{\frac{1}{2} g_W^2 g t^A}{(s_{e\nu} - M_W^2)}$$

the matrix element for two diagrams is

$$M \sim \left\{ \langle u - | \not{g} (\not{u} + \not{g}) \gamma^\mu | d - \rangle \frac{1}{[gu]\langle ug \rangle} - \langle u - | \gamma^\mu (\not{d} + \not{g}) \not{g} | d - \rangle \frac{1}{[gd]\langle dg \rangle} \right\}$$

$$\times \quad \langle e - | \gamma_\mu | \nu - \rangle$$

Using the Fierz and Charge conjugation identities

$$M \sim 2\Big\{ \langle u - |\not\epsilon(\not u + \not g)|e+\rangle \frac{[\nu d]}{[gu]\langle ug\rangle} - \langle \nu + |(\not d + \not g)\not\epsilon|d-\rangle \frac{\langle ue\rangle}{[gd]\langle dg\rangle} \Big\}$$

Choose the positive helicity for the emitted gluon

$$\not\epsilon = \frac{\sqrt{2}}{\langle bg\rangle}(|g-\rangle\langle b-| + |b+\rangle\langle g+|)$$

where b is an arbitrary gauge vector chosen to suit our convenience. Then the expression becomes,

$$\begin{aligned} M \quad &\sim \quad \frac{2\sqrt{2}}{\langle ug\rangle\langle dg\rangle\langle bg\rangle}\Big\{ \langle ub\rangle\langle dg\rangle\langle g+|(\not u + \not g)|e+\rangle \frac{[\nu d]}{[gu]} - \langle ug\rangle\langle ue\rangle\langle \nu+|(\not d + \not g)|b+\rangle \Big\} \\ &\sim \quad \frac{2\sqrt{2}}{\langle ug\rangle\langle dg\rangle\langle bg\rangle}\Big\{ \langle ub\rangle\langle dg\rangle\langle ue\rangle[\nu d] - \langle ug\rangle\langle ue\rangle([\nu d]\langle db\rangle + [\nu g]\langle gb\rangle) \Big\} \end{aligned}$$

Using the Schouten identity we may write

$$\langle ub\rangle\langle dg\rangle - \langle ug\rangle\langle db\rangle = \langle ud\rangle\langle bg\rangle$$

$$M^{(+)} \sim \frac{2\sqrt{2}\langle ue\rangle}{\langle ug\rangle\langle dg\rangle}\Big\{\langle ud\rangle[\nu d] + \langle ug\rangle[\nu g]\Big\}$$

$$\sim -\frac{2\sqrt{2}\langle ue\rangle^2[\nu e]}{\langle ug\rangle\langle dg\rangle}\}$$

where we have used momentum conservation to obtain the last result. The final result is independent of the choice of the gauge vector b.

- Note that this result could have been obtained far more simply by setting $b = u$..
- In this case the whole result, including the singularity when $u \cdot g = 0$, would have come from the second graph.
- The corresponding result for the negative helicity is

$$M^{(-)} \sim \frac{2\sqrt{2}[\nu d]^2\langle \nu e\rangle}{[ug][dg]}\}$$

- These results make it clear why spinor techniques are so efficient for QED/QCD.
- Real physical amplitudes contain square root singularities in the region of collinear emission which are compactly described by spinor products.

  Restoring the overall factors and squaring and summing (averaging) over final (initial) colours and spins we obtain,

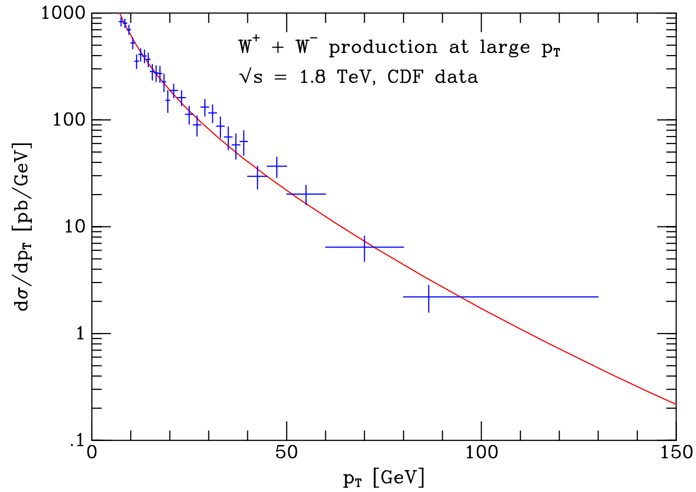
$$\overline{\sum}|M|^2 = \frac{g_W^4 g^2 C_F}{N} |V_{ud}|^2 \Big\{ \frac{\nu \cdot e \Big[(u \cdot e)^2 + (d \cdot \nu)^2 \Big]}{\Big[((e+\nu)^2 - M_W^2)^2 + M_W^2 \Gamma_W^2 \Big] u \cdot g \; d \cdot g} \Big\}$$

  where $N = 3$ is the number of colours.

- most W and Z bosons (collectively denoted by V) are produced with relatively little transverse momentum, *i.e.* $p_T \ll M_V$. However, part of the total cross section corresponds to the production of large transverse momentum bosons. The relevant mechanisms are the $2 \longrightarrow 2$ processes $q\bar{q} \longrightarrow V g$ and $qg \longrightarrow V q$.

$$\begin{aligned}
\overline{\sum}|\mathcal{M}^{q\bar{q}' \to W g}|^2 &= \frac{g^2 g_W^2}{16}|V_{qq'}|^2 \, \frac{8}{9} \, \frac{t^2+u^2+2M_W^2 s}{tu} \, , \\
\overline{\sum}|\mathcal{M}^{gq \to W q'}|^2 &= \frac{g^2 g_W^2}{16}|V_{qq'}|^2 \frac{1}{3} \, \frac{s^2+u^2+2tM_W^2}{-su} \, ,
\end{aligned}$$

with similar results for the Z boson. The transverse momentum distributions $d\sigma/dp_T^2$ are obtained by convoluting these matrix elements with parton distributions in the usual way.



- Data on the p_T distribution of the W boson compared with the next-to-leading-order QCD prediction.

  The poles at $t=0$ and $u=0$ in the matrix elements cause the leading-order theoretical cross section to diverge as $p_T \to 0$. The leading behaviour at small p_T comes from the emission of a soft ($k^\mu \to 0$) gluon in the process $q\bar{q} \to Vg$. Schematically (with $M = M_W$ or M_Z)

$$\frac{d\sigma^R}{dp_T^2} = \alpha_S \left(A \frac{\ln(M^2/p_T^2)}{p_T^2} + B \frac{1}{p_T^2} + C(p_T^2) \right) ,$$

where A and B are calculable coefficients and C is an integrable function.

Virtual corrections to $q\bar{q} \to V$ only contribute at $p_T = 0$, *i.e.* $d\sigma^V/dp_T^2 \propto \delta(p_T^2)$, and their contribution to the differential distribution essentially introduces a 'plus prescription' on the singular parts of the gluon emission cross section:

$$\frac{d\sigma^{R+V}}{dp_T^2} = \alpha_S \left(A \left[\frac{\ln(M^2/p_T^2)}{p_T^2} \right]_+ + B \left[\frac{1}{p_T^2} \right]_+ + \overline{C}(p_T^2) \right) ,$$

such that the integrated contribution is finite,[1]

$$\int dp_T^2 \, \frac{d\sigma^{R+V}}{dp_T^2} = \alpha_S \int dp_T^2 \, \overline{C}(p_T^2) \, .$$

An important ingredient missing from the above derivation is the *non-perturbative* contribution to the p_T distribution. At very small p_T, the intrinsic transverse motion of the quarks and gluons inside the colliding hadrons cannot be neglected. This non-perturbative contribution has to be combined with the perturbative large p_T tail. A simple convolution in transverse-momentum space of the perturbative distribution with non-perturbative intrinsic k_T distributions is one method of doing this.

[1] Recall that $\int_0^1 dx [f(x)]_+ = 0$.

For $p_T \gg \langle k_T \rangle$ a purely perturbative approach should be adequate. However, if at the same time $p_T \ll M$ then higher-order terms in the perturbation series cannot be neglected. In particular, the emission of multiple soft gluons becomes important and the leading contributions at each order have the form

$$\frac{1}{\sigma}\frac{d\sigma}{dp_T^2} \simeq \frac{1}{p_T^2}\left[A_1\alpha_S \ln\frac{M^2}{p_T^2} + A_2\alpha_S^2 \ln^3\frac{M^2}{p_T^2} + \ldots \right.$$
$$\left. + A_n\alpha_S^n \ln^{2n-1}\frac{M^2}{p_T^2} + \ldots\right],$$

where the A_i are calculable coefficients of order unity. The higher-order terms in the series are evidently important when

$$\alpha_S \ln^2\frac{M^2}{p_T^2} > 1 .$$

Taking into account the relative magnitude of the A_n coefficients, this corresponds to p_T values less than $10 - 15$ GeV.

Indeed the resummed double leading logarithms give

$$\frac{1}{\sigma}\frac{d\sigma}{dp_T^2} \simeq \frac{d}{dp_T^2} \exp\left(-\frac{\alpha_S}{2\pi} C_F \ln^2\frac{M^2}{p_T^2}\right),$$

which vanishes at $p_T = 0$.

- the production of a W or Z boson with $p_T \approx 0$ does *not* require that all emitted gluons are soft, merely that their *vector* transverse momentum sum is small.
- the double-leading-logarithm result omits contributions from the multiple emission of soft gluons with $k_{Ti} \sim p_T$ and $\sum_i \vec{k}_{Ti} = \vec{p}_T$. Such additional contributions 'fill in' the dip at $p_T \approx 0$. A more complete analysis of the small p_T distribution requires proper treatment of transverse momentum conservation in multiple gluon emission. This is achieved by introducing the two-dimensional impact parameter vector $\vec{b}$, which is the Fourier conjugate of $\vec{p}_T$, and writing

$$\delta^{(2)}\left(\sum_{i=1}^{n} \vec{k}_{Ti} - \vec{p}_T\right) = \frac{1}{(2\pi)^2} \int d^2b\, e^{-i\vec{b}\cdot\vec{p}_T} \prod_{i=1}^{n} e^{i\vec{b}\cdot\vec{k}_{Ti}} \,,$$

  for the emission of n soft gluons. The Sudakov form factor then appears in b-space after resumming large $\ln(b^2M^2)$ logarithms.
- a complete formalism has been developed for taking account of all large $\ln(M^2/p_T^2)$ logarithms in the perturbation series at small p_T. In particular, the result for the small $p_T(Z)$ distribution is

$$\frac{d\sigma}{dp_T^2} \simeq \sum_q \sigma_0^{q\bar{q}} \frac{1}{2}\int_0^\infty db\; b\; J_0(bp_T)\; \exp(-S(b, M_Z))$$

$$\int_0^1 dx_1\, dx_2\, \delta(x_1 x_2 - \frac{M_Z^2}{s}) \left[q(x_1,(b_0/b)^2)\, \bar{q}(x_2,(b_0/b)^2) + (q \leftrightarrow \bar{q})\right] ,$$

where

$$\sigma_0^{q\bar{q}} = \pi\sqrt{2} G_F M_Z^2 (V_q^2 + A_q^2)/(3s)$$

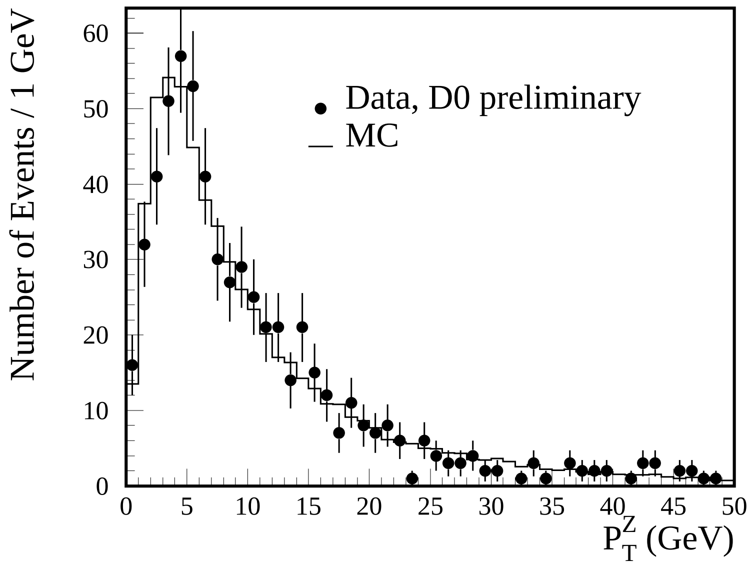
and $b_0 = 2\exp(-\gamma_E)$ ($\gamma_E = 0.5772\ldots$ is the Euler constant). The Sudakov form factor in b-space is $\exp(-S)$ where

$$\begin{aligned}
S(b,Q) &= \int_{(b_0/b)^2}^{Q^2} \frac{dq^2}{q^2} \left[\ln \frac{Q^2}{q^2} A(\alpha_S(q^2)) + B(\alpha_S(q^2))\right] , \\
A(\alpha_S) &= \sum_{n=1}^{\infty} \left(\frac{\alpha_S}{2\pi}\right)^n A^{(n)} , \\
B(\alpha_S) &= \sum_{n=1}^{\infty} \left(\frac{\alpha_S}{2\pi}\right)^n B^{(n)} .
\end{aligned}$$

The $A^{(n)}$ and $B^{(n)}$ coefficients for $n = 1, 2$ have been calculated.

The small-p_T distribution can, in principle, be used as a test of the resummed QCD perturbation series. In practice, however, there are some difficulties — for example, some non-perturbative cut-off or smearing must be included to make the b integral converge at large b and to avoid infra-red problems from evaluating α_S and the parton distributions at low scales. This introduces a significant theoretical uncertainty.

- The theoretical prediction incorporates both the resummed higher-order contributions at small p_T, and the exact next-to-leading-order correction at large p_T.



W mass measurement

- In the rest frame of the decaying W, the energy of the charged lepton is $M_W/2$. It can be used to measure the W mass.
- *transverse* momentum of the electron, also carries information on M_W.
- in the W rest frame the angular distribution of the electron is

$$\frac{1}{\sigma}\frac{d\sigma}{d\cos\theta*} = \frac{3}{8}\left(1+\cos^2\theta^*\right),$$

- If the W has zero transverse momentum

$$\cos\theta^* = \left(1-\frac{4p_{Te}^2}{M_W^2}\right)^{\frac{1}{2}},$$

so that

$$\frac{1}{\sigma}\frac{d\sigma}{dp_{Te}^2} = \frac{3}{M_W^2}\left(1-\frac{4p_{Te}^2}{M_W^2}\right)^{-\frac{1}{2}}\left(1-\frac{2p_{Te}^2}{M_W^2}\right).$$

- The distribution is strongly peaked at $p_{Te} = M_W/2$ (the *Jacobian peak*)

- to include the missing (*i.e.* neutrino) transverse momentum define the *transverse mass*

$$M_T^2 = 2|p_{Te}||p_{T\nu}|(1-\cos\Delta\phi_{e\nu}) \ .$$

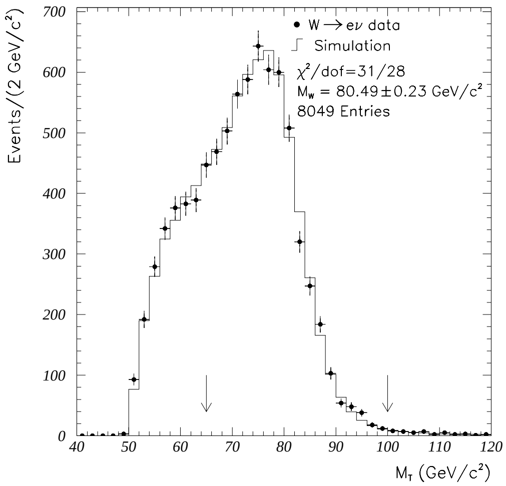
At leading order ($i.e.\ q\bar{q}' \to W \to e\nu$) and in the absence of any quark transverse momentum, we have $|p_{Te}| = |p_{T\nu}| = p^*$, $\Delta\phi_{e\nu} = \pi$, and so $M_T = 2|p_{Te}|$. The transverse mass distribution therefore also has a Jacobian peak, at $M_T = M_W$.

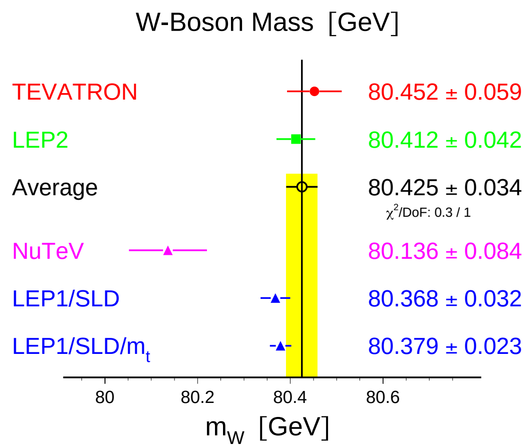
- the transverse mass distribution is that it is less sensitive to the transverse momentum (p_T^W) of the W boson. If p_T^W is small, the transverse momenta of the leptons in the laboratory and W centre-of-mass frames are related by a simple Galilean transformation:

$$\begin{aligned} p_{Te} &= p^* + \frac{1}{2}p_T^W \\ p_{T\nu} &= -p^* + \frac{1}{2}p_T^W \ . \end{aligned}$$

It is straightforward to show that, to leading order in p_T^W, the transverse mass is unchanged by such a transformation.

Current results





Results on W+jets

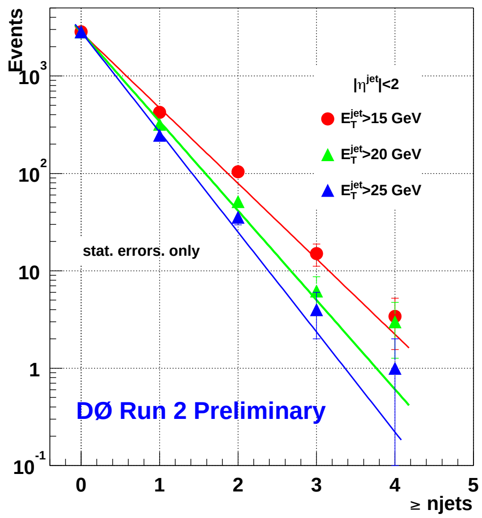
- One of the most important Standard Model processes in high-energy hadron-hadron collisions is the production of a W or Z with accompanying hadronic ($i.e.$ quark or gluon) jets.
- Most 'new physics' processes, for example the production of heavy quarks, Higgs bosons and supersymmetric particles, can be mimicked by the production of vector bosons in association with jets.
- decompose the total W cross section into its multijet components:

$$\sigma_W = \sigma_{W+0j} + \sigma_{W+1j} + \sigma_{W+2j} + \sigma_{W+3j} + \dots$$

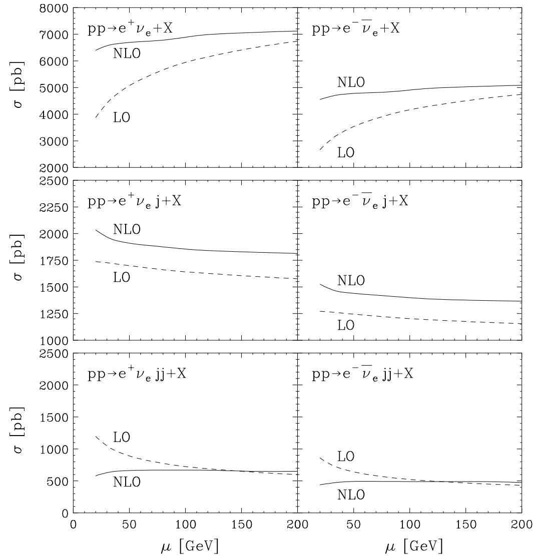
where, schematically,

$$\begin{aligned}
\sigma_{W+0j} &= a_0 + \alpha_S a_1 + \alpha_S^2 a_2 + \dots \\
\sigma_{W+1j} &= \alpha_S b_1 + \alpha_S^2 b_2 + \dots \\
\sigma_{W+2j} &= \alpha_S^2 c_2 + \dots \\
&\dots \quad .
\end{aligned}$$

- The $a_i, b_i, c_i, \ldots$ coefficients in these expansions are in general functions of the jet-definition parameters, in particular the cone size used to cluster the partons into jets, and the transverse momentum, rapidity and separation cuts imposed on the jets/clusters.
- The leading contributions to the cross sections, $a_0, b_1, c_2, \ldots$, can be calculated from the matrix elements for the 'tree-level' parton processes:



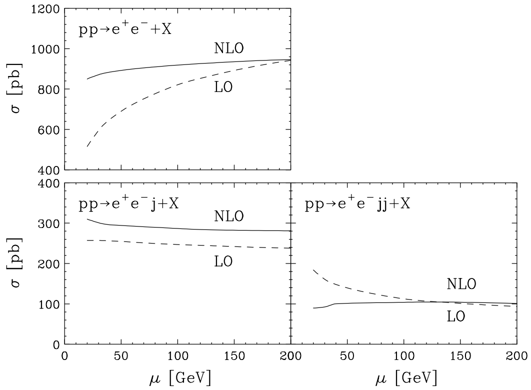
Vector boson + n jets



- Next-to-leading-order corrections are at present known only for the $W+0,1$ and 2 jet production cross sections
- NLO results for W production at LHC, showing improved stability under variations of the renormalization and factorization scale. Corrections are sometimes large.

Results on Z + jet production

- A similar pattern in the μ dependence for Z production



Recap

- Hard processes can be calculated using a factorized form, with parton distribution functions and short distance cross sections.
- Parton luminosities often control the size of cross sections.
- Spinor techniques give compact answers for QCD amplitudes, because QCD amplitudes contain square root singularities.
- W transverse momentum distribution requires resummation of multi-gluon emission.
- NLO predictions are only available for a limited number of background processes. There is much work to be done.